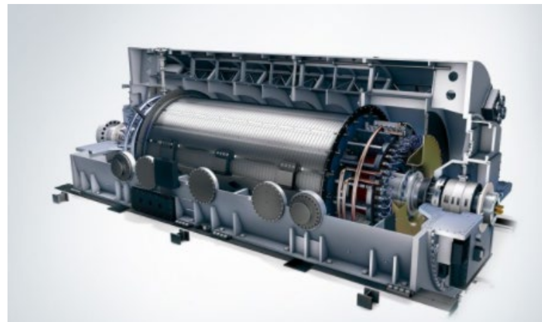
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 8: Rigid Body Dynamics
Due Friday April 29 2022



1 The 'Inertia constant' of a grid scale electric generator (or the combined generator and turbine) is the ratio of the kinetic energy stored in the spinning generator to the power output of the generator. It is an important metric in the design of control systems that synchronize AC generators connected to the grid. This NREL report has a nice overview of the challenges involved.

(In this problem the prefix G on a unit is 'Giga' or 10^9 – and the prefix M is 'Mega' or 10^6)

1.1 Calculate the mass moment of inertia of a 3000rpm, 1GW generator with an inertia constant of 5 seconds

The KE of the generator is $T = \frac{1}{2} I_{Gzz} \omega_z^2$ and we know $T/10^9 = 5$, so $I_{Gzz} = \frac{2 \times 5 \times 10^9}{(2\pi \times 3000/60)^2} = 101.32 \times 10^3 \, kgm^2$

[2 POINTS]

1.2 If the generator is driven by a turbine that provides a constant torque (unlikely in reality!) how long will it take for the (unloaded) generator to spin up from rest to 3000rpm?

The torque-power formula is $P = Q\omega$ so $Q = P/\omega = 10^9 / (2\pi.3000/60) = 3.183 \times 10^6 \, Nm$

The angular acceleration follows as $\alpha = Q / I_{zz} = 31.4 rad/s^2$

The constant acceleration formula gives $\omega = \alpha t \Rightarrow t = \frac{(3000 \times 2\pi / 60)}{31.4} = 10s$

[2 POINTS]

1.3 The grid frequency is required to stay within +/-0.5% of 60Hz (the frequency of an AC generator is proportional to its angular speed). Suppose that a 750MW load is connected to the generator while it runs at 3000rpm. If the turbine driving the generator loses power, and the load power remains constant, how long will it take for the generator's angular speed to drop by 0.5%?

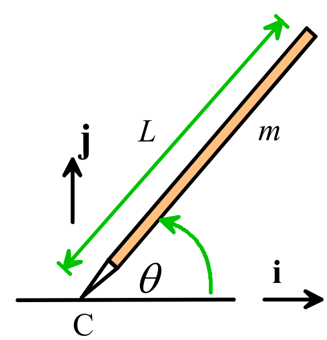
The change in KE of the generator must equal the work done on the system (-power output x time). Hence

$$\frac{1}{2}I_{Gzz}\left(\omega_1^2-\omega_0^2\right)=\frac{1}{2}I_{Gzz}\omega_0^2\left(0.95^2-1\right)=-Pt$$

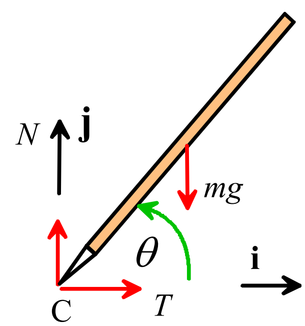
$$\Rightarrow t=-\frac{5\times10^9}{750\times10^6}\left(0.995^2-1\right)=0.065\,\text{sec}$$

[2 POINTS]

2. A pencil with length L starts at rest with a vertical orientation ($\theta=\pi/2$) on a surface with friction coefficient μ. A small disturbance causes it to tip over. The goal of this problem is to calculate the critical angle θ at the instant that the contact starts to slip.



2.1 Draw a free body diagram showing the forces acting on the pencil. Assume no slip.



[3 POINTS]

2.2 Write down the equations of translational and rotational motion for the pencil (i.e. $\mathbf{F}=m\mathbf{a}_G$, $\sum\mathbf{r}\times\mathbf{F}=m\mathbf{r}_G\times\mathbf{a}_G+I_{Gzz}\alpha\mathbf{k}$ or if you prefer $\sum\mathbf{r}\times\mathbf{F}=I_{Czz}\alpha\mathbf{k}$). Please state which point you choose to use to calculate the moment and angular momentum. Any choice will work, but some choices make the algebra simpler than others. You could try several points and then use the one that you find most helpful.

Newton: $T\mathbf{i}+(N-mg)\mathbf{j}=m\mathbf{a}_G$

Moments about C: $-mg\dfrac{L}{2}\cos\theta\mathbf{k}=I_{Czz}\alpha_z\mathbf{k}$ $\qquad I_{Czz}=\dfrac{mL^2}{12}+m\left(\dfrac{L}{2}\right)^2=m\dfrac{L^2}{3}$

Moments about COM $\left(T\dfrac{L}{2}\cos\theta-N\dfrac{L}{2}\sin\theta\right)\mathbf{k}=I_{Gzz}\alpha_z\mathbf{k}$ $\qquad I_{Gzz}=\dfrac{mL^2}{12}$

[3 POINTS]

2.3 Write down the kinematics equation relating the angular acceleration, angular velocity, and linear acceleration at the COM of the pencil (assume no slip).

$$\mathbf{a}_G - \mathbf{a}_C = \alpha_z \mathbf{k} \times \frac{L}{2}(\cos\theta \mathbf{i} + \sin\theta \mathbf{j}) - \omega^2 \frac{L}{2}(\cos\theta \mathbf{i} + \sin\theta \mathbf{j})$$

where $\mathbf{a}_C = \mathbf{0}$ since C is stationary.

[2 POINTS]

2.4 Use 2.2 (and if need be 2.3) to show that the angular speed of the pencil is related to the angle θ by

$$\omega \frac{d\omega}{d\theta} = -\frac{3g}{2L}\cos\theta$$

By separating variables and integrating, find an expression for the angular velocity as a function of θ. Check your answer using energy.

The moment – d**h**/dt relation about C gives directly

$$-mg\frac{L}{2}\cos\theta = m\frac{L^2}{3}\frac{d\omega}{dt} = m\frac{L^2}{3}\omega\frac{d\omega}{d\theta}$$

$$\Rightarrow \omega\frac{d\omega}{d\theta} = -\frac{3g}{2L}\cos\theta$$

Note that $\omega = 0$ at $\theta = \pi/2$; so separating variables and integrating

$$\int_0^\omega \omega d\omega = -\int_{\pi/2}^{\theta} \frac{3g}{2L}\cos\theta d\theta \Rightarrow \frac{1}{2}\omega^2 = -\frac{3g}{2L}(\sin\theta - \sin(\pi/2))$$

$$\Rightarrow \omega = \sqrt{\frac{3g}{L}(1-\sin\theta)}$$

We can also use energy – take the pencil + earth as system, so no external forces; the system is conservative therefore $T+U$=constant. The initial PE is mg$L$/2 and initial KE is zero; the final KE is $\frac{1}{2}I_{Czz}\omega^2$ and final PE is $mg(L/2)\sin\theta$. Hence

$$mgL/2 = \frac{1}{2}\frac{mL^2}{3}\omega^2 + mg(L/2)\sin\theta$$

which rearranges to the same answer.

[3 POINTS]

2.5 Use 2.2-2.4 to find formulas for the reaction forces at the contact as functions of θ. Hence, show that the contact will slip if

$$\frac{|3(3\sin\theta - 2)\cos\theta|}{4 - 3\cos^2\theta + 6\sin\theta(\sin\theta - 1)} > \mu$$

Hence, plot a graph showing the critical angle θ as a function of μ (assume $0 < \mu < 1$). It is easiest to do the plot by choosing a value of θ and then calculating the corresponding value of μ. You may find it helpful to plot T and N (and possibly |T|/N) as functions of θ to help understand the plot. Show that if μ exceeds a critical value, the slip direction of the pencil over the paper will be to the right (in the +**i** direction), while for μ below the critical value the pencil will slip to the left (in the –**i** direction). You should be able to verify this experimentally!

From 2.2 $T\mathbf{i} + (N - mg)\mathbf{j} = m\mathbf{a}_G$ and 2.3

$$T\mathbf{i} + (N - mg)\mathbf{j} = m\alpha_z \frac{L}{2}(\cos\theta\mathbf{j} - \sin\theta\mathbf{i}) - m\omega^2 \frac{L}{2}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

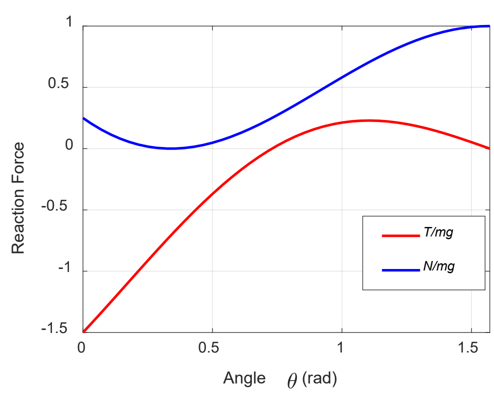
and finally using 2.4

$$T\mathbf{i} + (N - mg)\mathbf{j} = -m\frac{3g}{2L}\cos\theta\frac{L}{2}(\cos\theta\mathbf{j} - \sin\theta\mathbf{i}) + m\frac{3g}{L}(\sin\theta - 1)\frac{L}{2}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\Rightarrow T = \frac{3mg}{2}(\frac{3}{2}\sin\theta - 1)\cos\theta \qquad N = mg - \frac{3mg}{4}\cos^2\theta + \frac{3mg}{2}\sin\theta(\sin\theta - 1)$$

The contact will slip if, $|T| > \mu N$

$$\frac{|3(3\sin\theta - 2)\cos\theta|}{4 - 3\cos^2\theta + 6\sin\theta(\sin\theta - 1)} > \mu$$

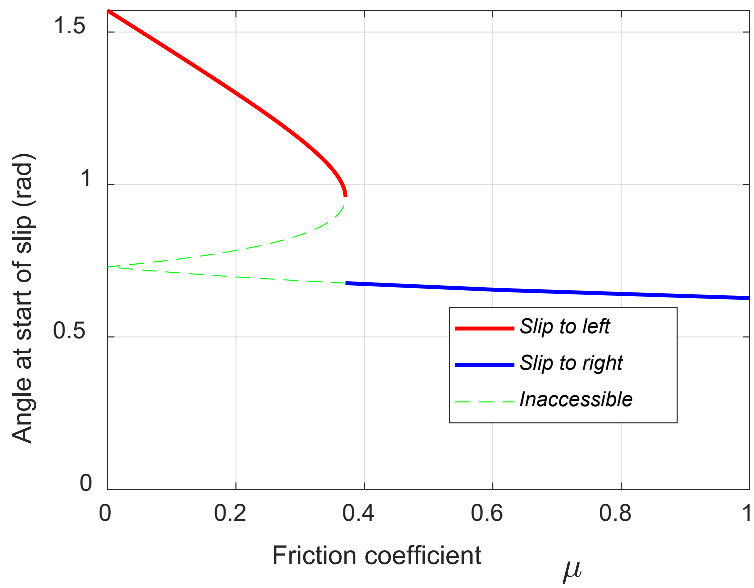


The top figure shows the reaction forces at the contact as a function of θ, the one below shows the ratio $|T|/N$ as a function of θ. Note that θ decreases with time, so at time $t=0$ $N=mg$ and $T=0$. As the pencil falls, the normal force decreases, and the tangential force increases. The tangential force is positive, meaning that it acts in the positive **I** direction. If the friction force is small enough (below the peak value of T/N), then $|T| > \mu N$ at some critical angle to the right of the peak value (about 0.86 radians). Since $T>0$ in this region the pencil will slip to the left (in the opposite direction to T). Once slip does start, the equations of motion for the pencil change, so we can no longer use the predictions of 2.2-2.4 – we have to re-analyze the motion with new EOM, which is quite difficult. But it's reasonable to assume that if slip starts, it will continue until the pencil hits the ground.

For values of μ above the peak value of T/N, slip will not start while $T>0$. But we notice that as θ.drops below about 0.6 radians, the ratio $|T|/N$ increases again – and since N drops to zero at about 0.4

radians eventually becomes infinite. This means the pencil will always slip for any finite value of μ. Since $T<0$, the pencil point will now slip to the right (again, in the opposite direction to T).

Finally, we can summarize these conclusions in the plot requested in the question.



[5 POINTS]

2.6 Would a pencil on a frictionless surface fall over more quickly than a pencil on a surface with a high friction coefficient?

The easiest way to answer this is to calculate the angular speed of a pencil on a frictionless surface. For this case there is no horizontal force acting on the pencil, so the COM must have zero horizontal component of acceleration and velocity. The height of the COM above the ground is $h=(L/2)\sin\theta$. Its vertical velocity is therefore $v_y = dh/dt = (L/2)\cos\theta\omega$. We can use energy to find the angular velocity. We can use the same method as in part 2.4, except that we have to use the general formula for the KE instead of the special formula for rotation about a stationary point. This gives

$$\frac{1}{2}\frac{m}{12}L^2\omega^2 + \frac{1}{2}mv_{Gy}^2 + mg(L/2)\sin\theta = mgL/2$$

$$\Rightarrow \frac{1}{2}\frac{m}{12}L^2\omega^2 + \frac{1}{2}m\left(\frac{L}{2}\cos\theta\omega\right)^2 = mg(L/2)(1-\sin\theta)$$

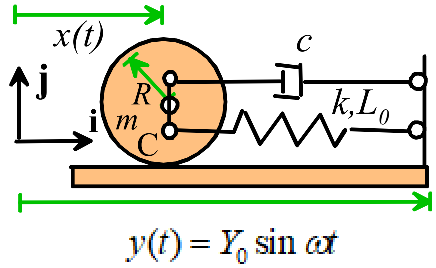
$$\Rightarrow \omega = -\sqrt{\frac{12g(1-\sin\theta)}{(1+3\cos^2\theta)L}}$$

(we took the negative root to match the direction the pencil is falling in the picture, but this is not important) Since $12/(1+3\cos^2\theta) \leq 3$ the magnitude of the angular velocity on the frictionless surface is greater than on the rough surface. The pencil on a frictionless surface therefore falls over more quickly

[3 POINTS]

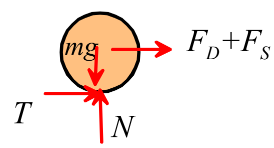
3. The figure shows a wheel with mass m and radius R attached by a spring and damper to a vibrating platform. The base vibrates horizontally with a harmonic displacement

$$y(t) = Y_0 \sin \omega t$$



The wheel rolls without slip. The goal of this problem is to find a formula for the steady state amplitude of vibration X_0 of the wheel.

3.1 Draw a free body diagram showing the forces acting on the wheel. Assume no slip at the contact at C.



[3 POINTS]

3.2 Write down the equations of translational and rotational motion for the wheel (i.e. $\mathbf{F} = m\mathbf{a}_G, \quad \sum \mathbf{r} \times \mathbf{F} = m\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha \mathbf{k}$). Please state which point you choose to use to calculate the moment and angular momentum.

$$(T + F_D + F_S)\mathbf{i} + (N - mg)\mathbf{j} = ma_{Gx}\mathbf{i}$$

Where $F_S = k(y - x - L_0)$ $\qquad F_D = c\left(\frac{dy}{dt} - \frac{dx}{dt}\right)$

Moments about C:

$$R\mathbf{j} \times ma_{Gx}\mathbf{i} + \frac{1}{2}mR^2\alpha_z\mathbf{k} = -(F_D + F_S)R\mathbf{k} \Rightarrow -mRa_{Gx} + \frac{1}{2}mR^2\alpha_z = -(F_D + F_S)R$$

Or, moments about COM $\frac{1}{2}mR^2\alpha_z\mathbf{k} = TR\mathbf{k}$

[3 POINTS]

3.3 Write down the kinematic equation relating the acceleration of the platform $a_{platform} = d^2y/dt^2$, the acceleration of the wheel, and the angular acceleration of the wheel.

Using the rolling wheel formulas

$$a_{Gx} = a_{platform} - \alpha_z R$$

[2 POINTS]

3.4 Use the results of 3.1-3.3 to show that the equation of motion for x can be arranged in the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C+K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right)$$

and find formulas for $K, C, \omega_n, \zeta, \lambda$

Solve 3.3 for α_z and substitute into the angular momentum about C from 3.2

$$\begin{aligned}
&-mRa_{Gx}+\frac{1}{2}mR^2\frac{(a_{platform}-a_{Gx})}{R}=-(F_D+F_S)R \\
&\Rightarrow \frac{3}{2}ma_{Gx}-(F_D+F_S)=\frac{1}{2}ma_{platform} \\
&\Rightarrow \frac{3}{2}\frac{m}{k}\frac{d^2x}{dt^2}+\frac{c}{k}\frac{dx}{dt}+x=\frac{1}{2}\frac{m}{k}\frac{d^2y}{dt^2}+\frac{c}{k}\frac{dy}{dt}+y-L_0 \\
&\Rightarrow \frac{3}{2}\frac{m}{k}\frac{d^2x}{dt^2}+\frac{2c}{\sqrt{6mk}}\sqrt{\frac{3m}{2k}}\frac{dx}{dt}+x=\frac{1}{3}\frac{3}{2}\frac{m}{k}\frac{d^2y}{dt^2}+\frac{2c}{\sqrt{6mk}}\sqrt{\frac{3m}{2k}}\frac{dy}{dt}+y-L_0
\end{aligned}$$

Matching coefficients with the equation given in the problem we have

$$\omega_n=\sqrt{\frac{2k}{3m}} \qquad \zeta=\frac{c}{\sqrt{6km}} \qquad K=1 \quad \lambda=\frac{1}{\sqrt{3}} \qquad C=-L_0$$

[3 POINTS]

3.5 Hence, write down a formula for the steady-state amplitude of vibration of the wheel in terms of $K, C, \omega_n, \zeta, \lambda$ and Y_0. If ζ is small, what is the anti-resonant frequency?

Using the formulas

$$X_0=KY_0M(\omega/\omega_n,\zeta) \qquad M(\omega/\omega_n,\zeta)=\frac{\left\{\left(1-\lambda^2\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}$$

The anti-resonance occurs when $\omega=\omega_n/\lambda=\sqrt{2k/m}$

[3 POINTS]

4. The figure (from this NASA report) shows an experiment designed to measure the mass moment of inertia (about the yaw axis) of a prototype for a crew return vehicle for the international space station. In the experiment, the vehicle and test stand vibrate through a small angle about the yaw axis, and the natural frequency is measured. The mass moment of inertia about the yaw axis can be calculated from the measured frequency. Our goal in this problem is to derive the relevant equation needed to do this.

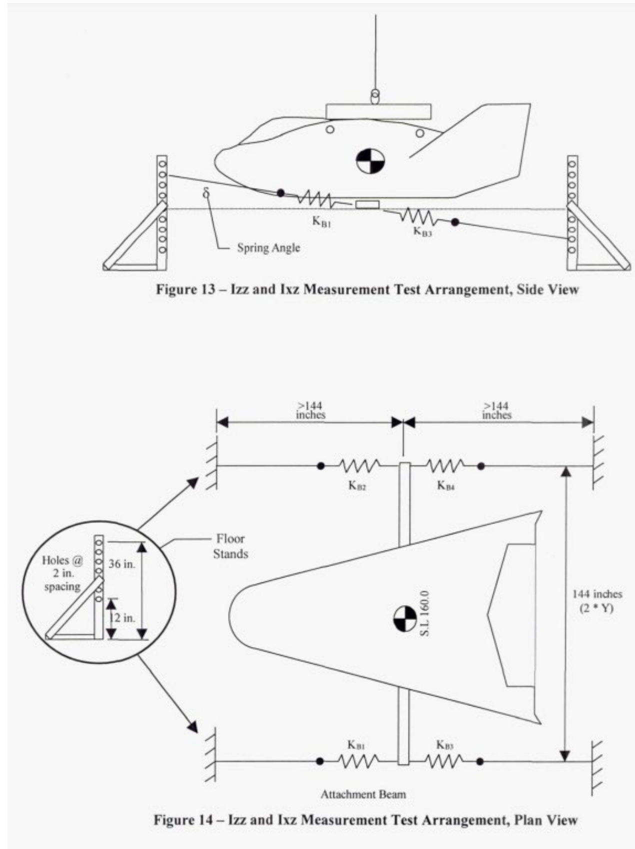
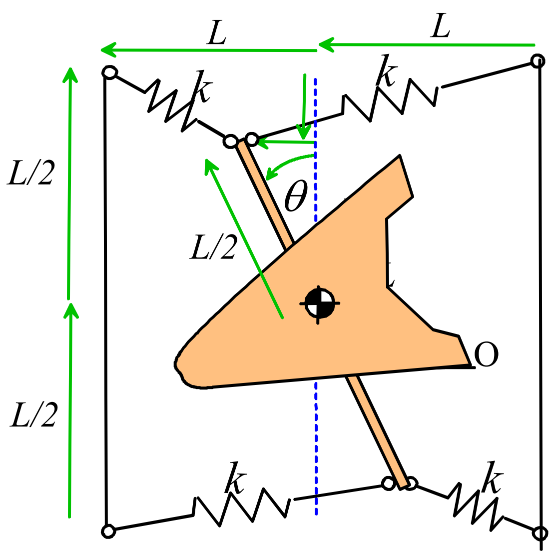


Figure 13 – Izz and Ixz Measurement Test Arrangement, Side View

Figure 14 – Izz and Ixz Measurement Test Arrangement, Plan View

In the real experiment the angle δ of the springs is adjusted until the vehicle vibrates only in yaw, with no pitch or roll, since the principal axes of inertia may not be perfectly aligned with the vertical. To keep things simple, assume that the principal axis is vertical, and $\delta = 0$. Assume also that the four springs have the same stiffness k, and that all the dimensions marked as 144 inches (or greater) are equal (denote the 144 inches by L in your calculation).

Suppose that the vehicle and test stand rotate through an angle θ about the vertical axis, with the COM of the vehicle fixed. Derive the equation of motion for θ. Linearize the equation for small θ, and hence find a formula relating the mass moment of inertia about the vertical axis to the natural frequency of vibration.



We can use the energy method to get the EOM.

The figure shows the geometry we need to calculate the stretched lengths of the springs, which are

$$\sqrt{\frac{L^2}{4}\left(1-\cos\theta\right)^2+L^2(1+\frac{1}{2}\sin\theta)^2}=L\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}$$ for the lengthened spring

and $\sqrt{\frac{L^2}{4}\left(1-\cos\theta\right)^2+L^2(1-\frac{1}{2}\sin\theta)^2}=\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}$ for the shortened one.

The kinetic energy is $T=\frac{1}{2}I_{Gzz}\left(\frac{d\theta}{dt}\right)^2$

The potential energy is $U=k\left\{L\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}-L_0\right\}^2+k\left\{L\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}-L_0\right\}^2$

where L_0 is the unstretched length of the springs.

We know $T+U$ is constant, and therefore (using the chain rule about a million times)

$$\frac{d}{dt}(T+U)=I_{Gzz}\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}+$$

$$kL\left\{L\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}-L_0\right\}\frac{\cos\theta+\frac{1}{2}\sin\theta}{\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}}\frac{d\theta}{dt}$$

$$-kL\left\{\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}-L_0\right\}\frac{\cos\theta-\frac{1}{2}\sin\theta}{\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}}\frac{d\theta}{dt}=0$$

Canceling the $d\theta/dt$ and rearranging gives

$$I_{Gzz}\frac{d^2\theta}{dt^2}+kL^2\left\{\frac{L_0}{L\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}}-\frac{L_0}{L\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}}\right\}\cos\theta$$

$$+\frac{kL^2}{2}\left\{2-\frac{L_0}{L\sqrt{\sin\theta-\frac{1}{2}\cos\theta+\frac{3}{2}}}-\frac{L_0}{L\sqrt{\frac{3}{2}-\sin\theta-\frac{1}{2}\cos\theta}}\right\}\sin\theta=0$$

[4 POINTS]

(Any algebraically equivalent result should get credit, of course)

Linearize with the usual small angle approximations $\cos\theta = 1, \quad \sin\theta = \theta$ use $(1+\varepsilon)^{-1/2} \approx 1 - \varepsilon / 2$ for small ε (or, if you prefer, just work out the full derivative) and drop terms of order θ^2 and higher

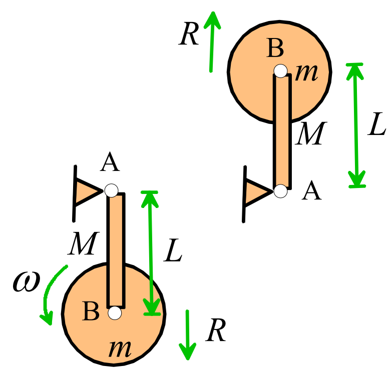
$$I_{Gzz}\frac{d^2\theta}{dt^2} + kL^2\left\{\frac{L_0}{L}(1+\frac{\theta}{2}) - \frac{L_0}{L}(1-\frac{\theta}{2})\right\} + \frac{kL^2}{2}\left\{2 - \frac{L_0}{L} - \frac{L_0}{L}\right\}\theta = 0$$

$$\Rightarrow I_{Gzz}\frac{d^2\theta}{dt^2} + kL^2\theta = 0 \Rightarrow \frac{I_{Gzz}}{kL^2}$$

The natural frequency is therefore $\omega_n = L\sqrt{k / I_{Gzz}} \Rightarrow I_{Gzz} = kL^2 / \omega_n^2$

[3 POINTS]

5 A disk of mass m and radius R is attached to a rod AB with mass M and length L, which is suspended from a frictionless pivot at A. For time $t<0$ the disk spins with angular speed ω and AB is stationary, with the center of the disk (B) vertically below the pivot (A). The disk is then braked so that rod and disk rotate with the same angular speed. The system is observed to come to rest with B vertically above A (both AB and the disk are stationary). The goal of this problem is to find the critical value of the angular speed ω necessary to achieve this.



5.1 Write down the initial angular momentum of the system about the pivot at A

Using the formula $\mathbf{h}_0 = \frac{1}{2}mR^2\omega\mathbf{k}$

[1 POINT]

5.2 Find a formula for the total mass moment of inertia I_A about O of the rigidly connected disk and link (after the wheel is braked)

Using the parallel axis theorem

$$I_O = \frac{1}{2}mR^2 + mL^2 + \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = m(\frac{1}{2}R^2 + L^2) + \frac{1}{3}ML^2$$

[1 POINT]

5.3 Let Ω denote the angular speed of link and disk just after the disk is braked. Write down the total angular momentum about A and energy of the system at this instant, in terms of Ω, I_O, M, L and m.

The angular momentum is $\mathbf{h}_1 = I_O \Omega \mathbf{k}$

The energy (taking the datum for PE at A) is $T + U = \frac{1}{2} I_O \Omega^2 - mgL - \frac{1}{2} MgL$

[2 POINTS]

5.4 Hence, use angular momentum and energy conservation after braking to show that the critical speed is

$$\omega = \frac{2}{R^2} \sqrt{2gL\left(\frac{1}{2}R^2 + L^2 + \frac{1}{3}\frac{M}{m}L^2\right)\left(2 + \frac{M}{m}\right)}$$

Energy conservation after braking gives

$$\frac{1}{2} I_O \Omega^2 - mgL - \frac{1}{2} MgL = mgL + \frac{1}{2} MgL \Rightarrow \Omega = \sqrt{\frac{2gL(2m+M)}{I_O}}.$$

Angular momentum is conserved about A during braking, which gives $\frac{1}{2} mR^2 \omega = I_0 \Omega$. Hence

$$\omega = \left(2I_0 / mR^2\right)\Omega = \frac{2}{R^2}\sqrt{\frac{2gLI_0(2m+M)}{m^2}} = \frac{2}{R^2}\sqrt{2gL\left(\frac{1}{2}R^2 + L^2 + \frac{1}{3}\frac{M}{m}L^2\right)\left(2 + \frac{M}{m}\right)}$$

[2 POINTS]