Chapter 3

Analyzing motion of systems of particles

In this chapter, we shall discuss

1. The concept of a particle
2. Position/velocity/acceleration relations for a particle
3. Newton's laws of motion for a particle
4. How to use Newton's laws to calculate the forces needed to make a particle move in a particular way
5. How to use Newton's laws to derive `equations of motion' for a system of particles
6. How to solve equations of motion for particles by hand or using a computer.

The focus of this chapter is on setting up and solving equations of motion – we will not discuss in detail the behavior of the various examples that are solved.

3.1 Equations of motion for a particle

We start with some basic definitions and physical laws.

3.1.1 Definition of a particle

A `Particle' is a point mass at some position in space. It can move about, but has no characteristic orientation or rotational inertia. It is characterized by its mass.

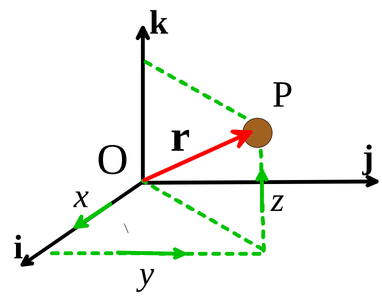
Examples of applications where you might choose to idealize part of a system as a particle include:

1. Calculating the orbit of a satellite – for this application, you don't need to know the orientation of the satellite, and you know that the satellite is very small compared with the dimensions of its orbit.
2. A molecular dynamic simulation, where you wish to calculate the motion of individual atoms in a material. Most of the mass of an atom is usually concentrated in a very small region (the nucleus) in comparison to inter-atomic spacing. It has negligible rotational inertia. This approach is also sometimes used to model entire molecules, but rotational inertia can be important in this case.

Obviously, if you choose to idealize an object as a particle, you will only be able to calculate its position. Its orientation or rotation cannot be computed.

3.1.2 Position, velocity, acceleration relations for a particle (Cartesian coordinates)

In most practical applications we are interested in the *position* or the *velocity* (or speed) of the particle as a function of time. But Newton's laws will only tell us its acceleration. We therefore need equations that relate the position, velocity and acceleration.



Position vector: In Newtonian physics we have to measure position and motion in a so-called 'Inertial Frame'. This concept will be discussed in more detail in Section 3.2. For now, suppose that we can identify

1. Three, mutually perpendicular, fixed directions in space: the three directions are described by unit vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
2. We choose a convenient stationary (or non-accelerating) point to use as origin.

The position vector (relative to the origin) is then specified by the three distances (*x*,*y*,*z*) shown in the figure.

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

In dynamics problems, *x*,*y*,*z* can all be functions of time, but $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are fixed.

Velocity vector: By definition, the velocity is the derivative of the position vector with respect to time (following the usual machinery of calculus)

$$\mathbf{v} = \lim_{\delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\delta t}$$

Velocity is a vector, and can therefore be expressed in terms of its Cartesian components

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

You can visualize a velocity vector as follows

- The *direction* of the vector is parallel to the direction of motion
- The *magnitude* of the vector $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the speed of the particle (in meters/sec, for example).

When both position and velocity vectors are expressed in terms Cartesian components, it is simple to calculate the velocity from the position vector. For this case, the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are *constant* (independent of time) and so

$$v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} = \frac{d}{dt}\left(x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

This is really three equations – one for each velocity component, i.e.

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

Acceleration vector: The acceleration is the derivative of the velocity vector with respect to time; or, equivalently, the second derivative of the position vector with respect to time.

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\mathbf{v}(t+\delta t) - \mathbf{v}(t)}{\delta t}$$

The acceleration is a vector, with Cartesian representation $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$.

Like velocity, acceleration has magnitude and direction. Sometimes it may be possible to visualize an acceleration vector – for example, if you know your particle is moving in a straight line, the acceleration vector must be parallel to the direction of motion; or if the particle moves around a circle at constant speed, its acceleration is towards the center of the circle. But sometimes you can't trust your intuition regarding the magnitude and direction of acceleration, and it can be best to simply work through the math.

The relations between Cartesian components of position, velocity and acceleration are

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

3.1.3 Review of some ideas from calculus

To analyze motion, we often have to calculate accelerations given position vectors as a function of time, or (more commonly) calculate velocity and position by integrating accelerations.

You should have been beaten into submission with this sort of problem in calculus courses (and maybe some physics courses as well) so we'll just review the most important procedures here as a reminder. We will focus on the generic one-dimensional problem: given a, calculate v and x; or vice-versa.

Calculating positions, velocities and accelerations graphically

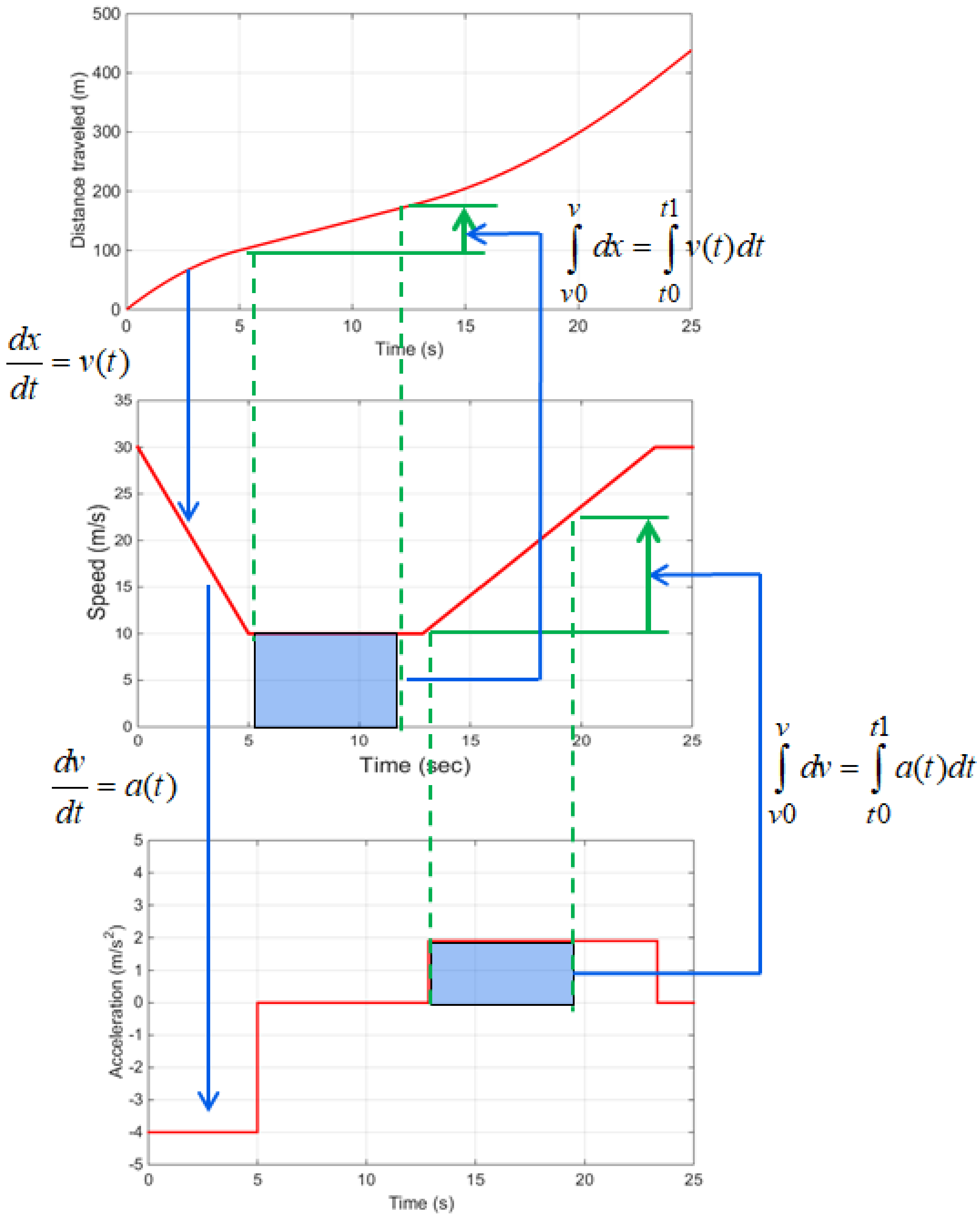
You will remember that:

- Speed is the slope of the distance-v-time curve
- Distance is the area under the speed-v-time curve

Or alternatively

- Acceleration is the slope of the speed-v-time curve
- Speed is the area under the acceleration-v-time curve

These ideas are illustrated in the figure below: if you can sketch a graph of acceleration, velocity or position, you can often use geometry to calculate all the other quantities of interest.



Rules for integrating accelerations and speeds

- Acceleration given as a function of time

$$\frac{dv}{dt} = a(t) \quad \Rightarrow \quad \int_{v0}^{v} dv = \int_{0}^{t} a(t)dt$$

Example:

$$\frac{dv}{dt} = \sqrt{t} \quad \Rightarrow \quad v = v_0 + \frac{2}{3} t^{3/2}$$

- Acceleration given as a function of speed

$$\frac{dv}{dt} = g(t) f(v) \quad \Rightarrow \quad \int_{v0}^{v} \frac{dv}{f(v)} = \int_0^t g(t) dt$$

Example:

$$\frac{dv}{dt} = -cv \quad \Rightarrow \quad \int_{v0}^{v} \frac{dv}{v} = -\int_0^t c dt \;\Rightarrow\; \log\left(\frac{v}{v_0}\right) = -ct$$

- Acceleration given as a function of distance

$$\frac{dv}{dt} = f(x) \quad \Rightarrow \quad \frac{dv}{dx}\frac{dx}{dt} = f(x) \Rightarrow \frac{dv}{dx} v = f(x) \Rightarrow \int_{v0}^{v} v dv = \int_0^x f(x) dx$$

Example

$$\frac{dv}{dt} = -kx \quad \Rightarrow \quad \int_{v0}^{v} v dv = \int_0^x -kx dx \quad \Rightarrow \quad \frac{1}{2} v^2 - \frac{1}{2} v_0^2 = -\frac{1}{2} kx^2 \;\Rightarrow v = \sqrt{v_0^2 - kx^2}$$

- Velocity given as a function of time

$$\frac{dx}{dt} = v(t) \quad \Rightarrow \quad \int_{x0}^{x} dx = \int_0^t v(t) dt$$

Example

$$v = v_0 + \frac{2}{3} t^{3/2} \Rightarrow x = x_0 + v_0 t + \frac{4}{15} t^{5/2}$$

- Velocity given as a separable function of position and time

$$\frac{dx}{dt} = g(t) f(x) \quad \Rightarrow \quad \int_{v0}^{v} \frac{dx}{f(x)} = \int_0^t g(t) dt$$

Example

$$\frac{dx}{dt} = \sqrt{v_0^2 - kx^2} \quad \Rightarrow \quad \int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 - kx^2}} = \int_0^t dt$$

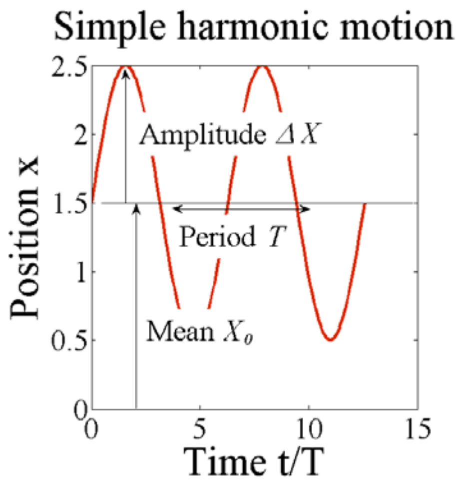
$$\Rightarrow \frac{1}{\sqrt{k}} \left\{ \sin^{-1}\left(\frac{\sqrt{k}\,x}{v_0}\right) - \sin^{-1}\left(\frac{\sqrt{k}\,x_0}{v_0}\right) \right\} = t$$

$$\Rightarrow x = \frac{v_0}{\sqrt{k}} \sin\left(\sqrt{k}\,t + \sin^{-1}\left(\frac{\sqrt{k}\,x_0}{v_0}\right)\right)$$

3.1.4 Examples using position-velocity-acceleration relations

It is important for you to be comfortable with calculating velocity and acceleration from the position vector of a particle. You will need to do this in nearly every problem we solve. In this section we provide a few examples. Each example gives a set of formulas that will be useful in practical applications.

Example 1: Constant acceleration along a straight line. There are many examples where an object moves along a straight line, with constant acceleration. Examples include free fall near the surface of a planet (without air resistance), the initial stages of the acceleration of a car, or and aircraft during takeoff roll, or a spacecraft during blastoff.



Suppose that

- The particle moves parallel to a unit vector **i**
- The particle has constant acceleration, with magnitude a
- At time $t = t_0$ the particle has speed v_0
- At time $t = t_0$ the particle has position vector $\mathbf{r} = x_0\mathbf{i}$

The position, velocity acceleration vectors are then

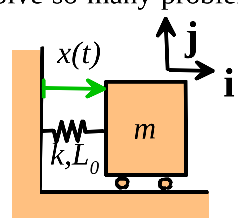
$$\mathbf{r} = \left(x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \right)\mathbf{i}$$

$$\mathbf{v} = (v_0 + at)\mathbf{i}$$

$$\mathbf{a} = a\mathbf{i}$$

Verify for yourself that the position, velocity and acceleration (i) have the correct values at $t=0$ and (ii) are related by the correct expressions (i.e. differentiate the position and show that you get the correct expression for the velocity, and differentiate the velocity to show that you get the correct expression for the acceleration).

HEALTH WARNING: These results can *only* be used if the ***acceleration is constant.*** In many problems acceleration is a function of time, or position – in this case these formulas cannot be used. People who have taken high school physics classes have used these formulas to solve so many problems that they automatically apply them to everything – this works for high school problems but not always in real life!



Example 2: Simple Harmonic Motion: The vibration of a very simple spring-mass system is an example of *simple harmonic motion*.

In simple harmonic motion (i) the particle moves along a straight line; and (ii) the position, velocity and acceleration are all trigonometric functions of time.

For example, the position vector of the mass might be given by

$$\mathbf{r} = x(t)\mathbf{i} = \left(X_0 + \Delta X \sin(2\pi t / T) \right)\mathbf{i}$$

Here X_0 is the average length of the spring, $X_0 + \Delta X$ is the maximum length of the spring, and T is the time for the mass to complete one complete cycle of oscillation (this is called the `period' of oscillation).

Harmonic vibrations are also often characterized by the *frequency* of vibration:

- The frequency in cycles per second (or Hertz) is related to the period by *f=1/T*
- The *angular* frequency is related to the period by $\omega = 2\pi / T$

The motion is plotted in the figure on the right.

The velocity and acceleration can be calculated by differentiating the position, as follows

$$\mathbf{v} = \frac{dx(t)}{dt}\mathbf{i} = \left(\frac{2\pi\Delta X}{T} \cos(2\pi t / T) \right)\mathbf{i}$$

$$\mathbf{a} = \frac{d^2 x(t)}{dt^2}\mathbf{i} = \left(-\frac{4\pi^2 \Delta X}{T^2} \sin(2\pi t / T) \right)\mathbf{i}$$

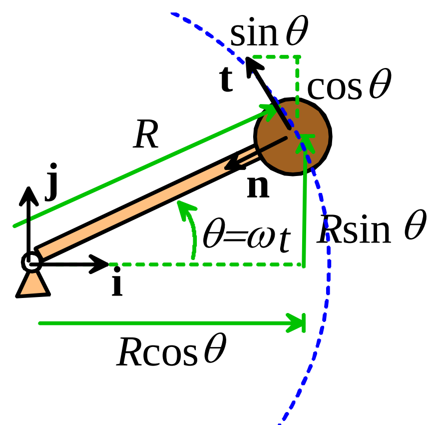
Note that:

- The velocity and acceleration are also harmonic, and have the same period and frequency as the displacement.
- If you know the frequency, and amplitude and of either the displacement, velocity, or acceleration, you can immediately calculate the amplitudes of the other two. For example, if ΔX , ΔV , ΔA denote the amplitudes of the displacement, velocity and acceleration, we have that

$$\Delta V = \frac{2\pi}{T}\Delta X \qquad \Delta A = \left(\frac{2\pi}{T} \right)^2 \Delta X = \frac{2\pi}{T}\Delta V$$

Example 3: Motion at constant speed around a circular path

Circular motion is also very common – examples include any rotating machinery, vehicles traveling around a circular path, and so on.



The simplest way to make an object move at constant speed along a circular path is to attach it to the end of a shaft (see the figure), and then rotate the shaft at a constant angular rate. Then, notice that

- The angle θ increases at constant rate. We can write $\theta = \omega t$, where ω is the (*constant*) angular speed of the shaft, **in radians/seconds**.
- The speed of the particle is related to ω by $V = R\omega$. To see this, notice that the circumferential distance traveled by the particle is $s = R\theta$. Therefore, $V = ds / dt = Rd\theta / dt = R\omega$.

For this example the position vector is

$$\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j}$$

The velocity can be calculated by differentiating the position vector.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\frac{d\theta}{dt}\sin\theta\mathbf{i} + R\frac{d\theta}{dt}\cos\theta\mathbf{j} = R\omega(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

Here, we have used the chain rule of differentiation, and noted that $d\theta / dt = \omega$.

The acceleration vector follows as

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=R\omega(-\frac{d\theta}{dt}\cos\theta\mathbf{i}-\frac{d\theta}{dt}\sin\theta\mathbf{j})=-R\omega^2(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})$$

Note that

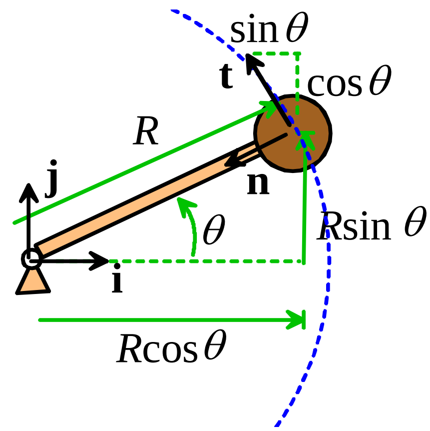
(i) The magnitude of the velocity is $V=R\omega$, and its direction is (obviously!) tangent to the path (to see this, visualize (using trig) the direction of the unit vector $\mathbf{t}=(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})$

(ii) The magnitude of the acceleration is $R\omega^2$ and its direction is towards the center of the circle. To see this, visualize (using trig) the direction of the unit vector $\mathbf{n}=-(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})$

We can write these mathematically as

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=R\omega\mathbf{t}=V\mathbf{t} \qquad \mathbf{a}=\frac{d\mathbf{v}}{dt}=R\omega^2\mathbf{n}=\frac{V^2}{R}\mathbf{n}$$

Example 4: More general motion around a circular path



We next look at more general circular motion, where the particle still moves around a circular path, but does not move at constant speed. The angle θ is now a general function of time.

We can write down some useful scalar relations:

- Angular rate: $\omega=\frac{d\theta}{dt}$
- Angular acceleration $\alpha=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$
- Speed $V=R\frac{d\theta}{dt}=R\omega$
- Rate of change of speed $\frac{dV}{dt}=R\frac{d^2\theta}{dt^2}=R\frac{d\omega}{dt}=R\alpha$

We can now calculate vector velocities and accelerations

$$\mathbf{r}=R\cos\theta\mathbf{i}+R\sin\theta\mathbf{j}$$

The velocity can be calculated by differentiating the position vector.

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-R\frac{d\theta}{dt}\sin\theta\mathbf{i}+R\frac{d\theta}{dt}\cos\theta\mathbf{j}=R\omega(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})$$

The acceleration vector follows as

$$\begin{aligned}\mathbf{a}&=\frac{d\mathbf{v}}{dt}=R\frac{d\omega}{dt}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})+R\omega(-\frac{d\theta}{dt}\cos\theta\mathbf{i}-\frac{d\theta}{dt}\sin\theta\mathbf{j})\\&=R\alpha(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})-R\omega^2(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\end{aligned}$$

It is often more convenient to re-write these in terms of the unit vectors **n** and **t** normal and tangent to the circular path, noting that $\mathbf{t}=(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})$, $\mathbf{n}=-(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})$. Then

$$\mathbf{v}=R\omega\mathbf{t}=V\mathbf{t} \qquad \mathbf{a}=R\alpha\mathbf{t}+R\omega^2\mathbf{n}=\frac{dV}{dt}\mathbf{t}+\frac{V^2}{R}\mathbf{n}$$

These are the famous *circular motion* formulas that you might have seen in physics class.

Using MATLAB 'live scripts'

If you find that your calculus is a bit rusty you can use MATLAB to do the tedious work for you. For example, to differentiate the vector

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

you would type

```
syms x(t) y(t) z(t) r(t) v(t) a(t)
r(t) = [x(t),y(t),z(t)]
v(t) = diff(r(t),t)
a(t) = diff(v(t),t)
```

r(t) = $\left(x(t) \quad y(t) \quad z(t)\right)$

v(t) =

$$\left(\frac{\partial}{\partial t}\, x(t) \quad \frac{\partial}{\partial t}\, y(t) \quad \frac{\partial}{\partial t}\, z(t)\right)$$

a(t) =

$$\left(\frac{\partial^2}{\partial t^2}\, x(t) \quad \frac{\partial^2}{\partial t^2}\, y(t) \quad \frac{\partial^2}{\partial t^2}\, z(t)\right)$$

It is essential to type in the (t) after x,y,and z – if you don't do this, MAPLE assumes that these variables are constants, and takes their derivative to be zero. You must enter (t) after _any_ variable that changes with time.

Here's how you would do the circular motion calculation if you only know that the angle θ is some arbitrary function of time, but don't know what the function is

```
syms R theta(t) r(t) v(t) a(t)
r(t) = [R*cos(theta(t)),R*sin(theta(t))]
v(t) = simplify(diff(r(t),t))
a(t) = simplify(diff(v(t),t))
```

r(t) = $\left(R\cos(\theta(t)) \quad R\sin(\theta(t))\right)$

v(t) =

$$\left(-R\sin(\theta(t))\frac{\partial}{\partial t}\,\theta(t) \quad R\cos(\theta(t))\frac{\partial}{\partial t}\,\theta(t)\right)$$

a(t) =

$$\left(-R\sin(\theta(t))\frac{\partial^2}{\partial t^2}\,\theta(t) - R\cos(\theta(t))\left(\frac{\partial}{\partial t}\,\theta(t)\right)^2 \quad R\cos(\theta(t))\frac{\partial^2}{\partial t^2}\,\theta(t) - R\sin(\theta(t))\left(\frac{\partial}{\partial t}\,\theta(t)\right)^2\right)$$

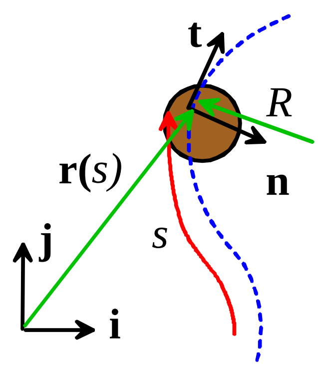
MATLAB can make very long and complicated calculations fairly painless. It is a godsend to engineers, who generally find that every real-world problem they need to solve is long and complicated. But of course it's important to know what the program is doing – so keep taking those math classes…

3.1.5 Velocity and acceleration in normal-tangential coordinates.

In some cases it is helpful to use special basis vectors to write down velocity and acceleration vectors, instead of a fixed {**i**,**j**,**k**} basis. If you see that this approach can be used to quickly solve a problem – go ahead and use it. If not, just use Cartesian coordinates – this will always work, and with MAPLE is not very hard. The only benefit of using the special coordinate systems is to save a couple of lines of rather tedious trigonometric algebra – which can be extremely helpful when solving an exam question, but is generally insignificant when solving a real problem.

Normal-tangential coordinates for particles moving along a prescribed planar path

In some problems, the formulas for velocity and acceleration look very complicated in (**i**,**j**) coordinates but become much simpler if they are expressed as components using a basis oriented parallel and perpendicular to the path. These basis vectors are called 'normal-tangential' coordinates.



For example, normal-tangential coordinates are nearly always used in vehicle dynamics problems, because the $\{\mathbf{t},\mathbf{n},\mathbf{k}\}$ directions point 'forwards' 'sideways' and 'vertically', so it is easy to understand the significance of accelerations along $\{\mathbf{t},\mathbf{n},\mathbf{k}\}$, while **{i,j,k}** components are generally very difficult to interpret.

To use normal-tangential coordinates we

- Specify the path by writing down the position vector of a point on the path in terms of the distance *s* travelled along the path. For a 2D curve:
$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$$
- Introduce two unit vectors **n** and **t**, with **t** pointing tangent to the path and **n** pointing normal to the path, towards the center of curvature (this sounds a bit scary, but **n** is just perpendicular to **t**, and if the path curves to the right **n** points to the right; if it curves to the left, it points to the left).
- Introduce the radius of curvature of the path *R* (in most problems we solve *R* is given, but we'll give some formulas later).
- Denote the speed of the particle by *V*. The speed can vary with time.

We then use the following formulas to calculate speed, velocity and acceleration
$$\mathbf{v} = V\mathbf{t}$$
$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$
We can also use the formula $V = ds/dt$ to write velocity and acceleration in terms of distance traveled along the path and its time derivatives
$$\mathbf{v} = \frac{ds}{dt}\mathbf{t}$$
$$\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{t} + \frac{1}{R}\left(\frac{ds}{dt}\right)^2\mathbf{n}$$

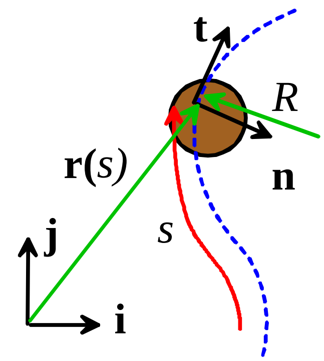
In words, these equations tell us:

(1) The direction of the velocity vector of a particle is tangent to its path.
(2) The magnitude of the velocity vector is equal to the speed.

(3) The speed is the derivative of distance traveled with respect to time
(4) The acceleration vector can be constructed by adding two components:

- the component of acceleration tangent to the particle's path is equal to dV/dt. We call this the 'tangential component' of acceleration.
- The component of acceleration perpendicular to the path (towards the center of curvature) is equal to V^2/R. We call this the 'normal component' of acceleration.

Deriving the normal-tangential formulas



In Newtonian physics, we have to start by defining an 'inertial frame', which (mathematically) is always a non-accelerating and non-rotating Cartesian {**i**,**j**,**k**} basis. The *x*,*y*,*z* coordinates of a particle and their derivatives tell us the Newtonian definitions of position, velocity and acceleration.

As we will see with specific examples below, some problems can be simplified by replacing the *x*,*y*,*z* coordinates with some simpler set of coordinates (these might be angles, or distances, or some combination of both that describe our system). Whenever we use new coordinated, we proceed by writing down the *x*,*y*,*z* coordinates in terms of our new ones. We usually also replace the {**i**,**j**,**k**} basis with a new set of directions that are defined by our new coordinate system. As we make small changes to our new coordinates, we will move around in space – the directions we move usually have some special significance, so it is helpful to write down all vector quantities of interest as components in the basis defined by these new directions.

This sounds very abstract, so let's see how it works for normal-tangential coordinates.

For problems where a particle moves along a known path, we can always write down the position of a point on the path in terms of the distance travelled along the path. For a 2D curve:

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$$

Here *s* is the arc-length traveled along the path. Since the path is known (*x*,*y* are given functions) we only need one coordinate (*s*) to specify where we are.

We now generate some basis vectors by working out how position vector changes if we make small changes to our new coordinate *s*. It is convenient to define

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \qquad \mathbf{n} = R\frac{d\mathbf{t}}{ds} \qquad R = \frac{1}{\sqrt{\frac{d\mathbf{t}}{ds}\cdot\frac{d\mathbf{t}}{ds}}} = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}}$$

The vectors $\{\mathbf{t},\mathbf{n},\mathbf{k}\}$ have unit length, are mutually perpendicular, and therefore define a new Cartesian basis. We can define any vector in this basis in the usual way.

With these definitions we are now ready to derive our formulas for velocity and acceleration:

$$\mathbf{v} = \frac{d\mathbf{r}(s)}{dt} = \frac{d\mathbf{r}(s)}{ds}\frac{ds}{dt} = V\mathbf{t}$$

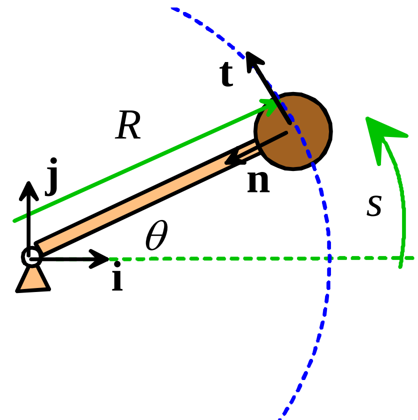
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dV}{dt}\mathbf{t} + V\frac{d\mathbf{t}}{dt} = \frac{dV}{dt}\mathbf{t} + V\frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \frac{dV}{dt}\mathbf{t} + V\frac{\mathbf{n}}{R}V = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

These are all just repeated applications of the chain rule….

Examples using normal-tangential coordinates

We'll work through a few examples below that show how to work with normal-tangential coordinates. These are all quite hard: you need to be able to use lots of basic ideas from vectors and calculus to be able to answer them.

Example: Circular motion in normal-tangential coordinates



We already analyzed circular motion as a special case. We'll revisit that example as an example of motion along a general curved path.

The distance traveled around the circle is the arc length *s*.

The arc-length formula gives $\theta = s/R$

Therefore $\mathbf{r} = R\cos\frac{s}{R}\mathbf{i} + R\sin\frac{s}{R}\mathbf{j}$ (just use trig)

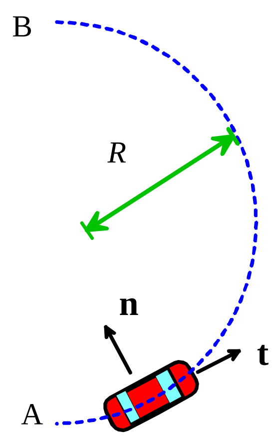
The tangent vector is $\mathbf{t} = \frac{d\mathbf{r}}{ds} = -\sin\frac{s}{R}\mathbf{i} + \cos\frac{s}{R}\mathbf{j}$ (this agrees with our earlier formula)

The normal vector is $\mathbf{n} = R\frac{d\mathbf{t}}{ds} = -\cos\frac{s}{R}\mathbf{i} - \sin\frac{s}{R}\mathbf{j}$ (also the same as our earlier formula)

The formula says velocity vector is $\mathbf{v} = \frac{ds}{dt}\mathbf{t}$ (the same as our earlier formula)

The formula for acceleration vector is $\mathbf{a} = \frac{d^2 s}{dt^2}\mathbf{t} + \frac{1}{R}\left(\frac{ds}{dt}\right)^2\mathbf{n}$ (also agrees with the earlier result)

Example (a simple exam problem):



A vehicle starts at rest at A and travels with constant tangential acceleration a_t around a circular path. To avoid skidding, the magnitude of the acceleration must not exceed μg, where μ is the friction coefficient between tires and road, and *g* is the gravitational acceleration. Find a formula for the shortest possible time to reach B.

We can use the formula: in normal-tangential coordinates

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} = a_t\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Since we are told a_t is constant and the car is at rest at *t=0* we can use the constant acceleration formula to find *V*

$$\frac{dV}{dt} = a_t \Rightarrow V = a_t t$$

Furthermore

$$\frac{ds}{dt} = a_t t \Rightarrow s = \frac{1}{2}a_t t^2$$

The arc-length from A to B is πR so the time to travel from A to B follows as

$$T = \sqrt{2\pi R / a_t}$$

We can find a_t from the condition that $|\mathbf{a}| \le \mu g$

$$|\mathbf{a}| = \left| a_t\mathbf{t} + \frac{V^2}{R}\mathbf{n} \right| = \sqrt{a_t^2 + \frac{V^4}{R^2}} = \sqrt{a_t^2 + \frac{(a_t t)^4}{R^2}} \le \mu g$$

(notice that we just used the usual Cartesian formula to find the magnitude of the vector). The maximum acceleration occurs at B, so we can substitute for *t* and solve for a_t

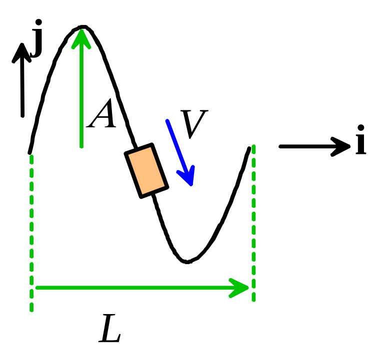
$$|\mathbf{a}| = \sqrt{a_t^2 + 4\pi^2 a_t^2} \le \mu g$$

$$\Rightarrow a_t \le \mu g / \sqrt{1 + 4\pi^2}$$

Finally remember $T = \sqrt{2\pi R / a_t}$ so

$$T \ge \sqrt{\frac{2\pi R}{\mu g}}\left(1 + 4\pi^2\right)^{1/4}$$

Example: Design speed limit for a curvy road: As a consulting firm specializing in highway design, we have been asked to develop a design formula that can be used to calculate the speed limit for cars that travel along a curvy road.



The following procedure will be used:

- The curvy road will be approximated as a sine wave $y = A\sin(2\pi x / L)$ as shown in the figure – for a given road, engineers will measure values of *A* and *L* that fit the path.
- Vehicles will be assumed to travel at constant speed *V* around the path – your mission is to calculate the maximum allowable value of *V*
- For safety, the magnitude of the acceleration of the car at any point along the path must be less than 0.2g, where g is the gravitational acceleration. (**Again, note that constant speed does not mean constant acceleration, because the car's direction is changing with time).**

Our goal, then, is to calculate a formula for the magnitude of the acceleration in terms of *V*, *A* and *L*. The result can be used to deduce a formula for the speed limit.

Calculation:

We can solve this problem quickly using normal-tangential coordinates. Since the speed is constant, the acceleration vector is

$$\mathbf{a} = \frac{V^2}{R}\mathbf{n}$$

Our only problem is that we don't know R – but we can use what we know about vectors and normal-tangential coordinates to figure it out.

The position vector is

$$\mathbf{r} = x\mathbf{i} + A\sin(2\pi x / L)\mathbf{j},$$

where x is some unknown function of distance s traveled along the path. We can calculate the tangent from the formula for the tangent (and the chain rule)

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dx}\frac{dx}{ds} = \left(\mathbf{i} + \frac{2\pi}{L}A\cos\left(2\pi x / L\right)\mathbf{j}\right)\frac{dx}{ds}$$

We know that **t** must be a unit vector, therefore

$$\mathbf{t}\cdot\mathbf{t} = 1 \Rightarrow \left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)\left(\frac{dx}{ds}\right)^2$$

$$\Rightarrow \frac{dx}{ds} = \left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)^{-1/2}$$

We could separate variables and integrate this to calculate *x(s)* if we need it – but in practice we don't need to bother.

So now we know that

$$\mathbf{t} = \frac{\left(\mathbf{i} + \frac{2\pi}{L}A\cos\left(2\pi x / L\right)\mathbf{j}\right)}{\sqrt{\left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)}}$$

The normal vector **n** must be perpendicular to both **t** and **k**, so we can create it using

$$\mathbf{n} = \pm\mathbf{k}\times\mathbf{t} = \pm\mathbf{k}\times\frac{\left(\mathbf{i} + \frac{2\pi}{L}A\cos\left(2\pi x / L\right)\mathbf{j}\right)}{\sqrt{\left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)}} = \pm\frac{\left(-\frac{2\pi}{L}A\cos\left(2\pi x / L\right)\mathbf{i} + \mathbf{j}\right)}{\sqrt{\left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)}}$$

(the $\pm$ is because there are two vectors normal to both **t** and **k**). Finally we know that

$$\mathbf{n} = R\frac{d\mathbf{t}}{ds} = \frac{\left(-\frac{4\pi^2}{L^2}A\sin\left(2\pi x / L\right)\mathbf{j}\right)}{\sqrt{\left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)}}\frac{dx}{ds} + \left(\mathbf{i} + \frac{2\pi}{L}A\cos\left(2\pi x / L\right)\mathbf{j}\right)\frac{d}{ds}\left\{\frac{1}{\sqrt{\left(1 + \frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x / L\right)\right)}}\right\}$$

Since **n** is a unit vector, we can take the dot product of both sides of this expression with **n** (from above)

$$\mathbf{n}\cdot\mathbf{n}=1=\pm R\frac{\left(-\frac{4\pi^2}{L^2}A\sin\left(2\pi x/L\right)\mathbf{j}\right)}{\left(1+\frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x/L\right)\right)}\frac{dx}{ds}$$

Therefore (using our earlier expression for dx/ds, and noting that R is positive by definition)

$$\frac{1}{R}=\frac{\left(\frac{4\pi^2}{L^2}A\sin\left(2\pi x/L\right)\mathbf{j}\right)}{\left(1+\frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x/L\right)\right)^{3/2}}$$

So now we know the acceleration vector is

$$\mathbf{a}=\frac{A(2\pi V/L)^2\sin(2\pi x/L)}{\left(1+\frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x/L\right)\right)^{3/2}}\mathbf{n}$$

We are interested in the magnitude of the acceleration…

$$|\mathbf{a}|=\frac{A(2\pi V/L)^2\sin(2\pi x/L)}{\left(1+\frac{4\pi^2}{L^2}A^2\cos^2\left(2\pi x/L\right)\right)^{3/2}}$$

We see from this that the car has the biggest acceleration when $x=L/2$. The maximum acceleration follows as

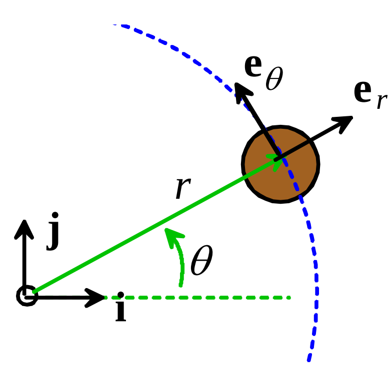
$$a_{\max}=A(2\pi V/L)^2$$

The formula for the speed limit is therefore $V<(L/2\pi)\sqrt{0.2g/A}$

Now we send in a bill for a big consulting fee…

3.1.6 Position, Velocity and Acceleration in cylindrical-polar coordinates.

When solving problems involving central forces (forces that attract particles towards a fixed point) it is often convenient to describe motion using *polar coordinates.*



Polar coordinate formulas

Polar coordinates are related to x,y coordinates through

$$r=\sqrt{x^2+y^2}\qquad \theta=\tan^{-1}(y/x)$$
$$x=r\cos\theta\qquad y=r\sin\theta$$

We can also specify height out of the plane of the picture using the usual z coordinate.

Suppose that the position of a particle is specified by its 'polar coordinates' (r,θ) relative to a fixed origin, as shown in the figure. Let $\mathbf{e}_r$ be a unit vector pointing in the radial direction, and let $\mathbf{e}_\theta$ be a unit vector pointing in the tangential direction, i.e

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

The position, velocity and acceleration of the particle can then be expressed as

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{k}$$
$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dz}{dt}\mathbf{k}$$
$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta + \frac{d^2z}{dt^2}\mathbf{k}$$

In most problems we solve, we just substitute known information about (r,θ) into these formulas.

Deriving the polar coordinate formulas

The formulas for polar coordinate can be derived using the same ideas we used to set up normal-tangential coordinates. The general procedure to set up any new coordinate system in Newtonian physics is:

(1) Choose an inertial frame – this defines a stationary Cartesian $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ basis

(2) Choose some new coordinates – here we use (r,θ,z) and write down position vector in the inertial frame in terms of these new coordinates. For the cylindrical-polar system

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$$

(3) We now think about making very small changes to each of r,θ,z . As we do so, $\mathbf{r}$ will change. We define unit vectors that point in each of the direction associated with making changes to r,θ,z : mathematically this operation is

$$\mathbf{e}_r = \frac{1}{|d\mathbf{r}/dr|}\frac{d\mathbf{r}}{dr} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathbf{e}_\theta = \frac{1}{|d\mathbf{r}/d\theta|}\frac{d\mathbf{r}}{d\theta} = \frac{-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}}{r} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$
$$\mathbf{e}_z = \frac{1}{|d\mathbf{r}/dz|}\frac{d\mathbf{r}}{dz} = \mathbf{k}$$

For the polar coordinate system it turns out that $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{k}\}$ are mutually perpendicular and so are a Cartesian basis. But note that the directions $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{k}\}$ are functions of r,θ,z . Regardless, we can express any vectors as components in our new basis using the usual ideas – the vector will be made up of contributions parallel to each of the new basis vectors.

The polar coordinate formulas now follow by simple calculus. Since we have chosen to work with r,θ,z instead of x,y,z , and $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{k}\}$ instead of $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, we would like to write down position, velocity and acceleration in the $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{k}\}$ basis, terms of time derivatives of r,θ,z . Before we work through the details we have to do some busy-work. Since $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{k}\}$ are functions of r,θ,z we will need to know how to differentiate them with respect to r,θ,z . To do this we go back to their original definitions:

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \Rightarrow \frac{d\mathbf{e}_r}{dr} = \frac{d\mathbf{e}_r}{dz} = \mathbf{0} \quad \frac{d\mathbf{e}_r}{d\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} = \mathbf{e}_\theta$$

$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \Rightarrow \frac{d\mathbf{e}_\theta}{dr} = \frac{d\mathbf{e}_\theta}{dz} = \mathbf{0} \quad \frac{d\mathbf{e}_\theta}{d\theta} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} = -\mathbf{e}_r$$

$$\mathbf{e}_z = \mathbf{k} \Rightarrow \frac{d\mathbf{e}_z}{dr} = \frac{d\mathbf{e}_z}{d\theta} = \frac{d\mathbf{e}_z}{dz} = \mathbf{0}$$

The rest is just a tedious exercise in using the chain rule. The position vector can be written down by inspection as $\mathbf{r} = r\mathbf{e}_r + z\mathbf{k}$. Then

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{k}$$

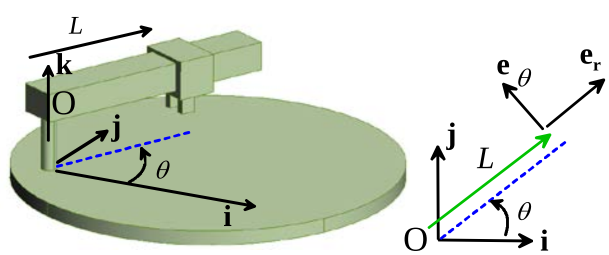
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{dz}{dt}\mathbf{k} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dz}{dt}\mathbf{k}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d\mathbf{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\frac{d\mathbf{e}_\theta}{d\theta}\frac{d\theta}{dt} + \frac{d^2z}{dt^2}\mathbf{k}$$

$$= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta + \frac{d^2z}{dt^2}\mathbf{k}$$

Examples using polar coordinates

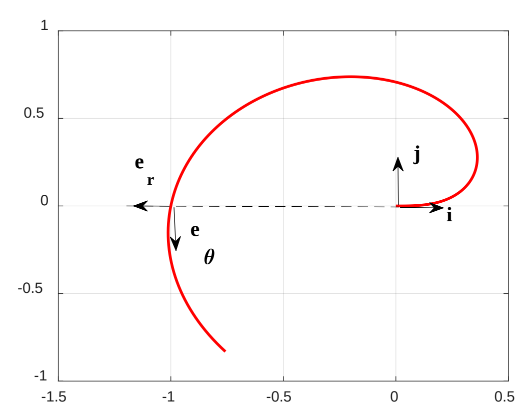
Example The robotic manipulator shown in the figure rotates with constant angular speed ω about the $\mathbf{k}$ axis. Find a formula for the maximum allowable (constant) rate of extension dL/dt if the acceleration of the gripper may not exceed g.



We can simply write down the acceleration vector, using polar coordinates. We identify $\omega = d\theta/dt$ and r=L, so that

$$\mathbf{a} = \left(-L\omega^2\right)\mathbf{e}_r + \left(2\frac{dL}{dt}\omega\right)\mathbf{e}_\theta \Rightarrow |\mathbf{a}|^2 = L^2\omega^4 + 4\left(\frac{dL}{dt}\omega\right)^2 < g^2 \Rightarrow \frac{dL}{dt} < \frac{1}{4}\sqrt{g^2/\omega^2 - L^2\omega^2}$$

Example: The position of a particle in polar coordinates is given by $\theta = t^2 \quad r = t/\sqrt{\pi}$ (meters). At the instant when $\theta = \pi$, calculate the following quantities:



- The position vector in $\mathbf{i},\mathbf{j}$ components and in $\mathbf{e}_r, \mathbf{e}_\theta$ components

  When $\theta = \pi, r = 1$ so

$$\mathbf{r} = -\mathbf{i} \;\; (meters)$$
$$\mathbf{r} = \mathbf{e}_r \;\; (meters)$$

- The velocity vector in $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{i}, \mathbf{j}$

  The polar coordinate formula is $\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = \frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2t\mathbf{e}_\theta = \frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2\sqrt{\pi}\mathbf{e}_\theta$ m/s

  By inspection we see that $\mathbf{i} = -\mathbf{e}_r, \mathbf{j} = -\mathbf{e}_\theta$ at the instant of interest, so $\mathbf{v} = -\frac{1}{\sqrt{\pi}}\mathbf{i} - 2\sqrt{\pi}\mathbf{j}$

- The acceleration vector in $\mathbf{e}_r, \mathbf{e}_\theta$ components

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta \quad \text{m/s}^2$$
$$= -4t^2\mathbf{e}_r + (2 + 4\frac{1}{\sqrt{\pi}}t)\mathbf{e}_\theta = -4\pi\mathbf{e}_r + 6\mathbf{e}_\theta$$

- Unit vectors $\mathbf{t}, \mathbf{n}$ tangent and normal to the path, in $\mathbf{e}_r, \mathbf{e}_\theta$. (Choose **n** to point towards the center of curvature)

  We know **t** is parallel to **v** so

$$\mathbf{t} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{4\pi + 1/\pi}}\left(\frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2\sqrt{\pi}\mathbf{e}_\theta\right) = \frac{1}{\sqrt{4\pi^2 + 1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right)$$

  We can find **n** in three ways: first, we know **n** must be perpendicular to **t** and must lie in the **i,j** plane (and hence is perpendicular to **k**). Remember that you can create a vector perpendicular to two others using a cross product, so $\mathbf{n} = \pm\mathbf{k} \times \mathbf{t}$. The positive choice points towards the center of curvature (by inspection), and note $\mathbf{k} \times \mathbf{e}_r = \mathbf{e}_\theta \quad \mathbf{k} \times \mathbf{e}_\theta = -\mathbf{e}_r$ so $\mathbf{n} = \frac{1}{\sqrt{4\pi^2 + 1}}\left(-2\pi\mathbf{e}_r + \mathbf{e}_\theta\right)$

  You can also use the condition $\mathbf{t} \cdot \mathbf{n} = 0$ - if we assume $\mathbf{n} = n_r\mathbf{e}_r + n_\theta\mathbf{e}_\theta$ then

$$\mathbf{t} \cdot \mathbf{n} = 0 \Rightarrow \frac{1}{\sqrt{4\pi^2 + 1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right) \cdot \left(n_r\mathbf{e}_r + n_\theta\mathbf{e}_\theta\right) = 0$$
$$\Rightarrow n_r + 2\pi n_\theta = 0$$

  Any n_r, n_θ that satisfies this (eg $n_r = -2\pi, n_\theta = 1$) is perpendicular to **t**. But **n** must be a unit vector, and we know we want the vector to point towards the origin (because the center of curvature of the path is inside the turn). So we have to choose

$$\mathbf{n} = \frac{1}{\sqrt{4\pi^2 + 1}}\left(-2\pi\mathbf{e}_r + \mathbf{e}_\theta\right)$$

The last (cumbersome, but general) way to do the calculation is to note that $\mathbf{a}-(\mathbf{a}\cdot\mathbf{t})\mathbf{t}$ must be parallel to **n**.

$$-4\pi\mathbf{e}_r+6\mathbf{e}_\theta-\left[\left(-4\pi\mathbf{e}_r+6\mathbf{e}_\theta\right)\cdot\frac{1}{\sqrt{4\pi^2+1}}\left(\mathbf{e}_r+2\pi\mathbf{e}_\theta\right)\right]\frac{1}{\sqrt{4\pi^2+1}}\left(\mathbf{e}_r+2\pi\mathbf{e}_\theta\right)$$

$$=-4\pi\mathbf{e}_r+6\mathbf{e}_\theta-\frac{8\pi}{4\pi^2+1}\left(\mathbf{e}_r+2\pi\mathbf{e}_\theta\right)=-\frac{4\pi(3+4\pi^2)}{4\pi^2+1}\mathbf{e}_r+\frac{2(3+4\pi^2)}{4\pi^2+1}\mathbf{e}_\theta$$

Dividing by the magnitude of this vector (to create a unit vector) gives the same answer as before.

- Tangential and normal components of acceleration a_t, a_n : we know (from ENGN30) that we can find the component of a vector in a basis by dotting it with the basis vectors, so

$$a_t=\mathbf{a}\cdot\mathbf{t}=\frac{8\pi}{\sqrt{1+4\pi^2}}\qquad a_n=\mathbf{a}\cdot\mathbf{n}=\frac{8\pi^2+6}{\sqrt{1+4\pi^2}}\ \text{m/s}^2$$

You can also do this problem using the formula

$\mathbf{a}=\frac{dV}{dt}\mathbf{t}+\frac{V^2}{R}\mathbf{n}$. This shows $a_t=\frac{dV}{dt}$, which is not too hard to calculate (but is a pain so I can't be bothered). To find $a_n=V^2/R$ we would either have to use our MA0200 super-powers to find the radius of curvature of the path, or else use $\mathbf{a}-\frac{dV}{dt}\mathbf{t}=+\frac{V^2}{R}\mathbf{n}=a_n\mathbf{n}$ and then take the magnitude of the vector on the left to get a_n .

3.1.7 Measuring position, velocity and acceleration

If you are designing a control system, you will need some way to detect the motion of the system you are trying to control. A vast array of different sensors is available for you to choose from: see for example the list at http://www.sensorland.com/HowPage001.html . A very short list of common sensors is given below

1. GPS – determines position on the earth's surface by measuring the time for electromagnetic waves to travel from satellites in known positions in space to the sensor. Can be accurate down to cm distances, but the sensor needs to be left in position for a long time for this kind of accuracy. A few m is more common.
2. Optical or radio frequency position sensing – measure position by (a) monitoring deflection of laser beams off a target; or measuring the time for signals to travel from a set of radio emitters with known positions to the sensor. Precision can vary from cm accuracy down to light wavelengths.
3. Capacitative displacement sensing – determine position by measuring the capacitance between two parallel plates. The device needs to be physically connected to the object you are tracking and a reference point. Can only measure distances of mm or less, but precision can be down to micron accuracy.

4. Electromagnetic displacement sensing – measures position by detecting electromagnetic fields between conducting coils, or coil/magnet combinations within the sensor. Needs to be physically connected to the object you are tracking and a reference point. Measures displacements of order cm down to microns.
5. Radar velocity sensing – measures velocity by detecting the change in frequency of electromagnetic waves reflected off the traveling object.
6. Inertial accelerometers: measure accelerations by detecting the deflection of a spring acting on a mass.

Accelerometers are also often used to construct an '*inertial platform,*' which uses gyroscopes to maintain a fixed orientation in space, and has three accelerometers that can detect motion in three mutually perpendicular directions. These accelerations can then be integrated to determine the position. They are used in aircraft, marine applications, and space vehicles where GPS cannot be used or where a backup is needed for GPS. They are often combined with GPS receivers as well: accelerometers are very good for measuring changes in velocity and position over a short time interval, and GPS is very good over long time intervals, so you can use 'sensor fusion' to make a sensor that uses both signals to get the best possible measurement.

3.2 Calculating forces required to cause prescribed motion of a particle

3.2.1 The Newtonian Inertial Frame.

Newton's laws are very familiar, and it is easy to write them down without much thought. They do have a flaw, however.

When we use Newton's laws, we assume that:
(1) We can identify some point in the universe that is not accelerating
(2) We can identify three mutually perpendicular directions that are 'fixed' in the sense that they do not rotate.

Together, these define a so-called 'inertial frame' – a Cartesian coordinate system in which motion obeys Newton's laws.

For engineering calculations, identifying a suitable origin and fixed directions usually poses no difficulty. If we are solving problems involving terrestrial motion over short distances compared with the earth's radius, we simply take a point on the earth's surface as fixed, and take three directions relative to the earth's surface to be fixed. If we are solving problems involving motion in space near the earth, or modeling weather, we take the center of the earth as a fixed point, (or for more complex calculations the center of the sun); and choose axes to have a fixed direction relative to nearby stars. Experiments show that Newton's laws predict motion sufficiently accurately for our needs. But there will always be some very small error.

In reality, an unambiguous inertial frame does not exist. We can only describe the *relative* motion of the mass in the universe, not its absolute motion. The general theory of relativity gives us a framework that avoids having to choose an inertial frame. More elaborate calculations show that Newton's laws are rigorous approximations to the general equations (in the sense of a Taylor expansion of the more general equations for low particle speeds compared with the speed of light), and would also (in principle) tell us the best choice of directions to set up a Newtonian frame at any point in space.

3.2.2 Newton's laws of motion for a particle

If we ignore these conceptual difficulties, Newton's laws for a particle are very simple. Let

1. *m* denote the mass of the particle
2. **F** denote the *resultant force* acting on the particle (as a vector, in the inertial frame)
3. **a** denote the *acceleration* of the particle (again, as a vector in the inertial frame). Then

$$\mathbf{F} = m\mathbf{a}$$

Occasionally, we use a particle idealization to model systems which, strictly speaking, are not particles. These are:

1. A large mass, which moves without rotation (e.g. a car moving along a straight line)
2. A single particle which is attached to a rigid frame with negligible mass (e.g. a person on a bicycle)

In these cases it may be necessary to consider the *moments* acting on the mass (or frame) in order to calculate unknown reaction forces.

1. For a large mass which moves without rotation, the resultant moment of external forces **about the center of mass** must vanish.
2. For a particle attached to a massless frame, the resultant moment of external forces acting on the frame **about the particle** must vanish.

$$\mathbf{M}_C = \mathbf{0}$$

We will see where this equation comes from when we analyze rigid body dynamics, and we'll also understand when it is no longer correct.

It is very important to take moments about the correct point in dynamics problems! Forgetting this is the most common reason to screw up a dynamics problem…

If you need to solve a problem where more than one particle is attached to a massless frame, you have to draw a separate free body diagram for each particle, and for the frame. The particles must obey Newton's laws $\mathbf{F} = m\mathbf{a}$. The forces acting on the frame must obey $\mathbf{F} = \mathbf{0}$ and $\mathbf{M}_C = \mathbf{0}$, (because the frame has no mass).

Newton's laws of motion can be used to calculate the forces required to make a particle move in a particular way.

We use the following general procedure to solve problems like this

(1) Decide how to idealize the system (what are the particles?)
(2) Draw a free body diagram showing the forces acting on each particle
(3) Consider the **kinematics** of the problem. The goal is to calculate the acceleration of each particle in the system – you may be able to start by writing down the position vector and differentiating it, or you may be able to relate the accelerations of two particles (eg if two particles move together, their accelerations must be equal).
(4) Write down **F**=*m***a** for each particle.
(5) If you are solving a problem involving a massless frames (see, e.g. Example 3, involving a bicycle with negligible mass) you also need to write down $\mathbf{M}_C = \mathbf{0}$ about the particle.
(5) Solve the resulting equations for any unknown components of force or acceleration (this is just like a statics problem, except the right hand side is not zero).

It is best to show how this is done by means of examples.

Example 1: Estimate the minimum thrust that must be produced by the engines of an aircraft in order to take off from the deck of an aircraft carrier (the figure is from **www.lakehurst.navy.mil/NLWeb/media-library.asp**)

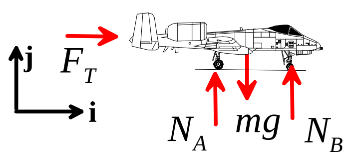
We will estimate the acceleration required to reach takeoff speed, assuming the aircraft accelerates from zero speed to takeoff speed along the deck of the carrier, and then use Newton's laws to deduce the force.

Data/ Assumptions:

1. The flight deck of a Nimitz class aircraft carrier is about 300m long (http://www.naval-technology.com/projects/nimitz/) but only a fraction of this is used for takeoff (the angled runway is used for landing). We will take the length of the runway to be d=200m
2. We will assume that the acceleration during takeoff roll is constant.
3. We will assume that the aircraft carrier is not moving (this is wrong – actually the aircraft carrier always moves at high speed during takeoff. We neglect motion to make the calculation simpler)
4. The FA18 Super Hornet is a typical aircraft used on a carrier – it has max catapult weight of m=15000kg http://www.boeing.com/defense-space/military/fa18ef/docs/EF_overview.pdf
5. The manufacturers are somewhat reticent about performance specifications for the Hornet but $v_t = 150$ knots (77 m/s) is a reasonable guess for a minimum controllable airspeed for this aircraft.

Calculations:

1. ***Idealization:*** We will idealize the aircraft as a particle. We can do this because the aircraft is not rotating during takeoff.



2. ***FBD*:** The figure shows a free body diagram. F_T represents the (unknown) force exerted on the aircraft due to its engines.
3. ***Kinematics:*** We must calculate the acceleration required to reach takeoff speed. We are given (i) the distance to takeoff d, (ii) the takeoff speed v_t and (iii) the aircraft is at rest at the start of the takeoff roll. We can therefore write down the position vector **r** and velocity **v** of the aircraft at takeoff, and use the straight line motion formulas for **r** and **v** to calculate the time t to reach takeoff speed and the acceleration a. Taking the origin at the initial position of the aircraft, we have that, at the instant of takeoff
$$\mathbf{r} = d\mathbf{i} = \frac{1}{2}at^2\mathbf{i} \qquad \mathbf{v} = v_t\mathbf{i} = at\mathbf{i}$$
This gives two scalar equations which can be solved for a and t
$$d = \frac{1}{2}at^2 \quad v_t = at \quad \Rightarrow \quad a = \frac{v_t^2}{2d} \quad t = \frac{2d}{v_t}$$
4. ***EOM:*** The vector equation of motion for this problem is

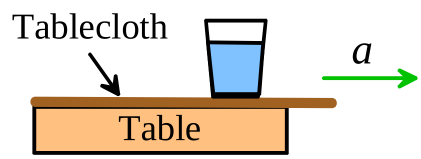
$$F_T \mathbf{i} = ma\mathbf{i} = m\frac{v_t^2}{2d}\mathbf{i}$$

5. ***Solution:*** The **i** component of the equation of motion gives an equation for the unknown force in terms of known quantities

$$F_T = m\frac{v_t^2}{2d}$$

Substituting numbers gives the magnitude of the force as F=222 kN. This is very close, but slightly greater than, the 200kN (44000lb) thrust quoted on the spec sheet for the Hornet. Using a catapult to accelerate the aircraft, speeding up the aircraft carrier, and increasing thrust using an afterburner buys a margin of safety.

Example 2: Mechanics of Magic! You have no doubt seen the simple `tablecloth trick' in which a tablecloth is whipped out from underneath a fully set table (if not, you can watch it at http://wm.kusa.gannett.edgestreams.net/news/1132187192333-11-16-05-spangler-2p.wmv)

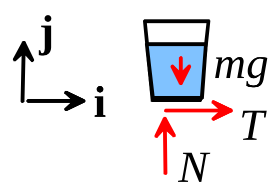


In this problem we shall estimate the critical acceleration that must be imposed on the tablecloth to pull it from underneath the objects placed upon it.

We wish to determine conditions for the tablecloth to slip out from under the glass. We can do this by calculating the reaction forces acting between the glass and the tablecloth, and see whether or not slip will occur. It is best to calculate the forces required to make the glass move with the tablecloth (i.e. to prevent slip), and see if these forces are big enough to cause slip.

1. ***Idealization*:** We will assume that the glass behaves like a particle (again, we can do this because the glass does not rotate)

2. ***FBD.*** The figure shows a free body diagram for the glass. The forces include (i) the weight; and (ii) the normal and tangential components of reaction at the contact between the tablecloth and the glass. The normal and tangential forces must act somewhere inside the contact area, but their position is unknown. For a more detailed discussion of contact forces see Sects 2.4 and 2.5.



3. ***Kinematics*** We are assuming that the glass has the same acceleration as the tablecloth. The table cloth is moving in the **i** direction, and has magnitude a. The acceleration vector is therefore $\mathbf{a} = a\mathbf{i}$.
4. ***EOM***. Newton's laws of motion yield

$$\mathbf{F} = m\mathbf{a} \Rightarrow T\mathbf{i} + (N - mg)\mathbf{j} = ma\mathbf{i}$$

5. ***Solution:*** The **i** and **j** components of the vector equation must each be satisfied (just as when you solve a statics problem), so that

$$T = ma \qquad N - mg = 0 \Rightarrow N = mg$$

Finally, we must use the friction law to decide whether or not the tablecloth will slip from under the glass. Recall that, for no slip, the friction force must satisfy

$$|T| < \mu N$$

where μ is the friction coefficient. Substituting for T and N from (5) shows that for no slip

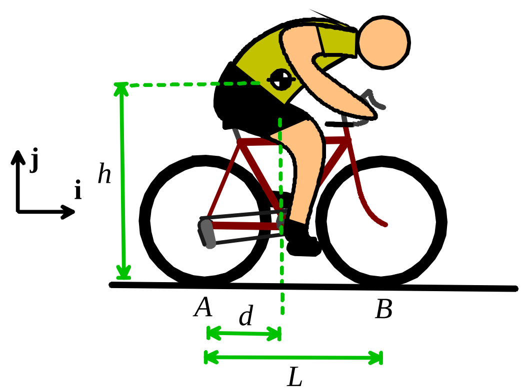
$$|a| < \mu g$$

To do the trick, therefore, the acceleration must exceed μg. For a friction coefficient of order 0.1, this gives an acceleration of order $1 m/s^2$. There is a special trick to pulling the tablecloth with a large acceleration – but that's a secret.

Example 3: Bicycle Safety. If a bike rider brakes too hard on the front wheel, his or her bike will tip over (the figure is from http://www.thosefunnypictures.com/picture/7658/bike-flip.html). In this example we investigate the conditions that will lead the bike to capsize, and identify design variables that can influence these conditions.

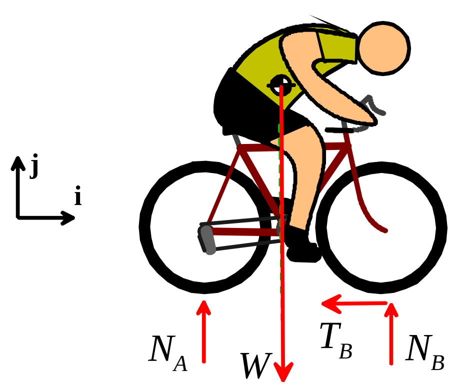
If the bike tips over, the rear wheel leaves the ground. If this happens, the reaction force acting on the wheel must be zero – so we can detect the point where the bike is just on the verge of tipping over by calculating the reaction forces, and finding the conditions where the reaction force on the rear wheel is zero.

1. ***Idealization***:
   a. We will idealize the rider as a particle (apologies to bike racers – but that's how we think of you…). The particle is located at the center of mass of the rider. The figure shows the most important design parameters- these are the height of the rider's COM, the wheelbase L and the distance of the COM from the rear wheel.



   b. We assume that the bike is a massless frame. The wheels are also assumed to have no mass. This means that the forces acting on the wheels must satisfy $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$ - and can be analyzed using methods of statics. If you've forgotten how to think about statics of wheels, you should re-read the notes on this topic – in particular, make sure you understand the nature of the forces acting on a freely rotating wheel (Section 2.4.6 of the reference notes).
   c. We assume that the rider brakes so hard that the front wheel is prevented from rotating. It must therefore skid over the ground. Friction will resist this sliding. We denote the friction coefficient at the contact point B by μ.
   d. The rear wheel is assumed to rotate freely.
   e. We neglect air resistance.

2. ***FBD.*** The figure shows a free body diagram for the rider and for the bike together. Note that
   a. A normal and tangential force acts at the contact point on the front wheel (in general, both normal and tangential forces always



act at contact points, unless the contact happens to be frictionless). Because the contact is slipping it is essential to draw the friction force in the correct direction – the force must resist the motion of the bike;

b. Only a normal force acts at the contact point on the rear wheel *because it is freely rotating, and behaves like a 2-force member*.

3. ***Kinematics*** The bike is moving in the $\mathbf{i}$ direction. As a vector, its acceleration is therefore $\mathbf{a} = a\mathbf{i}$, where a is unknown.

4. ***EOM*:** Because this problem includes a massless frame, we must use two equations of motion ($\mathbf{F} = m\mathbf{a}$ and $\mathbf{M}_C = \mathbf{0}$). *It is essential to take moments about the particle (i.e. the rider's COM).*

$\mathbf{F} = m\mathbf{a}$ gives $-T_B\mathbf{i} + (N_A + N_B - W)\mathbf{j} = ma\mathbf{i}$

$\mathbf{M}_C = \mathbf{0}$ gives $N_B(L-d)\mathbf{k} - N_A d\mathbf{k} - T_B h\mathbf{k} = \mathbf{0}$

The two nonzero components of $\mathbf{F} = m\mathbf{a}$ and the one nonzero component of $\mathbf{M}_C = \mathbf{0}$ give us three scalar equations

$$-T_B = ma$$
$$(N_A + N_B - W) = 0$$
$$N_B(L-d) - N_A d - T_B h = 0$$

We have *four* unknowns – the reaction components N_A, N_B, T_B and the acceleration a so we need another equation. The missing equation is the *friction law*

$$T_B = \mu N_B$$

5. **Solution:** (tedious algebra – you could avoid this by using Matlab)

The third equation and the friction law show that

$$N_B(L - d - \mu h) - N_A d = 0$$

Multiply $(N_A + N_B - W) = 0$ by $(L - d - \mu h)$ and subtract it from this equation:

$$-N_A d - (L - d - \mu h)(N_A - W) = 0$$
$$\Rightarrow N_A = W\frac{L - d - \mu h}{L - \mu h}$$

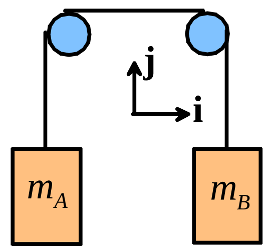
We are interested in finding what makes the reaction force at A go to zero (that's when the bike is about to tip). So

$$N_A = W\frac{L - d - \mu h}{L - \mu h} \le 0 \Rightarrow \mu \le (L-d)/h$$

This tells us that the bike will tip if the friction coefficient exceeds a critical magnitude, which depends on the geometry of the bike. The simplest way to design a tip-resistant bike is to make the height of the center of mass *h* small, and the distance (*L-d)* between the front wheel and the COM as large as possible.

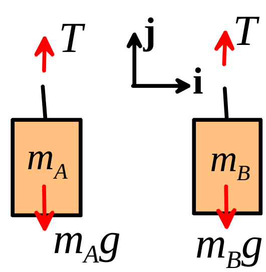
A `recumbent' bike is one way to achieve this – the figure (from http://en.wikipedia.org/wiki/Recumbent_bicycle) shows an example. The recumbent design offers many other significant advantages over the classic bicycle besides tipping resistance.

Example 4: A stupid problem that you might find in the FE professional engineering exam. The purpose of this problem is to show what you need to do to solve problems involving more than one particle.



Two weights of mass m_A and m_B are connected by a cable passing over two freely rotating pulleys as shown. They are released, and the system begins to move. Find an expression for the tension in the cable connecting the two weights.

1. **Idealization** – The masses will be idealized as particles; the cable is inextensible and the mass of the pulleys is neglected. This means the internal forces in the cable, and the forces acting between cables/pulleys must satisfy $\mathbf{F}=\mathbf{0}$ and $\mathbf{M}=\mathbf{0}$, and we can treat them as though they were in static equilibrium.



2. ***FBD*** – we have to draw a separate FBD for each particle. Since the pulleys and cable are massless, the tension *T* in the cable is constant.

3. ***Kinematics*** We know that both masses must move in the **j** direction. We also know that the masses always move at the same speed but in opposite directions. Therefore, their accelerations must be equal and opposite. We can express this mathematically as
$$a_A\mathbf{j}=-a_B\mathbf{j}$$

4. ***EOM*****:** We must write down two equations of motion, as there are two masses
$$(T-m_A g)\mathbf{j}=m_A a_A\mathbf{j}$$
$$(T-m_B g)\mathbf{j}=m_B a_B\mathbf{j}$$
We now have three equations for three unknowns (the unknowns are a_A, a_B and *T*).

5. ***Solution:*** More algebra. We can eliminate a_B so that the last two equations are:

$$(T - m_A g) = m_A a_A$$

$$(T - m_B g) = -m_B a_A$$

Now we can multiply the first equation by m_B and the second by m_A and add them

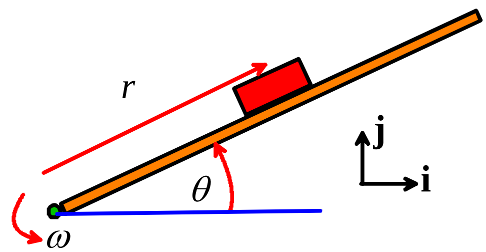
$$(T m_A + T m_B - 2 m_A m_B g) = 0$$

So

$$T = \frac{2 m_A m_B}{m_A + m_B} g$$

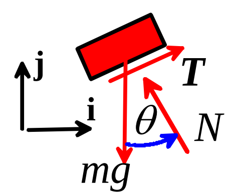
We pass!

Example 5: Another stupid FE exam problem: The figure shows a small block on a rotating bar. The contact between the block and the bar has friction coefficient μ. The bar rotates at constant angular speed ω. Find the critical angular velocity that will just make the block start to slip when $\theta = 0$. Which way does the block slide?



The general approach to this problem is the same as for the Magic trick example – we will calculate the reaction force exerted by the bar on the block, and see when the forces are large enough to cause slip at the contact. We analyze the motion assuming the slip does *not* occur, and then find out the conditions where this can no longer be the case.

1. ***Idealization*** – We will idealize the block as a particle. This is dangerous, because the block is clearly rotating. We hope that because it rotates at constant rate, the rotation will not have a significant effect – but we can only check this once we know how to deal with rotational motion.
2. ***FBD*****:** The figure shows a free body diagram for the block. The block is subjected to a vertical gravitational force, and reaction forces at the contact with the bar. Since we have assumed that the contact is not slipping, we can choose the direction of the tangential component of the reaction force arbitrarily. The resultant force on the block is



$$\mathbf{F} = \left(T\cos\theta - N\sin\theta\right)\mathbf{i} + (N\cos\theta + T\sin\theta - mg)\mathbf{j}$$

3. ***Kinematics*** We can use the circular motion formula to write down the acceleration of tbe block (see section 3.1.3)

$$\mathbf{a} = -r\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

4. ***EOM*****:** The equation of motion is

$$\left(T\cos\theta - N\sin\theta\right)\mathbf{i} + (N\cos\theta - mg)\mathbf{j} = -mr\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

5. ***Solution:*** The **i** and **j** components of the equation of motion can be solved for N and T. Doing this by hand is a pain, but Matlab makes it painless

```
syms T N theta m r omega g real
eq1 = T*cos(theta)-N*sin(theta) == -m*r*omega^2*cos(theta);
eq2 = N*cos(theta)+T*sin(theta)-m*g == -m*r*omega^2*sin(theta);
[N,T] = (solve([eq1,eq2],[N,T]));
N = simplify(N)
T = simplify(T)
```

$$N = g\,m\cos(\theta)$$

$$T = -m\,\left(\omega^2 r - g\sin(\theta)\right)$$

To find the point where the block just starts to slip, we use the friction law. Recall that, at the point of slip

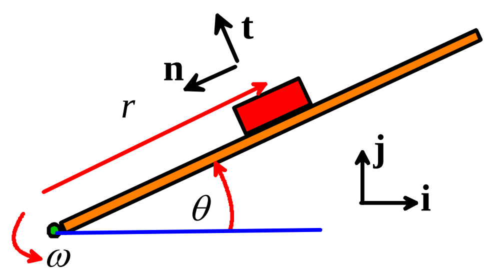
$$|T| = \mu N$$

For the block to slip with $\theta = 0$

$$\left|-r\omega^2\right| = \mu g$$

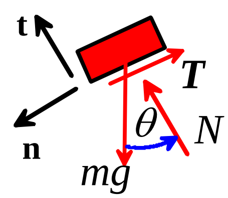
so the critical angular velocity is $\omega = \sqrt{\mu g / r}$. Since the tangential traction T is negative, and the friction force must *oppose* sliding, the block must slide outwards, i.e. r is increasing during slip.

Alternative method of solution using normal-tangential coordinates



We will solve this problem again, but this time we'll use the short-cuts described in Section 3.1.4 to write down the acceleration vector, and we'll write down the vectors in Newton's laws of motion in terms of the unit vectors **n** and **t** normal and tangent to the object's path.

(i) ***Acceleration vector*** If the block does not slip, it moves with speed $V = \omega r$ around a circular arc with radius r. Its acceleration vector has magnitude V^2 / r and direction parallel to the unit vector **n**.



(ii) The force vector can be resolved into components parallel to **n** and **t**. Simple trig on the free body diagram shows that

$$\mathbf{F} = (N - mg\cos\theta)\mathbf{t} + (mg\sin\theta - T)\mathbf{n}$$

(iii) Newton's laws then give

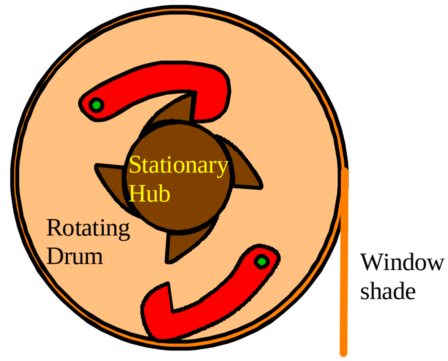
$$\mathbf{F} = m\mathbf{a} = (N - mg\cos\theta)\mathbf{t} + (mg\sin\theta - T)\mathbf{n} = m\omega^2 r\mathbf{n}$$

The components of this vector equation parallel to **t** and **n** yield two equations, with solution

$$N = mg\cos\theta \qquad T = mg\sin\theta - m\omega^2 r$$

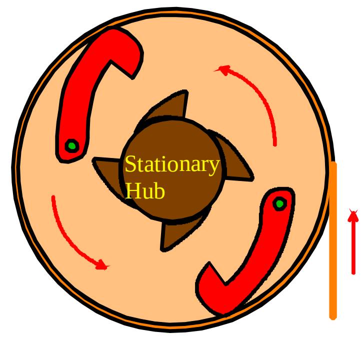
This is the same solution as before. Normal-tangential coordinates makes the equations and algebra much simpler, however.

Example 6: Window blinds. Have you ever wondered how window shades work? You give the shade a little downward jerk, let it go, and it winds itself up. If you pull the shade down slowly, it stays down.



The figure shows the mechanism (which probably only costs a few cents to manufacture) that achieves this remarkable feat of engineering. It's called an `inertial latch' – the same principle is used in the inertia reels on the seatbelts in your car.

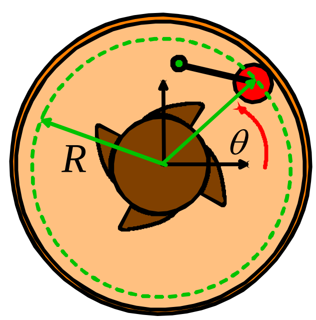
The picture shows an enlarged end view of the window shade. The hub, shown in brown, is fixed to the bracket supporting the shade and cannot rotate. The drum, shown in peach, rotates as the shade is pulled up or down. The drum is attached to a torsional spring, which tends to cause the drum to rotate counterclockwise, so winding up the shade. The rotation is prevented by the small dogs, shown in red, which engage with the teeth on the hub. You can pull the shade downwards freely, since the dogs allow the drum to rotate counterclockwise.



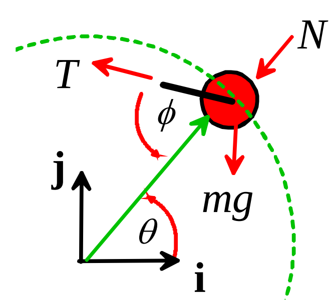
To raise the shade, you need to give the end of the shade a jerk downwards, and then release it. When the drum rotates sufficiently quickly (we will calculate how quickly shortly) the dogs open up, as shown on the right. They remain open until the drum slows down, at which point the topmost dog drops and engages with the teeth on the hub, thereby locking up the shade once more.

We will estimate the critical rotation rate required to free the rotating drum.

1. ***Idealization*** – We will idealize the topmost dog as a particle on the end of a massless, inextensible rod, as shown in the figure.
    a. We will assume that the drum rotates at constant angular rate ω. Our goal is to calculate the critical speed where the dog is just on the point of dropping down to engage with the hub.
    b. When the drum spins fast, the particle is contacts the outer rim of the drum – a normal force acts at the contact. When the dog is on the point of dropping this contact force goes to zero. So our goal is to calculate the contact force, and then to find the critical rotation rate where the force will drop to zero.
    c. We neglect friction.



2. ***FBD.*** The figure shows a free body diagram for the particle. The particle is subjected to: (i) a reaction force N where it contacts the rim; (ii) a tension T in the link, and (iii) gravity. The resultant force is
$$\mathbf{F} = \left(-T\cos(\phi-\theta) - N\cos\theta\right)\mathbf{i} + (-N\sin\theta + T\sin(\phi-\theta) - mg)\mathbf{j}$$



3. ***Kinematics*** We can use the circular motion formula to write down the

acceleration of the particle(see section 3.1.3)

$$\mathbf{a} = -R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

4. ***EOM*:** The equation of motion is

$$\left(-T\cos(\phi-\theta) - N\cos\theta\right)\mathbf{i} + (-N\sin\theta + T\sin(\phi-\theta) - mg)\mathbf{j} = \mathbf{a} = -R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

5. ***Solution:*** The **i** and **j** components of the equation of motion can be solved for *N* and *T* – MAPLE makes this painless

```
syms T N theta m R omega g phi real
eq1 = -T*cos(phi-theta) - N*cos(theta) == -R*omega^2*cos(theta);
eq2 = -N*sin(theta)+T*sin(phi-theta)-m*g == -R*omega^2*sin(theta);
[T,N] = solve([eq1,eq2],[T,N]);
T = simplify(T)
N = simplify(N)
```

T =

$$\frac{g\,m\cos(\theta)}{\sin(\varphi)}$$

N =

$$-\frac{g\,m\cos(\varphi-\theta) - R\,\omega^2\sin(\varphi)}{\sin(\varphi)}$$

The normal reaction force is therefore

$$N = -mg\cos(\theta-\phi)/\sin\phi + mR\omega^2$$

We are looking for the point where this can first become zero or negative. Note that $\max\{\cos(\theta-\phi)\} = 1$ at the point where $\theta-\phi$ =0. The smallest value of *N* therefore occurs at this point, and has magnitude

$$N_{\min} = -mg/\sin\phi + mR\omega^2$$

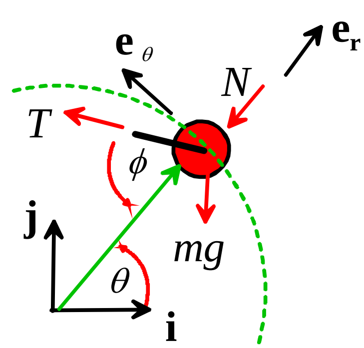
The critical speed where *N=0* follows as

$$\omega = \sqrt{g/(R\sin\phi)}$$

Changing the angle ϕ and the radius *R* gives a convenient way to control the critical speed in designing an inertial latch.

Alternative solution using polar coordinates

We'll work through the same problem again, but this time handle the vectors using polar coordinates.



1. ***FBD.*** The figure shows a free body diagram for the particle. The particle is subjected to: (i) a reaction force *N* where it contacts the rim; (ii) a tension *T* in the link, and (iii) gravity. The resultant force is

$$\mathbf{F} = -(N + T\cos\phi + mg\sin\theta)\mathbf{e}_r + (T\sin\phi - mg\cos\theta)\mathbf{e}_\theta$$

2. ***Kinematics*** The acceleration vector is now

$$\mathbf{a} = -R\omega^2 \mathbf{e}_r$$

3. ***EOM*:** The equation of motion is

$$-(N + T\cos\phi + mg\sin\theta)\mathbf{e}_r + (T\sin\phi - mg\cos\theta)\mathbf{e}_\theta = -R\omega^2 \mathbf{e}_r$$

4. ***Solution:*** The $\mathbf{e}_r, \mathbf{e}_\theta$ components of the equation of motion can be solved for N and T. If we use polar coordinates we can do this by hand – the $\mathbf{e}_\theta$ component shows that

$$T = mg\cos\theta / \sin\theta$$

We can substitute this back into the $\mathbf{e}_r$ component to get

$$N = -mg\cos(\theta - \phi) / \sin\phi + mR\omega^2$$

We are looking for the point where this can first become zero or negative. Note that $\max\{\cos(\theta - \phi)\} = 1$ at the point where $\theta - \phi = 0$. The smallest value of N therefore occurs at this point, and has magnitude

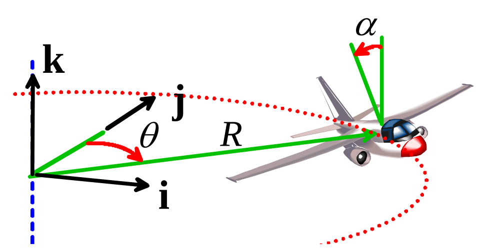
$$N_{\min} = -mg / \sin\phi + mR\omega^2$$

The critical speed where $N=0$ follows as

$$\omega = \sqrt{g / (R\sin\phi)}$$

Changing the angle ϕ and the radius R gives a convenient way to control the critical speed in designing an inertial latch.

Example 7: Aircraft Dynamics Aircraft performing certain instrument approach procedures (such as holding patterns or procedure turns) are required to make all turns at a standard rate, so that a complete 360 degree turn takes 2 minutes. All turns must be made at constant altitude and constant speed, V.



People who design instrument approach procedures need to know the radius of the resulting turn, to make sure the aircraft won't hit anything. Engineers designing the aircraft are interested in the forces needed to complete the turn – specifically, the *load factor*, which is the ratio of the lift force on the aircraft to its weight.

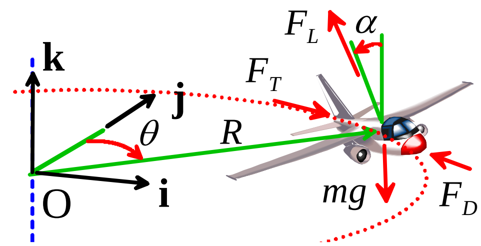
In this problem we will calculate the radius of the turn R and the bank angle required, as well as the load factor caused by the maneuver, as a function of the aircraft speed V.

Before starting the calculation, it is helpful to understand what makes an aircraft travel in a circular path. Recall that

1. If an object travels at constant speed around a circle, its acceleration vector has constant magnitude, and has direction towards the center of the circle
2. A force must act on the aircraft to produce this acceleration – i.e. the resultant force on the aircraft must act towards the center of the circle. The necessary force comes from *the horizontal component of the lift force* – the pilot banks the wings, so that the lift acts at an angle to the vertical.

With this insight, we expect to be able to use the equations of motion to calculate the forces.

1. ***Idealization*** – The aircraft is idealized as a particle – it's not obvious that this is accurate, because the aircraft clearly rotates as it travels around the curve. However, the forces we wish to calculate turn out to be fully determined by **F**=m**a** and are not influenced by the rotational motion.



2. ***FBD***. The figure shows a free body diagram for the aircraft. It is subjected to (i) a gravitational force (mg); (ii) a thrust from the engines F_T, (iii) a drag force F_D, acting perpendicular to the direction of motion, and (iv) a lift force F_L, acting perpendicular to the plane of the wings.

   The resultant force is

   $$\left[(F_T - F_D)\cos\theta - F_L\sin\alpha\sin\theta\right]\mathbf{i} + \left[(F_D - F_T)\sin\theta - F_L\sin\alpha\cos\theta\right]\mathbf{j} + \left(F_L\cos\alpha - mg\right)\mathbf{k}$$

   (you may find the components of the lift force difficult to visualize – to see where these come from, note that the lift force can be projected onto components along OR and the **k** direction as $\mathbf{F}_L = F_L\sin\alpha\overrightarrow{RO} + F_L\cos\alpha\mathbf{k}$. Then note that $\overrightarrow{RO} = -\sin\theta\mathbf{i} - \cos\theta\mathbf{j}$.)

3. ***Kinematics***
   a. The aircraft moves at constant speed around a circle, so the angle $\theta = \omega t$, where ω is the (*constant*) angular speed of the line OP. Since the aircraft completes a turn in two minutes, we know that $\omega = 2\pi/(2\times 60) = \pi/60$ rad/sec
   b. The position vector of the plane is
   $$\mathbf{r} = R\sin\omega t\mathbf{i} + R\cos\omega t\mathbf{j}$$
   We can differentiate this expression with respect to time to find the velocity
   $$\mathbf{v} = R\omega(\cos\omega t\mathbf{i} - \sin\omega t)\mathbf{j}$$
   c. The magnitude of the velocity is $V = R\omega$, so if the aircraft flies at speed *V*, the radius of the turn must be $R = V/\omega$
   d. Differentiating the velocity gives the acceleration
   $$\mathbf{a} = -R\omega^2(\sin\omega t\mathbf{i} + \cos\omega t)\mathbf{j}$$

4. ***EOM*:** The equation of motion is

   $$\left[(F_T - F_D)\cos\theta - F_L\sin\alpha\sin\theta\right]\mathbf{i} + \left[(F_D - F_T)\sin\theta - F_L\sin\alpha\cos\theta\right]\mathbf{j} + \left(F_L\cos\alpha - mg\right)\mathbf{k}$$
   $$= -mR\omega^2(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$
   $$= -mV\omega(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

5. ***Solution:*** The **i j** and **k** components of the equation of motion give three equations that can be solved for F_T, F_L and α. We assume that the drag force is known, since this is a function of the aircraft's speed.

```
syms FT FD FL alpha theta R m omega g V real
eq1 = (FT-FD)*cos(theta)-FL*sin(alpha)*sin(theta)==-m*V*omega*sin(theta);
eq2 = (FT-FD)*cos(theta)-FL*sin(alpha)*cos(theta)==-m*V*omega*cos(theta);
eq3 = FL*cos(alpha)-m*g ==0;
[FL,alpha,FT] = solve([eq1,eq2,eq3],[FL,alpha,FT]);
FL = simplify(FL)
alpha = simplify(alpha)
FT = simplify(FT)
```

```
Warning: The solutions are valid under the following conditions: V ~= 0 & omega ~= 0;
V ~= 0 & omega ~= 0.
 To include parameters and conditions in the solution, specify the 'ReturnConditions'
option.
```

FL =

$$\begin{pmatrix} m\sqrt{V^2\omega^2+g^2} \\ -m\sqrt{V^2\omega^2+g^2} \end{pmatrix}$$

alpha =

$$\begin{pmatrix} -2\,\mathrm{atan}\left(\dfrac{g-\sqrt{V^2\omega^2+g^2}}{V\omega}\right) \\ -2\,\mathrm{atan}\left(\dfrac{g+\sqrt{V^2\omega^2+g^2}}{V\omega}\right) \end{pmatrix}$$

FT =

$$\begin{pmatrix} \mathrm{FD} \\ \mathrm{FD} \end{pmatrix}$$

(The two solutions are a bit weird, but to a mathematician having the airplane fly upside down and generate negative lift is a perfectly acceptable solution)

$$\alpha = -2\tan^{-1}((g-\sqrt{V^2\omega^2+g^2})/V\omega) \qquad F_L = mg\sqrt{1+V^2\omega^2/g^2} \qquad F_T = F_D$$

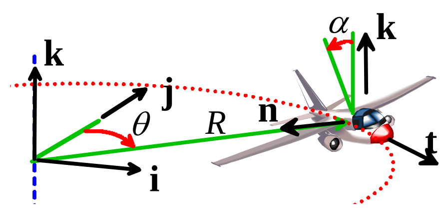
We can calculate values of α, $R = V/\omega$ and the load factor F_L/mg for a few aircraft

a. Cessna 150 – V=70knots (36 m/s) : $\alpha = 11^0$ R=690m, $F_L/mg = 1.02$

b. Boeing 747: V=200 knots (102 m/s) $\alpha = 28^0$ R=1950m, $F_L/mg = 1.14$

c. F111 V=300 knots (154 m/s) $\alpha = 39^0$ R=2950m, $F_L/mg = 1.3$

Alternative solution using normal-tangential coordinates



This problem can also be solved rather more quickly using normal and tangential basis vectors.

(i) **Acceleration vector**. The aircraft travels around a circular path at constant speed, so its acceleration is

$$\mathbf{a} = \frac{V^2}{R}\mathbf{n} = V\omega\mathbf{n}$$

where **n** is a unit vector pointing towards the center of the circle.

(ii) **Force vector**. The force vector can be written in terms of the unit vectors **n,t,k** as

$$\mathbf{F} = (F_T - F_D)\mathbf{t} + F_L\sin\alpha\,\mathbf{n} + (F_L\cos\alpha - mg)\mathbf{k}$$

(iii) **Newton's law** $\mathbf{F} = (F_T - F_D)\mathbf{t} + F_L\sin\alpha\,\mathbf{n} + (F_L\cos\alpha - mg)\mathbf{k} = mV\omega\mathbf{n}$

The **n**, **t** and **k** components of this equation give three equations that can be solved for F_T, F_L and α. We can easily do this by hand

$$\alpha = \tan^{-1}(V\omega/g) \qquad F_L = mg\sqrt{1+V^2\omega^2/g^2} \qquad F_T = F_D$$

3.3 Deriving and solving equations of motion for systems of particles

We next turn to the more difficult problem of predicting the motion of a system that is subjected to a set of forces.

3.3.1 General procedure for deriving and solving equations of motion for systems of particles

It is very straightforward to analyze the motion of systems of particles. You should always use the following procedure

1. Introduce a set of variables that can describe the motion of the system. Don't worry if this sounds vague – it will be clear what this means when we solve specific examples.
2. Write down the position vector of each particle in the system in terms of these variables
3. Differentiate the position vector(s), to calculate the velocity and acceleration of each particle in terms of your variables;
4. Draw a free body diagram showing the forces acting on each particle. You may need to introduce variables to describe reaction forces. Write down the resultant force vector.
5. Write down Newton's law $\mathbf{F} = m\mathbf{a}$ for each particle. This will generate up to 3 equations of motion (one for each vector component) for each particle.
6. If you wish, you can eliminate any unknown reaction forces from Newton's laws. If you are trying to solve the equations by hand, you should always do this; of you are using MATLAB, it's not usually necessary – you can have MATLAB calculate the reactions for you. The result will be a set of differential equations for the variables defined in step (1)
7. If you find you have fewer equations than unknown variables, you should look for any *constraints* that restrict the motion of the particles. The constraints must be expressed in terms of the unknown accelerations.
8. Identify the *initial conditions* for the variables defined in (1). These are usually the values of the unknown variables, their time derivatives, at time $t=0$. If you happen to know the values of the variables at some other instant in time, you can use that too. If you don't know their values at all, you should just introduce new (unknown) variables to denote the initial conditions.
9. Solve the differential equations, subject to the initial conditions.

Steps (3) (6) and (8) can usually be done on the computer, so you don't actually have to do much calculus or math.

Sometimes, you can avoid solving the equations of motion completely, by using *conservation laws* – conservation of energy, or conservation of momentum – to calculate quantities of interest. These short-cuts will be discussed in the next chapter.

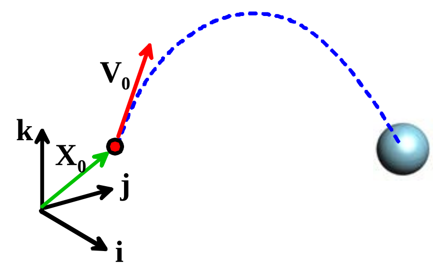
3.3.2 Simple examples of equations of motion and their solutions

The general process described in the preceding section can be illustrated using simple examples. In this section, we derive equations of motion for a number of simple systems, and find their solutions.

The purpose of these examples is to illustrate the straightforward, step-by-step procedure for analyzing motion in a system. Although we solve several problems of practical interest, we will simply set up and solve the equations of motion with some arbitrary values for system parameter, and won't attempt to explore their behavior in detail. More detailed discussions of the behavior of dynamical systems will follow in later chapters.

Example 1: Trajectory of a particle near the earth's surface (no air resistance)

At time *t=0*, a projectile with mass *m* is launched from a position $\mathbf{X}_0 = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ with initial velocity vector $\mathbf{V}_0 = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$. Calculate its trajectory as a function of time.



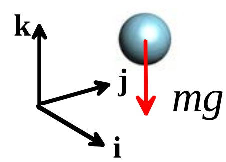
1. ***Introduce variables to describe the motion***: We can simply use the Cartesian coordinates of the particle $(x(t), y(t), z(t))$

2. ***Write down the position vector in terms of these variables***: $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

3. ***Differentiate the position vector with respect to time to find the acceleration***. For this example, this is trivial

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

4. ***Draw a free body diagram***. The only force acting on the particle is gravity – the free body diagram is shown in the figure. The force vector follows as $\mathbf{F} = -mg\mathbf{k}$.



5. ***Write down Newton's laws of motion.*** This is easy

$$\mathbf{F} = m\mathbf{a} \Rightarrow -mg\mathbf{k} = m\left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}\right)$$

The vector equation actually represents three separate differential equations of motion

$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = 0 \qquad \frac{d^2z}{dt^2} = -g$$

6. ***Eliminate reactions*** – this is not needed in this example.

7. ***Identify initial conditions.*** The initial conditions were given in this problem – we have that

$$\left\{x = X_0 \quad \frac{dx}{dt} = V_x\right\} \quad \left\{y = Y_0 \quad \frac{dy}{dt} = V_y\right\} \quad \left\{z = Z_0 \quad \frac{dz}{dt} = V_z\right\}$$

8. ***Solve the equations of motion***. In general we will use MAPLE or matlab to do the rather tedious algebra necessary to solve the equations of motion. Here, however, we will integrate the equations by hand, just to show that there is no magic in MAPLE.

The equations of motion are

$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = 0 \qquad \frac{d^2z}{dt^2} = -g$$

It is a bit easier to see how to solve these if we define

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y \qquad \frac{dz}{dt} = v_z$$

The equation of motion can be re-written in terms of (v_x, v_y, v_z) as

$$\frac{dv_x}{dt} = 0 \qquad \frac{dv_y}{dt} = 0 \qquad \frac{dv_z}{dt} = -g$$

We can separate variables and integrate, using the initial conditions as limits of integration

$$\int_{V_x}^{v_x} dv_x = \int_0^t 0dt \qquad \int_{V_y}^{v_y} dv_x = \int_0^t 0dt \qquad \int_{V_z}^{v_z} dv_z = \int_0^t -gdt$$

$$v_x = V_x \qquad v_y = V_y \qquad v_z = V_z - gt$$

Now we can re-write the velocity components in terms of *(x,y,z)* as

$$\frac{dx}{dt} = V_x \qquad \frac{dy}{dy} = V_y \qquad \frac{dz}{dt} = V_z - gt$$

Again, we can separate variables and integrate

$$\int_{X_0}^{x} dx = \int_0^t V_x dt \qquad \int_{Y_0}^{y} dy = \int_0^t V_y dt \qquad \int_{Z_0}^{z} dz = \int_0^t (V_z - gt)\, dt$$

$$x = X_0 + V_x t \qquad y = Y_0 + V_y t \qquad z = Z_0 + V_z t - \frac{1}{2} g t^2$$

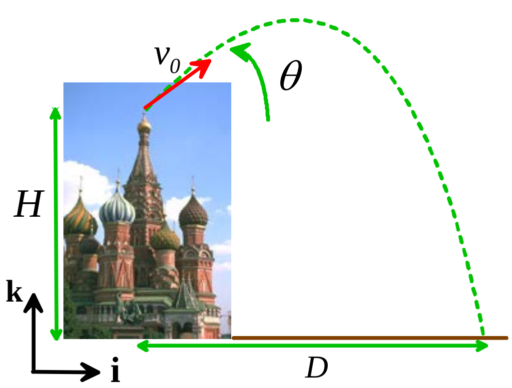
so the position and velocity vectors are

$$\mathbf{r} = (X + V_x t)\mathbf{i} + (Y + V_y t)\mathbf{j} + \left(Z + V_z t - \frac{1}{2} g t^2\right)\mathbf{k}$$

$$\mathbf{v} = V_x \mathbf{i} + V_y \mathbf{j} + (V_z - gt)\mathbf{k}$$

Applications of trajectory problems: It is traditional in elementary physics and dynamics courses to solve vast numbers of problems involving particle trajectories. These invariably involve being given some information about the trajectory, which you must then use to work out something else. These problems are all somewhat tedious, but we will show a couple of examples to uphold the fine traditions of a 19th century education.

Estimate how far you could throw a stone from the top of the Kremlin palace.



Note that

1. The horizontal and vertical components of velocity at time *t=0* follow as
   $$V_x = v_0 \cos\theta \qquad V_y = 0 \quad V_z = v_0 \sin\theta$$
2. The components of the position of the particle at time *t=0* are $X = 0, Y = 0, Z = H$
3. The trajectory of the particle follows as
   $$\mathbf{r} = (v_0 \cos\theta t)\mathbf{i} + \left(H + v_0 \sin\theta t - \frac{1}{2} g t^2\right)\mathbf{k}$$
4. When the particle hits the ground, its position vector is $\mathbf{r} = D\mathbf{i}$. This must be on the trajectory, so

$$(v_0\cos\theta t_I)\mathbf{i}+\left(H+v_0\sin\theta t_I-\frac{1}{2}gt_I^2\right)\mathbf{k}=D\mathbf{i}$$

where t_I is the time of impact.

5. The two components of this vector equation gives us two equations for the two unknowns $\{t_I, D\}$, which can be solved

```
clear all
syms v0 theta tI g D H real
eq1 = v0*cos(theta)*tI==D;
eq2 = H+v0*sin(theta)*tI - g*tI^2/2;
[D,tI] = solve([eq1,eq2],[D,tI]);
D = simplify(D)
```

D =

$$\begin{pmatrix} \dfrac{v_0\cos(\theta)\left(\sqrt{v_0^2\sin(\theta)^2+2\,H\,g}+v_0\sin(\theta)\right)}{g} \\ -\dfrac{v_0\cos(\theta)\left(\sqrt{v_0^2\sin(\theta)^2+2\,H\,g}-v_0\sin(\theta)\right)}{g} \end{pmatrix}$$

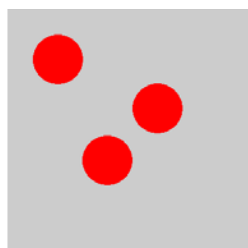
For a rough estimate of the distance we can use the following numbers

1. Height of Kremlin palace – 71m
2. Throwing velocity – maybe 25mph? (pretty pathetic, I know - you can probably do better, especially if you are on the baseball/softball teams).
3. Throwing angle – 45 degrees.

Substituting numbers gives 36m (118ft).

If you want to go wild, you can maximize D with respect to θ, but this won't improve your estimate much.

Silicon nanoparticles with radius 20nm *are in thermal motion near a flat surface. At the surface, they have an average velocity* $\sqrt{2kT/m}$, *where m is their mass, T is the temperature and k=1.3806503 × 10-23 is the Boltzmann constant. Estimate the maximum height above the surface that a typical particle can reach during its thermal motion, assuming that the only force acting on the particles is gravity*



1. The particle will reach its maximum height if it happens to be traveling vertically, and does not collide with any other particles.
2. At time *t=0* such a particle has position $X=0, Y=0, Z=0$ and velocity $V_x=0 \quad V_y=0 \;\; V_z=\sqrt{2kT/m}$
3. For time *t>0* the position vector of the particle follows as

$$\mathbf{r}=\left(\left(\sqrt{2kT/m}\right)t-\frac{1}{2}gt^2\right)\mathbf{k}$$

Its velocity is

$$\mathbf{v}=\left(\sqrt{2kT/m}-gt\right)\mathbf{k}$$

4. When the particle reaches its maximum height, its velocity must be equal to zero (if you don't see this by visualizing the motion of the particle, you can use the mathematical statement that if

$\mathbf{r} = y\mathbf{k}$ is a maximum, then $d\mathbf{r}/dt = \mathbf{v} = (dy/dt)\mathbf{k} = 0$). Therefore, at the instant of maximum height $\mathbf{v} = \left(\sqrt{2kT/m} - gt_{\max}\right)\mathbf{k} = \mathbf{0}$

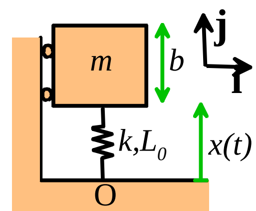
5. This shows that the instant of max height occurs at time $t_{\max} = \left(\sqrt{2kT/m}\right)/g$
6. Substituting this time back into the position vector shows that the position vector at max height is

$$\mathbf{r} = \left(\frac{2kT}{mg} - \frac{1}{2}\frac{2kT}{mg}\right)\mathbf{k} = \frac{kT}{mg}\mathbf{k}$$

7. Si has a density of about 2330 kg/m^3. At room temperature (293K) we find that the distance is surprisingly large: 10mm or so. Gravity is a very weak force at the nano-scale – surface forces acting between the particles, and the particles and the surface, are much larger.

Example 2: Free vibration of a suspension system.

A vehicle suspension can be idealized as a mass m supported by a spring. The spring has stiffness k and un-stretched length L_0. To test the suspension, the vehicle is constrained to move vertically, as shown in the figure. It is set in motion by stretching the spring to a length L and then releasing it (from rest). Find an expression for the motion of the vehicle after it is released.



As an aside, it is worth noting that a particle idealization is usually too crude to model a vehicle – a rigid body approximation is much better. In this case, however, we assume that the vehicle does not rotate. Under these conditions the equations of motion for a rigid body reduce to $\mathbf{F} = m\mathbf{a}$ and $\mathbf{M} = \mathbf{0}$, and we shall find that we can analyze the system as if it were a particle.

1. ***Introduce variables to describe the motion***: The length of the spring $x(t)$ is a convenient way to describe motion.

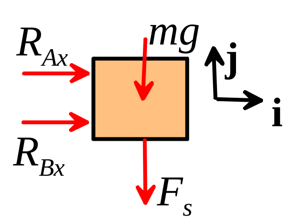
2. ***Write down the position vector in terms of these variables***: We can take the origin at O as shown in the figure. The position vector of the center of mass of the block is then

$$\mathbf{r} = \left[x(t) + \frac{b}{2}\right]\mathbf{j}$$

3. ***Differentiate the position vector with respect to time to find the acceleration***. For this example, this is trivial

$$\mathbf{v} = \frac{dx}{dt}\mathbf{j} \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{j}$$

4. ***Draw a free body diagram***. The free body diagram is shown in the figure: the mass is subjected to the following forces



- Gravity, acting at the center of mass of the vehicle
- The force due to the spring
- Reaction forces at each of the rollers that force the vehicle to move vertically.

Recall the spring force law, which says that the forces exerted by a spring act parallel to its length, tend to shorten the spring, and are proportional to the difference between the length of the spring and its un-stretched length.

5. ***Write down Newton's laws of motion.*** This is easy

$$\mathbf{F} = m\mathbf{a} \Rightarrow (R_{Ax} + R_{Bx})\mathbf{i} - \left(mg + k(x - L_0)\right)\mathbf{j} = m\frac{d^2x}{dt^2}\mathbf{j}$$

The **i** and **j** components give two scalar equations of motion

$$(R_{Ax} + R_{Bx}) = 0$$

$$\frac{d^2x}{dt^2} = -\left(g + \frac{k}{m}(x - L_0)\right)$$

6. ***Eliminate reactions*** – this is not needed in this example.

7. ***Identify initial conditions.*** The initial conditions were given in this problem – at time *t=0*, we know that $x = L$ and $dx/dt = 0$

8. ***Solve the equations of motion***. Again, we will first integrate the equations of motion by hand, and then repeat the calculation with Matlab. The equation of motion is

$$\frac{d^2x}{dt^2} = -\left(g + \frac{k}{m}(x - L_0)\right)$$

We can re-write this in terms of

$$\frac{dx}{dt} = v_x$$

This gives

$$\frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx} = -\left(g + \frac{k}{m}(x - L_0)\right)$$

We can separate variables and integrate

$$\int_0^{v_x} v_x dv_x = \int_L^x -\left(g + \frac{k}{m}(x - L_0)\right)dx$$

$$\Rightarrow \frac{1}{2}v_x^2 = -g(x - L) - \frac{k}{2m}\left(x^2 - L^2\right) + \frac{k}{m}L_0(x - L)$$

$$\Rightarrow v_x = \sqrt{\frac{k}{m}\left[\left(L - L_0 + \frac{mg}{k}\right)^2 - \left(x - L_0 + \frac{mg}{k}\right)^2\right]}$$

Don't worry if the last line looks mysterious – writing the solution in this form just makes the algebra a bit simpler. We can now integrate the velocity to find *x*

$$v_x = \frac{dx}{dt} = \sqrt{\frac{k}{m}\left[\left(L - L_0 + \frac{mg}{k}\right)^2 - \left(x - L_0 + \frac{mg}{k}\right)^2\right]}$$

$$\Rightarrow \int_L^x \frac{dx}{\sqrt{\frac{k}{m}\left[\left(L - L_0 + \frac{mg}{k}\right)^2 - \left(x - L_0 + \frac{mg}{k}\right)^2\right]}} = \int_0^t dt$$

The integral on the left can be evaluated using the substitution

$$\frac{(x - L_0 + mg/k)}{(L - L_0 + mg/k)} = \cos\theta$$

so that

$$\int_0^{\theta_0} \frac{-\sin\theta d\theta}{\sqrt{\frac{k}{m}\left[1-\cos^2\theta\right]}} = \int_0^t dt \qquad \theta_0 = \cos^{-1}\frac{(x - L_0 + mg/k)}{(L - L_0 + mg/k)}$$

$$\Rightarrow \theta_0 = -t\sqrt{\frac{k}{m}}$$

$$\Rightarrow \frac{(x - L_0 + mg/k)}{(L - L_0 + mg/k)} = \cos\left(-t\sqrt{\frac{k}{m}}\right)$$

$$\Rightarrow x = (L - L_0 + mg/k)\cos\left(-t\sqrt{\frac{k}{m}}\right) + L_0 - mg/k$$

Here's the Matlab solution

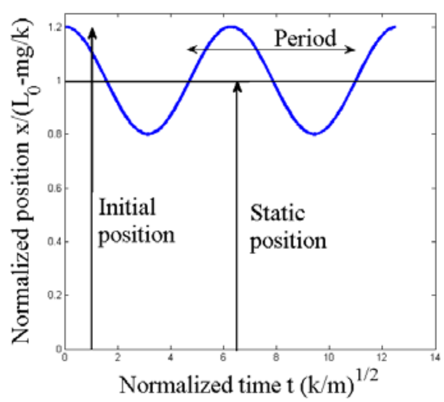
```
clear all
syms g k m L L0 real
assume(k>0); assume(m>0);
syms x(t) v(t)
eq = diff(x(t),t,2) == -(g+k*(x(t)-L0)/m);
v(t) = diff(x(t),t);
IC = [x(0)==L, v(0)==0];
x(t) = simplify(dsolve([eq,IC]))
```

x(t) =

$$\cos\left(\frac{\sqrt{k}\,t}{\sqrt{m}}\right)\left(L - \frac{L_0 k - g m}{k}\right) + \frac{L_0 k - g m}{k}$$

It doesn't quite look the same as the hand calculation – but of course $\cos\theta = \cos(-\theta)$ so they really are the same.

The solution is plotted in the figure. The behavior of vibrating systems will be discussed in more detail later in this course, but it is worth noting some features of the solution:



1. The average position of the mass is $\bar{x} = L_0 - mg/k$. Here, *mg/k* is the *static deflection* of the spring i.e. the deflection of the spring due to the weight of the vehicle (without motion). This means that the car vibrates symmetrically about its static deflection.
2. The *amplitude* of vibration is $L - L_0 + mg/k$. This corresponds to the distance of the mass above its average position at time *t=0*.
3. The *period* of oscillation (the time taken for one complete cycle of vibration) is $T = 2\pi\sqrt{m/k}$

4. The *frequency* of oscillation (the number of cycles per second) is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (note *f=1/T)*.

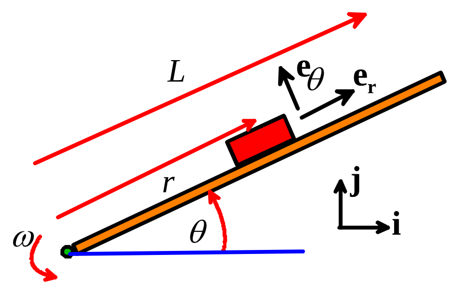
   Frequency is also sometimes quoted as *angular frequency*, which is related to f by $\omega = 2\pi f = \sqrt{k/m}$. Angular frequency is in radians per second.

An interesting feature of these results is that the static deflection is related to the frequency of oscillation - so if you measure the static deflection $\delta = mg/k$, you can calculate the (angular) vibration frequency as $\omega = \sqrt{g/\delta}$

Example 3: Silly FE exam problem

This example shows how polar coordinates can be used to analyze motion.

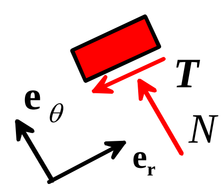
The rod shown in the picture rotates at constant angular speed in the horizontal plane. The interface between block and rod has friction coefficient μ. The rod pushes a block of mass *m*, which starts at *r=0* with radial speed *V*. Find an expression for *r(t)*.



1. **Introduce variables to describe the motion** – the polar coordinates r, θ work for this problem

2. **Write down the position vector and differentiate to find acceleration** – we don't need to do this – we can just write down the standard result for polar coordinates

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\mathbf{e}_r + 2\frac{dr}{dt}\omega\mathbf{e}_\theta$$



3. **Draw a free body diagram –** shown in the figure – note that it is important to draw the friction force in the correct direction. The block will slide radially outwards, and friction opposes the slip.

4. **Write down Newton's law**

$$-T\mathbf{e}_r + N\mathbf{e}_\theta = m\left\{\left(\frac{d^2r}{dt^2} - r\omega^2\right)\mathbf{e}_r + 2\frac{dr}{dt}\omega\mathbf{e}_\theta\right\}$$

5. **Eliminate reactions**

   The $\mathbf{e}_\theta$ component of **F**=m**a** shows that

$$N = 2\frac{dr}{dt}\omega$$

The friction law gives $T = \mu N$. Substituting this into the $\mathbf{e}_r$ component of **F**=m**a** and simplifying shows that

$$\frac{d^2 r}{dt^2} + 2\mu\omega \frac{dr}{dt} - r\omega^2 = 0$$

6. **Identify initial conditions** Here, *r=0* *dr/dt=V* at time *t=0*.

7. **Solve the equation:** If you've taken AM33 you will know how to solve this equation… If not you can use Matlab:

```
clear all
syms mu omega V0 real
syms r(t) v(t)
diffeq = diff(r(t),t,2) + 2*mu*omega*diff(r(t),t) - r(t)*omega^2 ==0;
v(t) = diff(r(t),t);
IC = [r(0)==0,v(0)==V0];
r(t) = simplify(dsolve(diffeq,IC))
```

r(t) =

$$\frac{V_0\, e^{-\omega t\,\left(\mu - \sqrt{\mu^2+1}\right)} - V_0\, e^{-\omega t\,\left(\mu + \sqrt{\mu^2+1}\right)}}{2\,\omega\,\sqrt{\mu^2+1}}$$

This can be simplified slightly by hand:

$$r(t) = \frac{V}{\omega\sqrt{1+\mu^2}} e^{-\mu\omega t} \sinh(\sqrt{1+\mu^2}\,\omega t)$$

3.3.3 Numerical solutions to equations of motion using MATLAB

In the preceding section, we were able to solve all our equations of motion exactly, and hence to find *formulas* that describe the motion of the system. This should give you a warm and fuzzy feeling – it appears that with very little work, you can predict *everything* about the motion of the system. You may even have visions of running a consulting business from your yacht in the Caribbean, with nothing more than your chef, your masseur (or masseuse) and a laptop with a copy of MAPLE.

Unfortunately real life is not so simple. Equations of motion for most engineering systems cannot be solved exactly. Even very simple problems, such as calculating the effects of air resistance on the trajectory of a particle, cannot be solved exactly.

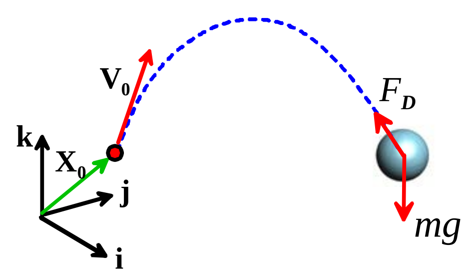
For nearly all practical problems, the equations of motion need to be solved *numerically*, by using a computer to calculate values for the position, velocity and acceleration of the system as functions of time. Vast numbers of computer programs have been written for this purpose – some focus on very specialized applications, such as calculating orbits for spacecraft (STK); calculating motion of atoms in a material (LAMMPS); solving fluid flow problems (e.g. fluent, CFDRC); or analyzing deformation in solids (e.g. ABAQUS, ANSYS, NASTRAN, DYNA); others are more general purpose equation solving programs.

In this course we will use a general purpose program called MATLAB, which is widely used in all engineering applications. You should complete the MATLAB tutorial before proceeding any further.

In the remainder of this section, we provide a number of examples that illustrate how MATLAB can be used to solve dynamics problems. Each example illustrates one or more important technique for setting up or solving equations of motion.

Example 1: Trajectory of a particle near the earth's surface (with air resistance)

As a simple example we set up MATLAB to solve the particle trajectory problem discussed in the preceding section. We will extend the calculation to account for the effects of air resistance, however. We will assume that our projectile is spherical, with diameter *D*, and we will assume that there is no wind. You may find it helpful to review the discussion of aerodynamic drag forces in Section 2.1.7 before proceeding with this example.



1. ***Introduce variables to describe the motion***: We can simply use the Cartesian coordinates of the particle $(x(t), y(t), z(t))$

2. ***Write down the position vector in terms of these variables***: $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

3. ***Differentiate the position vector with respect to time to find the acceleration***. Simple calculus gives

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

4. ***Draw a free body diagram***. The particle is now subjected to two forces, as shown in the picture.

Gravity – as always we have $\mathbf{F}_g = -mg\mathbf{k}$.

Air resistance.

The *magnitude* of the air drag force is given by $F_D = \frac{1}{2}\rho C_D D V^2$, where

- ρ is the air density,
- C_D is the drag coefficient,
- D is the projectile's diameter, and
- V is the magnitude of the projectile's velocity relative to the air. Since we assumed the air is stationary, V is simply the magnitude of the particle's velocity, i.e.

$$V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The *Direction* of the air drag force is always opposite to the direction of motion of the projectile through the air. In this case the air is stationary, so the drag force is simply opposite to the direction of the particle's velocity. Note that $\mathbf{v}/V$ is a unit vector parallel to the particle's velocity. The drag force vector is therefore

$$\mathbf{F}_D = -\frac{1}{2}\rho C_D \frac{\pi D^2}{4} V^2 \frac{\mathbf{v}}{V} = -\frac{1}{2}\rho C_D \frac{\pi D^2}{4} V \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right)$$

The total force vector is therefore

$$\mathbf{F} = -mg\mathbf{k} - \frac{1}{2}\rho C_D \frac{\pi D^2}{4} V \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right)$$

5. ***Write down Newton's laws of motion.***

$$\mathbf{F} = m\mathbf{a} \Rightarrow -mg\mathbf{k} - \frac{1}{2}\rho C_D \frac{\pi D^2}{4} V \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) = m\left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \right)$$

It is helpful to simplify the equation by defining a *specific drag coefficient* $c = \dfrac{\pi}{8m}\rho D^2 C_D$, so that

$$-g\mathbf{k} - cV\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) = \left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \right)$$

The vector equation actually represents three separate differential equations of motion

$$\frac{d^2x}{dt^2} = -cV\frac{dx}{dt} \qquad \frac{d^2y}{dt^2} = -cV\frac{dy}{dt} \qquad \frac{d^2z}{dt^2} = -g - cV\frac{dz}{dt}$$

6. ***Eliminate reactions*** – this is not needed in this example.

7. ***Identify initial conditions.*** The initial conditions were given in this problem - we have that

$$\left\{ x = X_0 \quad \frac{dx}{dt} = V_x \right\} \qquad \left\{ y = Y_0 \quad \frac{dy}{dt} = V_y \right\} \quad \left\{ z = Z_0 \quad \frac{dz}{dt} = V_z \right\}$$

8. ***Solve the equations of motion***. We can't use the magic 'dsolve' command in MAPLE to solve this equation – it has no known exact solution. So instead, we use MATLAB to generate a numerical solution.

This takes two steps. First, we must turn the equations of motion into a form that MATLAB can use. This means we must convert the equations into first-order vector valued differential equation of the general form $\dfrac{d\mathbf{y}}{dt} = f(t, \mathbf{y})$. Then, we must write a MATLAB script to integrate the equations of motion.

Converting the equations of motion: We can't solve directly for (*x*,*y*,*z*), because these variables get differentiated more than once with respect to time. To fix this, we *introduce the time derivatives of* (*x*,*y*,*z*) as new unknown variables. In other words, we will solve for (x, y, z, v_x, v_y, v_z), where

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dx}{dt} \qquad v_z = \frac{dx}{dt}$$

These definitions are three new equations of motion relating our unknown variables.   In addition, we can re-write our original equations of motion as

$$\frac{dv_x}{dt} = -cVv_x \qquad \frac{dv_y}{dt} = -cVv_y \qquad \frac{dv_z}{dt} = -g - cVv_z$$

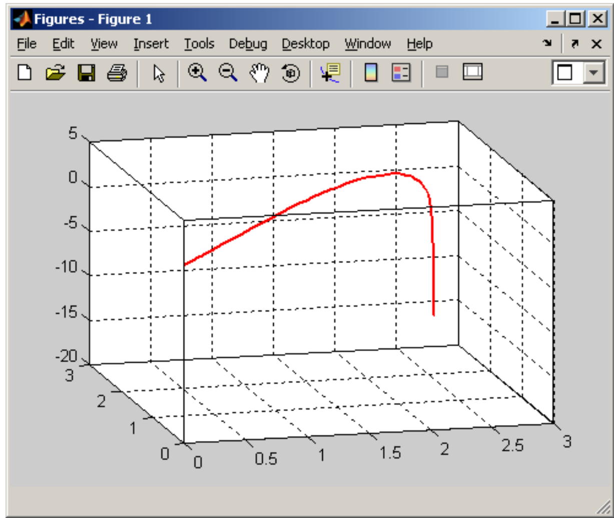
So, expressed as a vector valued differential equation, our equations of motion are

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -cVv_x \\ -cVv_y \\ -g - cVv_z \end{bmatrix}$$

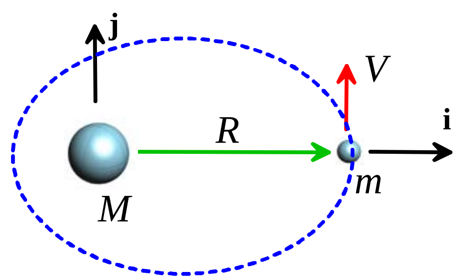
MATLAB script. The procedure for solving these equations is discussed in the MATLAB tutorial. A basic MATLAB script is listed below.

```
function trajectory_3d
      % Function to plot trajectory of a projectile
      % launched from position X0 with velocity V0
      % with specific air drag coefficient c
      % stop_time specifies the end of the calculation

      g = 9.81; % gravitational accel
      c=0.5; % The constant c
      X0=0; Y0=0; Z0=0; % The initial position
      VX0=10; VY0=10; VZ0=20;
      stop_time = 5;

      initial_w = [X0,Y0,Z0,VX0,VY0,VZ0]; % The solution at t=0

      [times,sols] = ode45(@(t,w) eom(t,w,c,g),[0,stop_time],initial_w);

      plot3(sols(:,1),sols(:,2),sols(:,3)) % Plot the trajectory

end
function dwdt = eom(t,w,c,g)
      % Variables stored as follows w = [x,y,z,vx,vy,vz]
      % i.e. x = w(1), y=w(2), z=w(3), etc
         x = w(1); y=w(2); z=w(3);
         vx = w(4); vy = w(5); vz = w(6);
         vmag = sqrt(vx^2+vy^2+vz^2);
         dxdt = vx; dydt = vy; dzdt = vz;
         dvxdt = -c*vmag*vx;
         dvydt = -c*vmag*vy;
         dvzdt = -c*vmag*vz-g;
         dwdt = [dxdt;dydt;dzdt;dvxdt;dvydt;dvzdt];
end
```

This produces a plot that looks like this (the plot's been edited to add the grid,etc)



Example 2: Simple satellite orbit calculation



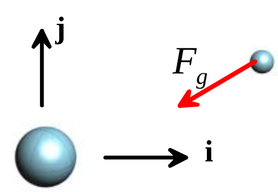
The figure shows satellite with mass m orbiting a planet with mass M. At time $t=0$ the satellite has position vector $\mathbf{r} = R\mathbf{i}$ and velocity vector $\mathbf{v} = V\mathbf{j}$. The planet's motion may be neglected (this is accurate as long as $M>>m$). Calculate and plot the orbit of the satellite.

1. ***Introduce variables to describe the motion***: We will use the (x,y) coordinates of the satellite.

2. ***Write down the position vector in terms of these variables***: $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

3. ***Differentiate the position vector with respect to time to find the acceleration***.

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$



4. ***Draw a free body diagram***. The satellite is subjected to a gravitational force.

The magnitude of the force is $F_g = \frac{GMm}{r^2}$, where

- G is the gravitational constant, and
- $r = \sqrt{x^2 + y^2}$ is the distance between the planet and the satellite

The direction of the force is always towards the origin: $-\mathbf{r}/r$ is therefore a unit vector parallel to the direction of the force. The total force acting on the satellite is therefore

$$\mathbf{F} = -\frac{GMm}{r^2}\frac{\mathbf{r}}{r} = -\frac{GMm}{r^3}(x\mathbf{i} + y\mathbf{j})$$

5. ***Write down Newton's laws of motion.***

$$\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{GMm}{r^3}(x\mathbf{i} + y\mathbf{j}) = m\left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}\right)$$

The vector equation represents two separate differential equations of motion

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x \qquad \frac{d^2y}{dt^2} = -\frac{GM}{r^3}y$$

6. ***Eliminate reactions*** – this is not needed in this example.

7. ***Identify initial conditions.*** The initial conditions were given in this problem - we have that

$$\left\{x = R \quad \frac{dx}{dt} = 0\right\} \qquad \left\{y = 0 \quad \frac{dy}{dt} = V\right\}$$

8. ***Solve the equations of motion***. We follow the usual procedure: (i) convert the equations into MATLAB form; and (ii) code a MATLAB script to solve them.

Converting the equations of motion: We introduce the time derivatives of (x,y) as new unknown variables. In other words, we will solve for (x, y, v_x, v_y), where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dx}{dt}$$

These definitions are new equations of motion relating our unknown variables. In addition, we can re-write our original equations of motion as

$$\frac{dv_x}{dt} = -\frac{GM}{r^3}x \qquad \frac{dv_y}{dt} = -\frac{GM}{r^3}y$$

So, expressed as a vector valued differential equation, our equations of motion are

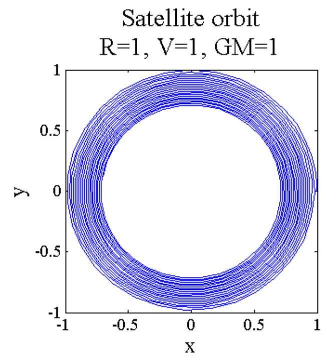
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ -GMx/r^3 \\ -GMy/r^3 \end{bmatrix}$$

Matlab script: Here's a simple script to solve these equations.

```
function satellite_orbit
% Function to plot orbit of a satellite
% launched from position (R,0) with velocity (0,V)

GM=1;
R=1;
V=1;
Time=100;
w0 = [R,0,0,V]; % Initial conditions

[t_values,w_values] = ode45(@(t,w) odefunc(t,w,GM),[0,time],w0);

plot(w_values(:,1),w_values(:,2))
```

```
end
function dwdt = odefunc(t,w,GM)
   x=w(1); y=w(2); vx=w(3); vy=w(4);
   r = sqrt(x^2+y^2);
   dwdt = [vx;vy;-GM*x/r^3;-GM*y/r^3];
end
```

Running the script produces the result shown (the plot was annotated by hand)



Do we believe this result? It is a bit surprising – the satellite seems to be spiraling in towards the planet. Most satellites don't do this – so the result is a bit suspicious. The First Law of Scientific Computing states that ` *if a computer simulation predicts a result that surprises you, it is probably wrong*.'

So how can we test our computation? There are two good tests:

1. Look for any features in the simulation that you can predict without computation, and compare your predictions with those of the computer.
2. Try to find a special choice of system parameters for which you can derive an exact solution to your problem, and compare your result with the computer

We can use both these checks here.

1. Conserved quantities For this particular problem, we know that (i) the total energy of the system should be constant; and (ii) the angular momentum of the system about the planet should be constant (these conservation laws will be discussed in the next chapter – for now, just take this as given). The total energy of the system consists of the potential energy and kinetic energy of the satellite, and can be calculated from the formula

$$E = -\frac{GMm}{r} + \frac{1}{2}mv^2 = -\frac{GMm}{r} + \frac{1}{2}m\left(v_x^2 + v_y^2\right)$$
$$\Rightarrow \frac{E}{m} = -\frac{GM}{r} + \frac{1}{2}\left(v_x^2 + v_y^2\right)$$

The total angular momentum of the satellite (about the origin) can be calculated from the formula

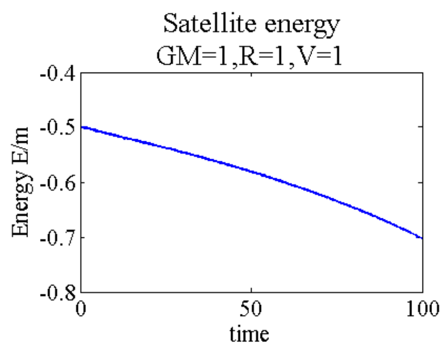
$$\mathbf{H} = \mathbf{r} \times m\mathbf{v} = \left(x\mathbf{i} + y\mathbf{j}\right) \times m(v_x\mathbf{i} + v_y\mathbf{j}) = m\left(xv_y - yv_x\right)\mathbf{k}$$
$$\Rightarrow \frac{|\mathbf{H}|}{m} = \left(xv_y - yv_x\right)$$

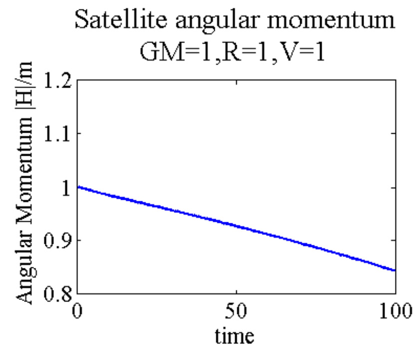
(If you don't know these formulas, don't panic – we will discuss energy and angular momentum in the next part of the course)

We can have MATLAB plot E/m and $|\mathbf{H}|/m$, and see if these are really conserved. The energy and momentum can be calculated by adding these lines to the MATLAB script

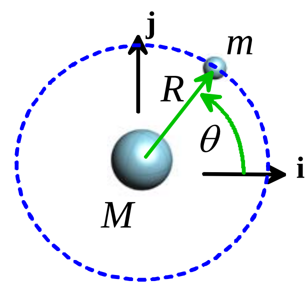
```
for i =1:length(t)
     r = sqrt(w_values(i,1)^2 + w_values(i,2)^2)
     vmag = sqrt(w_values(i,1)^2 + w_values(i,2)^2)
      energy(i) = -GM/r + vmag^2/2;
      angularm(i) = w_values(i,1)*w_values(i,4)-w_values(i,2)*w_values(i,3);
end
```

You can then plot the results (e.g. plot(t_values,energy)). The results are shown below.





These results look *really bad* – neither energy, nor angular momentum, are conserved in the simulation. Something is clearly very badly wrong.



Comparison to exact solution: It is not always possible to find a convenient exact solution, but in this case, we might guess that some special initial conditions could set the satellite moving on a *circular* path. A circular path might be simple enough to analyze by hand. So let's assume that the path is circular, and try to find the necessary initial conditions. If you still remember the circular motion formulas, you could use them to do this. But only morons use formulas – here we will derive the solution from scratch. Note that, for a circular path

(a) the particle's radius *r*=constant. In fact, we know *r*=*R*, from the position at time *t=0*.

(b) The satellite must move at constant speed, and the angle θ must increase linearly with time, i.e. $\theta = \omega t$ where $\theta = \omega$ is a constant (see section 3.1.3 to review motion at constant speed around a circle).

With this information we can solve the equations of motion. Recall that the position, velocity and acceleration vectors for a particle traveling at constant speed around a circle are

$$\mathbf{r} = R\cos\theta(t)\mathbf{i} + R\sin\theta(t)\mathbf{j}$$

$$\mathbf{v} = R\omega(-\sin\theta(t)\mathbf{i} + \cos\theta(t)\mathbf{j})$$

$$\mathbf{a} = -R\omega^2\left(\cos\theta(t)\mathbf{i} + R\sin\theta(t)\mathbf{j}\right)$$

We know that $|\mathbf{v}| = V$ from the initial conditions, and $|\mathbf{v}|$ is constant. This tells us that

$$V = R\omega$$

Finally, we can substitute this into Newton's law

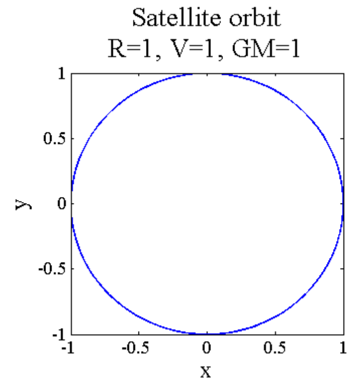
$$\mathbf{F} = m\mathbf{a} \Rightarrow -\frac{GMm}{R^2}\frac{\mathbf{r}}{R} = m\mathbf{a} \Rightarrow -\frac{GMm}{R^2}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -m\frac{V^2}{R}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

Both components of the equation of motion are satisfied if we choose

$$\frac{GM}{R^2} = \frac{V^2}{R}$$

So, if we choose initial values of GM, V, R satisfying this equation, the orbit will be circular. In fact, our original choice, $GM = 1, V = 1, R = 1$ should have given a circular orbit. It did not. Again, this means our computer generated solution is totally wrong.

Fixing the problem: In general, when computer predictions are suspect, we need to check the following

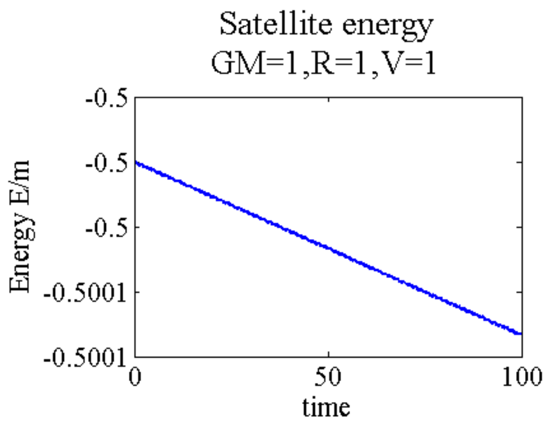


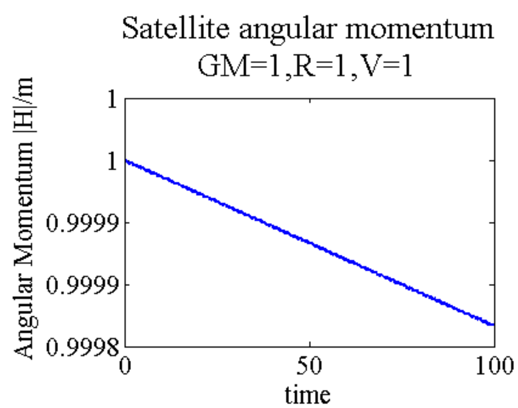
1. Is there an error in our MATLAB program? This is nearly always the cause of the problem. In this case, however, the program *is* correct (it's too simple to get wrong, even for me).
2. There may be something wrong with our equations of motion (because we made a mistake in the derivation). This would not explain the discrepancy between the circular orbit we predict and the simulation, since we used the same equations in both cases.
3. Is the MATLAB solution sufficiently accurate? Remember that by default the ODE solver tries to give a solution that has 0.1% error. This may not be good enough. So we can try solving the problem again, but with better accuracy. We can do this by modifying the MATLAB call to the equation solver as follows

```
options = odeset('RelTol',0.00001);
[t_values,w_vlues] = ode45(@(t,w) odefunc(t,w,GM),[0,time],w0,options);
```

4. Is there some feature of the equation of motion that makes them especially difficult to solve? In this case we might have to try a different equation solver, or try a different way to set up the problem.

The figure on the right shows the orbit predicted with the better accuracy. You can see there is no longer any problem – the orbit is perfectly circular. The figures below plot the energy and angular momentum predicted by the computer.

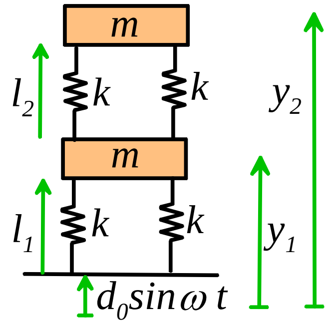




There is a small change in energy and angular momentum but the rate of change has been reduced dramatically. We can make the error smaller still by using improving the tolerances further, if this is needed. But the changes in energy and angular momentum are only of order 0.01% over a large number of orbits: this would be sufficiently accurate for most practical applications.

Most ODE solvers are purposely designed to lose a small amount of energy as the simulation proceeds, because this helps to make them more stable. But in some applications this is unacceptable – for example in a molecular dynamic simulation where we are trying to predict the entropic response of a polymer, or a free vibration problem where we need to run the simulation for an extended period of time. There are special ODE solvers that will conserve energy exactly.

Example 3: Earthquake response of a 2-storey building



The figure shows a very simple idealization of a 2-storey building. The roof and second floor are idealized as blocks with mass *m*. They are supported by structural columns, which can be idealized as springs with stiffness *k* and unstretched length *L*. At time *t=0* the floors are at rest and the columns have lengths $l_1 = L - mg/k$ $l_2 = L - mg/2k$ (can you show this?). We will neglect the thickness of the floors themselves, to keep things simple.

For time *t>0*, an earthquake makes the ground vibrate vertically. The ground motion can be described using the equation $d = d_0 \sin \omega t$. Horizontal motion may be neglected. Our goal is to calculate the motion of the first and second floor of the building.

It is worth noting a few points about this problem:

1. You may be skeptical that the floor of a building can be idealized as a particle (then again, maybe you couldn't care less…). If so, you are right – it certainly is not a `small' object. However, because the floors move vertically without rotation, the rigid body equations of motion simply reduce to $\mathbf{F} = m\mathbf{a}$ and **M=0**, where the moments are taken about the center of mass of the block. The floors behave as though they are particles, even though they are very large.
2. Real earthquakes involve predominantly *horizontal*, not vertical motion of the ground. In addition, structural columns resist extensional loading much more strongly than transverse loading. So we should really be analyzing horizontal motion of the building rather than vertical motion. However, the free body diagrams for horizontal motion are messy (see if you can draw them) and the equations of motion for vertical and horizontal motion turn out to be the same, so we consider vertical motion to keep things simple.
3. This problem could be solved analytically (e.g. using the `dsolve' feature of MAPLE) – a numerical solution is not necessary. Try this for yourself.

1. ***Introduce variables to describe the motion***: We will use the height of each floor (y_1, y_2) as the variables.

2. ***Write down the position vector in terms of these variables***: We now have to worry about two masses, and must write down the position vector of both

$$\mathbf{r}_1 = y_1 \mathbf{j} \qquad \mathbf{r}_2 = y_2 \mathbf{j}$$

Note that we must measure the position of each mass from a *fixed* point.

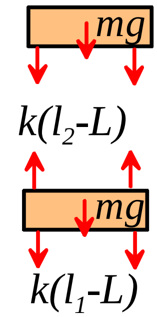
3. ***Differentiate the position vector with respect to time to find the acceleration***.

$$\mathbf{v}_1 = \left(\frac{dy_1}{dt}\right)\mathbf{j} \qquad \mathbf{a}_1 = \left(\frac{dy_1^2}{dt^2}\right)\mathbf{j}$$

$$\mathbf{v}_2 = \left(\frac{dy_2}{dt}\right)\mathbf{j} \qquad \mathbf{a}_2 = \left(\frac{dy_2^2}{dt^2}\right)\mathbf{j}$$

4. ***Draw a free body diagram***. We must draw a free body diagram for each mass. The resultant force acting on the bottom and top masses, respectively, are

$$\mathbf{F}_1 = \{-mg - 2k(l_1 - L) + 2k(l_2 - L)\}\,\mathbf{j} \qquad \mathbf{F}_2 = -mg - 2k(l_2 - L)\mathbf{j}$$



We will have to find the spring lengths l_1, l_2 in terms of our coordinates y_1, y_2 to solve the problem. Geometry shows that

$$l_2 = y_2 - y_1 \qquad l_1 = y_1 - d_0 \sin \omega t$$

5. ***Write down Newton's laws of motion.*** **F**=*m***a** for each mass gives

$$\{-mg - 2k(l_1 - L) + 2k(l_2 - L)\}\,\mathbf{j} = m\frac{d^2 y_1}{dt^2}\mathbf{j}$$

$$\{-mg - 2k(l_2 - L)\}\,\mathbf{j} = m\frac{d^2 y_2}{dt^2}\mathbf{j}$$

This is two equations of motion – we can substitute for l_1, l_2 and rearrange them as

$$\frac{dy_1^2}{dt^2} = \left(-g - 2\frac{k}{m}(y_1 - d_0 \sin \omega t - L) + 2\frac{k}{m}(y_2 - y_1 - L)\right)$$

$$\frac{dy_2^2}{dt^2} = \left(-g - 2\frac{k}{m}(y_2 - y_1 - L)\right)$$

6. ***Eliminate reactions*** – this is not needed in this example.

7. ***Identify initial conditions.*** We know that, at time *t=0*

$$y_1 = L - mg/k \qquad \frac{dy_1}{dt} = 0 \qquad y_2 = 2L - 3mg/2k \qquad \frac{dy_2}{dt} = 0$$

8. ***Solve the equations of motion***. We need to (i) reduce the equations to the standard MATLAB form and (ii) write a MATLAB script to solve them.

Converting the equations. We now need to do two things: (a) remove the second derivatives with respect to time, by introducing new variables; and (b) rearrange the equations into the form $d\mathbf{y}/dt = \mathbf{f}(t, \mathbf{y})$. We remove the derivatives by introducing $v_1 = \frac{dl_1}{dt} \quad v_2 = \frac{dl_2}{dt}$ as additional unknown variables, in the usual way. Our equations of motion can then be expressed as

$$\frac{dy_1}{dt} = v_1$$

$$\frac{dy_2}{dt} = v_2$$

$$\frac{dv_1}{dt^2} = \left(-g - 2\frac{k}{m}(y_1 - d_0 \sin \omega t - L) + 2\frac{k}{m}(y_2 - y_1 - L)\right)$$

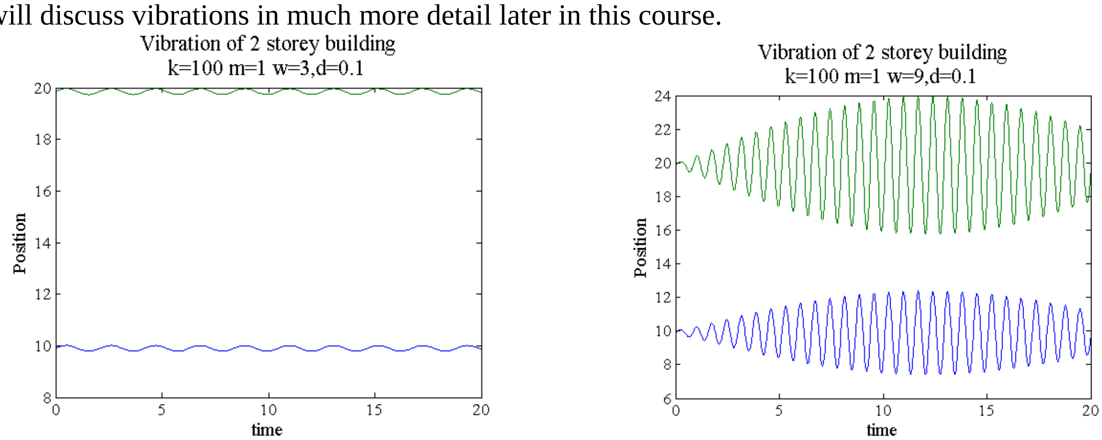
$$\frac{dv_2}{dt^2} = \left(-g - 2\frac{k}{m}(y_2 - y_1 - L)\right)$$

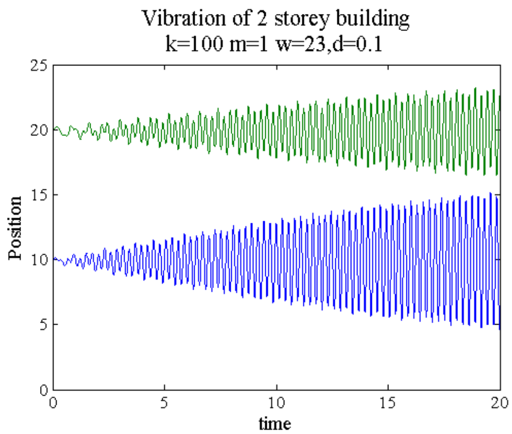
We can now code MATLAB to solve these equations directly for d**y**/d**t**. A script (which plots the position of each floor as a function of time) is shown below.

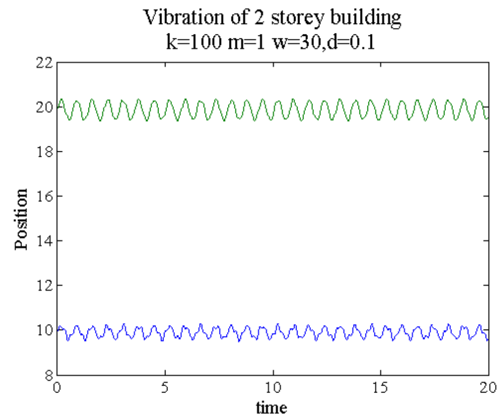
```
function building
      %
      k=100;
      m=1;
      omega=9;
      d=0.1;
      L=10;
      time=20;
      g = 9.81;
      w0 = [L-m*g/k,2*L-3*m*g/(2*k),0,0];
      [t_values,w_values] = ode45(@(t,w) eom(k,m,L,d,omega,g),[0,time],w0);
      plot(t_values,w_values(:,1:2));
end
function dwdt = eom(t,w,k,m,L,d,omega,g)
      y1=w(1);
      y2=w(2);
      v1=w(3);
      v2=w(4);

      dwdt = [v1;v2;...
     -2*k*(y1-d*sin(omega*t)-L)/m+2*k*(y2-y1-L)/m;...
                -g-2*k*(y2-y1-L)/m];
end
```

The figures below plot the height of each floor as a function of time, for various earthquake frequencies. For special earthquake frequencies (near the two resonant frequencies of the structure) the building vibrations are very severe. As long as the structure is designed so that its resonant frequencies are well away from the frequency of a typical earthquake, it will be safe.

We will discuss vibrations in much more detail later in this course.







Final remark: we made this calculation a bit more complicated than necessary by solving for the heights y_1, y_2. It is better to solve for the *deflections* of the floors instead of their heights. Define

$$z_1 = y_1 - (L - mg/k) \qquad z_2 = y_2 - (2L - 3mg/2k)$$

If we substitute for y_1, y_2 in our equations we find that

$$\frac{dz_1}{dt} = v_1$$

$$\frac{dz_2}{dt} = v_2$$

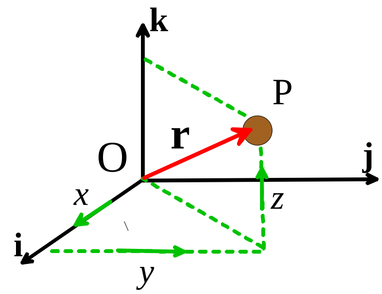
$$\frac{dv_1}{dt^2} = \left(-2\frac{k}{m}(z_1 - d_0 \sin \omega t) + 2\frac{k}{m}(z_2 - z_1) \right)$$

$$\frac{dv_2}{dt^2} = \left(-2\frac{k}{m}(z_2 - z_1) \right)$$

These are a lot simpler, and more importantly, tell us that the motion of the system does not depend on the spring length *L* or gravity.

3.4 Summary of main equations and definitions

Position-velocity-acceleration relations in a Cartesian Frame



$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$

$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k}$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

The direction of the velocity vector is tangent to its path.

The magnitude of the velocity vector $\sqrt{v_x^2 + v_y^2 + v_z^2}$ is the distance traveled along the path per unit time (speed).

A unit vector tangent to the path can be found as

$$\mathbf{t} = \frac{v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

Straight line motion with constant acceleration

$$\mathbf{r} = \left[X_0 + V_0 t + \frac{1}{2}at^2 \right]\mathbf{i} \qquad \mathbf{v} = (V_0 + at)\mathbf{i} \qquad \mathbf{a} = a\mathbf{i}$$

Here, a is the (constant) acceleration; X_0, V_0 are the position and speed at time $t=0$.

Straight line motion with time/position dependent acceleration

Acceleration given as a function of time: $\mathbf{r} = \left(X_0 + \int_0^t v(t)dt \right)\mathbf{i} \qquad \mathbf{v} = \left(V_0 + \int_0^t a(t)dt \right)\mathbf{i}$

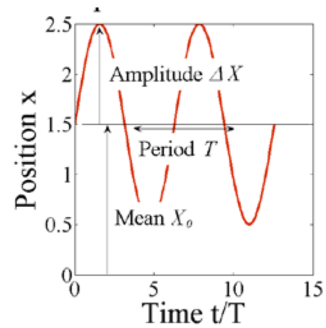
Acceleration given as a function of position $\int_{V_0}^{v(t)} v dv = \int_0^{x(t)} a(x)dx$

Separation of variables for one-dimensional motion

$$a = \frac{dv}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{V_0}^{v} f(v)dv = \int_0^t g(t)dt$$

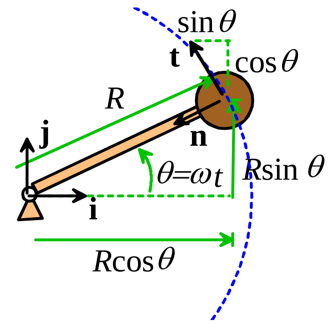
$$v = \frac{dx}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{X_0}^{x(t)} f(x)dv = \int_0^t v(t)dt$$

Simple Harmonic Motion



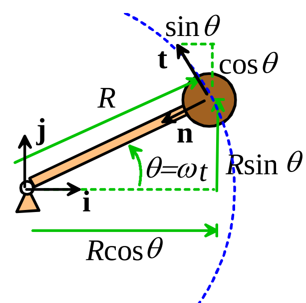
$$\mathbf{r} = \left[X_0 + \Delta X \sin(2\pi t / T)\right]\mathbf{i}$$
$$\mathbf{v} = V\cos(2\pi t / T)\mathbf{i}$$
$$\mathbf{a} = -A\sin(2\pi t / T)\mathbf{i}$$
$$V = \frac{2\pi\Delta X}{T} \qquad A = \frac{2\pi V}{T} = \frac{4\pi^2 \Delta X}{T^2}$$

Circular Motion at Constant Speed



$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$
$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$
$$\mathbf{a} = -\omega^2 R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \omega^2 R\mathbf{n} = \frac{V^2}{R}\mathbf{n}$$

General Circular Motion



$$\omega = d\theta / dt \quad \alpha = d\omega / dt = d^2\theta / dt^2$$
$$s = R\theta \quad V = ds / dt = R\omega$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$
$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$
$$\mathbf{a} = R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$
$$= \alpha R\mathbf{t} + \omega^2 R\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Note that the straight-line motion relations can be used to relate θ, ω, α, by exchanging $x \rightarrow \theta, v \rightarrow \omega, a \rightarrow \alpha$

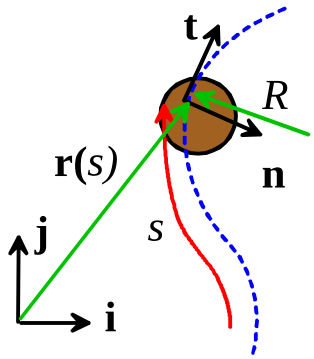
Motion along an arbitrary path in normal-tangential coordinates

$$\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$$

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \qquad \mathbf{n} = R\frac{d\mathbf{t}}{ds} \qquad R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}}$$

$$\mathbf{v} = \frac{ds}{dt}\mathbf{t} = V\mathbf{t}$$

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{t} + \frac{1}{R}\left(\frac{ds}{dt}\right)^2\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$



Position-velocity-acceleration relations in polar-coordinates

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \qquad \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

$$\mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta \qquad \mathbf{j} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta$$

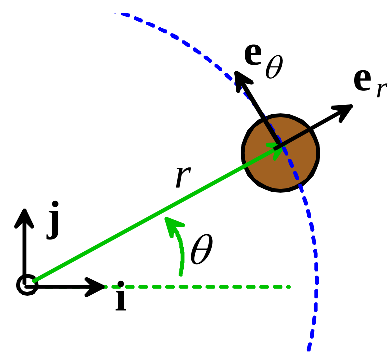
$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1} y/x$$

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

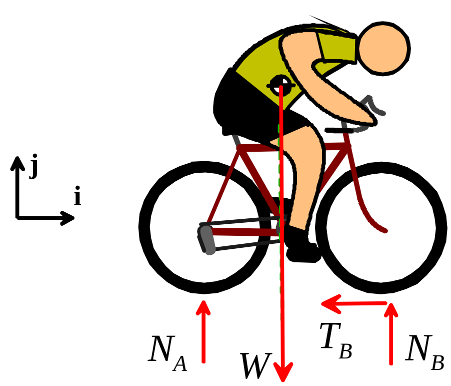
$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta$$



Newton's laws

For a particle $\mathbf{F} = m\mathbf{a}$

For a rigid body moving without rotation or rotating at fixed angular rate about a fixed axis $\mathbf{M}_C = 0$ (*you must take moments about the center of mass)*



Drawing free body diagrams:

1. Decide which part of a system you will idealize as a particle (you may need more than one particle)
2. Draw the part of the system you have idealized as a particle by itself (**very important**!). It is important to make sure that your particle is isolated – it can't be touching something else. You may need more than one drawing if you have more than one particle in your system.
3. Draw on any of the following external forces that apply. Make sure you draw them acting in the correct direction, acting on the correct part of the body:
    a. gravity (at the COM);
    b. air resistance or lift forces (and sometimes moments) – various conventions are used to locate these forces but in this course we usually put them at the COM and neglect moments;
    c. Buoyancy forces (act at the COM of the displaced fluid)
    d. Electrostatic or electromagnetic forces
4. Draw the forces exerted by springs attached to the particle. It is best to assume that springs always pull on the point they are connected to, and that the magnitude of the force in the spring is $F_s = k(l - l_0)$, where *l* is the length of the spring, and l_0 is its unstretched length.
5. Draw the forces exerted by dashpots or dampers (like springs, assume they pull on the object they are connected to, and exert a force magnitude $F_d = \lambda dl / dt$ where *l* is the length of the dashpot.
6. Draw forces exerted by cables. Cables always pull, and exert a force parallel to the direction of the cable. The magnitude of the force has to be left as an unknown.
7. Draw any unknown reaction forces, with the following rules:
    a. Reaction forces must act at any point on any point of the body that is touching something outside the particle (i.e. a part of your system that you did not include in your drawing in step 2).
    b. If the connection between the two touching objects prevents them from rotating with respect to one another (or, like a motor, makes them rotate with some controllable angular speed), you will need to draw both reaction forces and moments. (Reaction moments do sometimes come up in dynamics problems, but they are not very common, so think carefully before including them).
    c. If friction acts at the contact point, and you don't know whether the two objects slide at the contact (or you know they do not slide), draw both a normal and a tangential force with unknown magnitudes *N*,*T* (or some suitable variable). The direction of the friction force is not important. DO NOT assume $T = \mu N$.
    d. If friction acts at the contact point, and you know the contact slips, draw both a tangential and a normal force. You must draw the tangential force so that it opposes the direction of sliding (ask a faculty member or TA if you don't understand this). If slip occurs you can assume $T = \mu N$.
    e. If the contact point is frictionless, draw only a normal force.
    f. If your particle is being touched by a two-force member (no, this is not a gender and sexuality class… a two force member is a massless rod, connected through freely rotating hinges at both ends. A massless freely rotating wheel can also be idealized as a two-force member) you can assume the reaction force acts parallel to the two-force member.
    g. If you have more than one particle in your system, make sure that any forces exerted by one particle on the other have equal and opposite reactions.

Calculating unknown forces or accelerations using Newton's laws:

1. Decide how to idealize the system (what are the particles?)
2. Draw a free body diagram showing the forces acting on each particle
3. Consider the **kinematics** of the problem. The goal is to calculate the acceleration of each particle in the system – you may be able to start by writing down the position vector and differentiating it, or you may be able to relate the accelerations of two particles (eg if two particles move together, their accelerations must be equal).
4. Write down **F**=*m***a** for each particle.
5. If you are solving a problem involving a massless frames (see, e.g. Example 3, involving a bicycle with negligible mass) you also need to write down $\mathbf{M}_C = \mathbf{0}$ about the particle.
6. Solve the resulting equations for any unknown components of force or acceleration (this is just like a statics problem, except the right hand side is not zero).

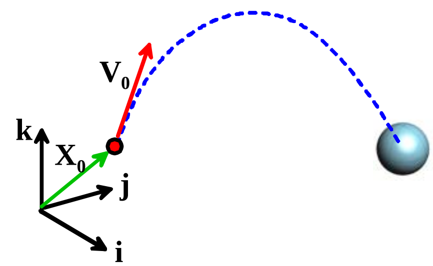
Problems like this will usually ask you to make some design prediction at the end, which might involve calculating critical conditions for something to slip, tip, break, etc.

- At the onset of slip at a contact $|T| = \mu|N|$
- At the critical point where an object tips over, a reaction force somewhere will go to zero. You will have to identify where this point is, find the reaction force, and set it to zero.

Deriving equations of motion for a system of particles

1. Introduce a set of variables that can describe the motion of the system. Don't worry if this sounds vague – it will be clear what this means when we solve specific examples.
2. Write down the position vector of each particle in the system in terms of these variables
3. Differentiate the position vector(s), to calculate the velocity and acceleration of each particle in terms of your variables;
4. Draw a free body diagram showing the forces acting on each particle. You may need to introduce variables to describe reaction forces. Write down the resultant force vector.
5. Write down Newton's law $\mathbf{F} = m\mathbf{a}$ for each particle. This will generate up to 3 equations of motion (one for each vector component) for each particle.
6. If you wish, you can eliminate any unknown reaction forces from Newton's laws. If you are trying to solve the equations by hand, you should always do this; of you are using MATLAB, it's not usually necessary – you can have MATLAB calculate the reactions for you. The result will be a set of differential equations for the variables defined in step (1)
7. If you find you have fewer equations than unknown variables, you should look for any *constraints* that restrict the motion of the particles. The constraints must be expressed in terms of the unknown accelerations.
8. Identify the *initial conditions* for the variables defined in (1). These are usually the values of the unknown variables, their time derivatives, at time $t=0$. If you happen to know the values of the variables at some other instant in time, you can use that too. If you don't know their values at all, you should just introduce new (unknown) variables to denote the initial conditions.
9. Solve the differential equations, subject to the initial conditions.

Trajectory equations for particle moving near earth's surface with no air resistance



$$\left.\begin{aligned} \mathbf{r} &= X_0\mathbf{i} + Y_0\mathbf{j} + Z_0\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= V_{x0}\mathbf{i} + V_{y0}\mathbf{j} + V_{z0}\mathbf{k} \end{aligned}\right\} t = 0$$

$$\mathbf{r} = \left(X_0 + V_{x0}t\right)\mathbf{i} + \left(Y_0 + V_{y0}t\right)\mathbf{j} + \left(Z_0 + V_{z0}t - \frac{1}{2}gt^2\right)\mathbf{k}$$

$$\mathbf{v} = \left(V_{x0}\right)\mathbf{i} + \left(V_{y0}\right)\mathbf{j} + \left(V_{z0} - gt\right)\mathbf{k}$$

$$\mathbf{a} = -g\mathbf{k}$$

Solving differential equations with Matlab:

Example: to solve

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = 0$$

with initial conditions $y = y_0$ $\frac{dy}{dt} = v_0$ at time *t=0*

```
clear all
syms x(t) t omega_n zeta x0 v0
diffeq = (1/omega_n^2)*diff(x(t),t,2) + 2*zeta/omega_n * diff(x(t),t) + x(t)==0
Dx = diff(x)
initial_condition = [x(0)==x0, Dx(0)==v0]
x(t) = simplify(dsolve(diffeq,initial_condition))
```

Re-writing a second-order differential equation as a pair of first-order equations for MATLAB

Example: to solve

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = 0$$

we introduce $v = dy/dt$ as an additional variable. This new equation, together with the original ODE can then be written in the following form

$$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} v \\ -2\zeta\omega_n v - \omega_n^2 y \end{bmatrix}$$

This is now in the form

$$\frac{d\mathbf{w}}{dt} = f(t, \mathbf{w}) \qquad \mathbf{w} = \begin{bmatrix} y \\ v \end{bmatrix}$$

as required.