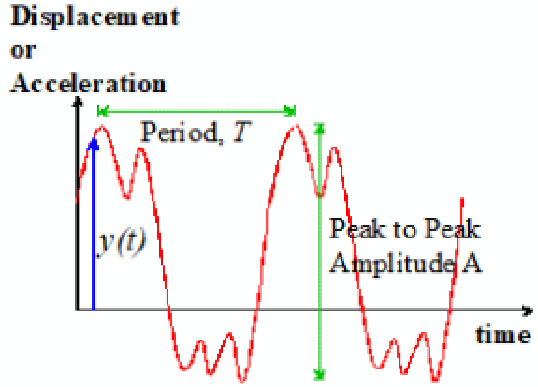
Free vibrations – concept checklist

1. Understand simple harmonic motion (amplitude, period, frequency, phase)
2. Understand the motion of a vibrating spring-mass system (and how the motion is predicted)
3. Calculate natural frequency of a 1 degree of freedom linear system (Derive EOM and use the solutions given on the handout)
4. Calculate the amplitude and phase of an undamped 1 DOF linear system from the initial conditions
5. Understand the concept of natural frequencies and mode shapes for vibration of a general undamped linear system
6. Use energy methods to derive an EOM
7. Use Taylor series to calculate natural frequencies for a nonlinear system
8. Understand forces exerted by a dashpot
9. How to combine springs and dashpots in series and parallel
10. Derive EOM for a damped spring-mass system
11. Be able to calculate natural frequency, damping factor, and damped natural frequency for a damped linear system
12. Understand (qualitatively) the motion of a damped linear system
13. Understand 'critical damping'

Free vibrations

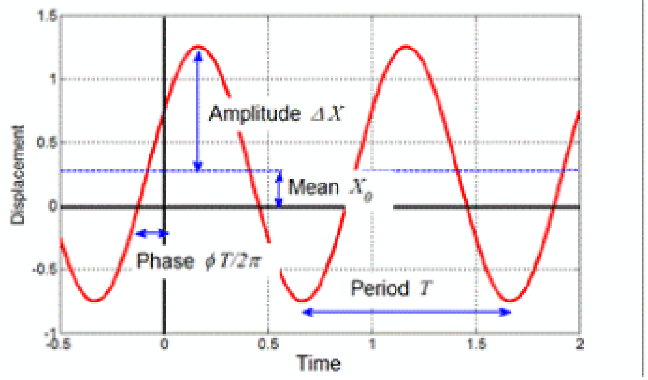
Typical vibration response

- Period, frequency, angular frequency amplitude



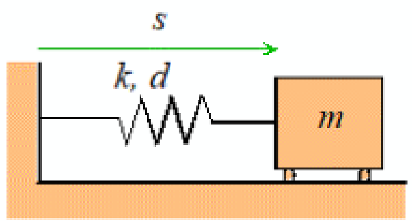
Simple Harmonic Motion

$$x(t) = X_0 + \Delta X \sin(\omega t + \phi)$$
$$v(t) = \Delta V \cos(\omega t + \phi)$$
$$a(t) = -\Delta A \sin(\omega t + \phi)$$
$$\Delta V = \omega \Delta X \qquad \Delta A = \omega \Delta V$$



Free Vibration of Undamped 1DOF systems

- Free -> No time dependent external forces
- Undamped -> No energy loss
- 1 DOF -> one variable describes system

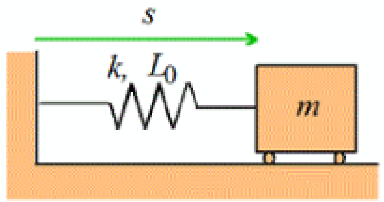


Free vibrations

Harmonic Oscillator

Derive EOM (F=ma) $\frac{m}{k}\frac{d^2 s}{dt^2} + s = L_0$

Canonical Vibration Problem: The spring mass system is released with velocity v_0 from position s_0 at time *t=0* . Find *s(t)* .



Compare with 'standard' differential equation

Equation $\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$ Initial Conditions $x = x_0$ $\frac{dx}{dt} = v_0$ $t = 0$

Solution $x = C + X_0 \sin(\omega_n t + \phi)$

$X_0 = \sqrt{(x_0 - C)^2 + v_0^2 / \omega_n^2}$ $\phi = \tan^{-1}\left(\frac{(x_0 - C)\omega_n}{v_0}\right)$

Or $x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$

$x = s$ $C = L_0$ $x_0 = s_0$

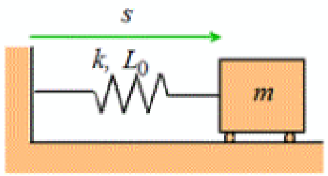
$\frac{1}{\omega_n^2} = \frac{m}{k}$

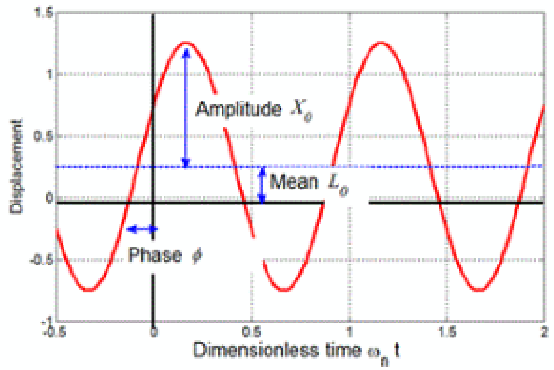
Solution

$$s(t) = L_0 + \sqrt{(s_0 - L_0)^2 + v_0^2 / \omega_n^2}\,\sin(\omega_n t + \phi)$$

$$\left.\begin{array}{l} s = s_0 \\ \frac{ds}{dt} = v_0 \end{array}\right\} t = 0$$

Natural Frequency $\omega_n = \sqrt{\frac{k}{m}}$





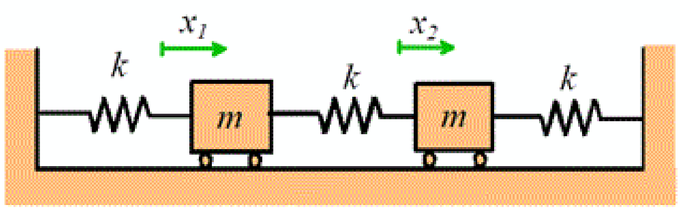
Free vibrations

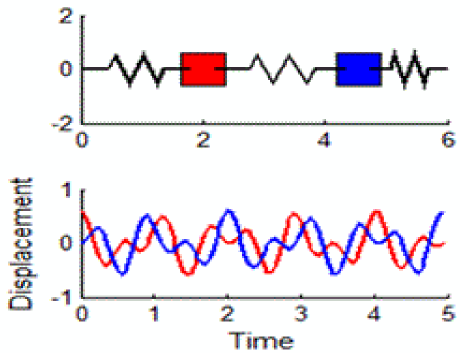
Natural Frequencies and Mode Shapes

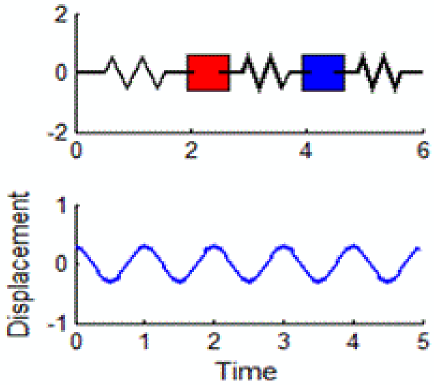
General system does not always vibrate harmonically

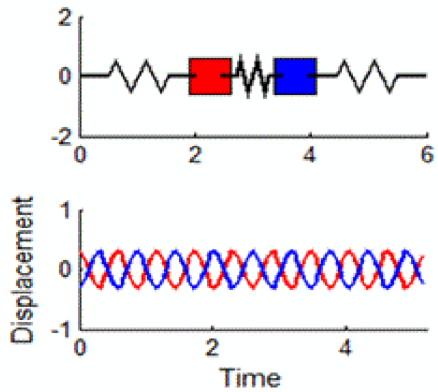
All unforced undamped systems vibrate harmonically at special frequencies, calle **Natural Frequencies** of the system

The system will vibrate harmonically if it is released from rest with a special set of initial displacements, called **Mode Shapes** or **Vibration Modes**.







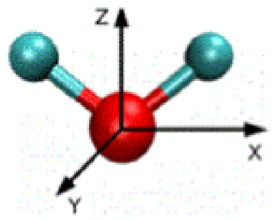


Counting degrees of freedom and vibration modes

DOF = no. coordinates required to describe motion

2D system # DOF = 2*p + 3*r-c

3D system # DOF = 3*p+6*r-c



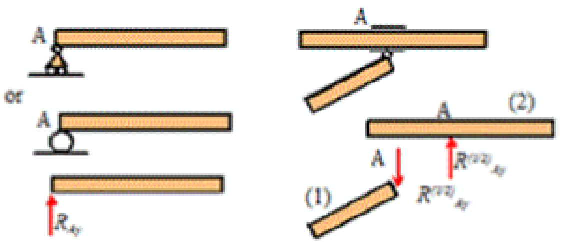
(d) Water molecule

Vibration modes = # DOF - # translation/rotation rigid body modes

Examples of 2D constraints

Roller joint

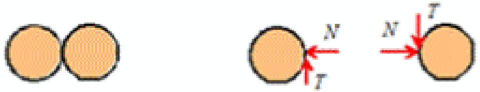
1 constraint (prevents motion in one direction)



Nonconformal contact (two bodies meet at a point)

No friction or slipping: 1 constraint (prevents interpenetration)

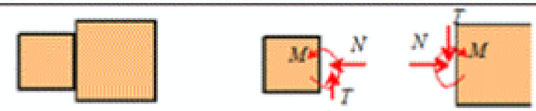
Sticking friction 2 constraints (prevents relative motion



Conformal contact (two rigid bodies meet along a line)

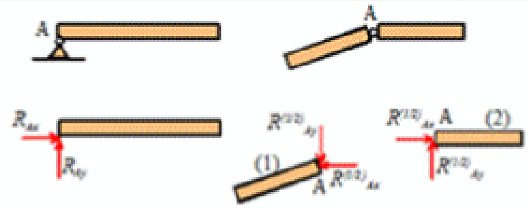
No friction or slipping: 2 constraint (prevents interpenetration and rotation)

Sticking friction 3 constraints (prevents relative motion)

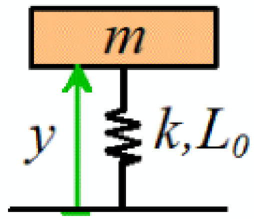


Pinned joint (generally only applied to a rigid body, as it would stop a particle moving completely)

2 constraints (prevents motion horizontally and vertically)



Calculating natural frequencies for 1DOF systems



- Use F=ma (or energy) to find the equation of motion
- For an undamped system the equation will look like

$$A\frac{d^2 y}{dt^2} + By = D$$

- Handout online gives solution to

$$\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$$

- Rearrange your equation to look like this

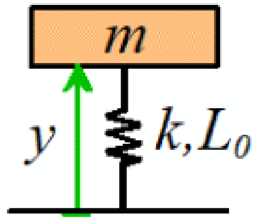
$$\frac{A}{B}\frac{d^2 y}{dt^2} + y = \frac{D}{B}$$

$$\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$$

$$\frac{1}{\omega_n^2} = \frac{A}{B} \Rightarrow \omega_n = \sqrt{\frac{B}{A}}$$

$$C = D / B$$

Calculating natural frequencies for 1DOF systems



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$$\frac{1}{\omega_n^2} = \frac{A}{B} \Rightarrow \omega_n = \sqrt{\frac{B}{A}}$$

$$C = D / B$$

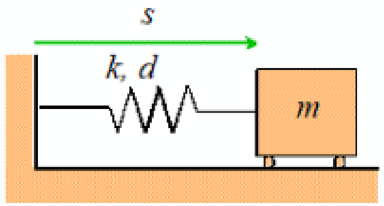
Tricks for calculating nat freqs of undamped systems

Using energy conservation to find EOM

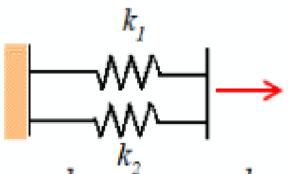
$$KE+PE=\frac{1}{2}m\left(\frac{ds}{dt}\right)^2+\frac{1}{2}k(s-L_0)^2=const$$

$$\Rightarrow \frac{d}{dt}(KE+PE)=m\left(\frac{ds}{dt}\right)\frac{d^2s}{dt^2}+k(s-L_0)\frac{ds}{dt}=0$$

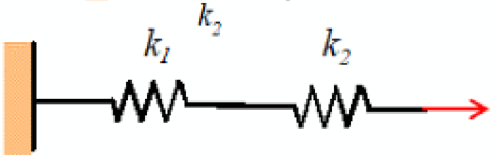
$$\Rightarrow m\frac{d^2s}{dt^2}+ks=kL_0$$



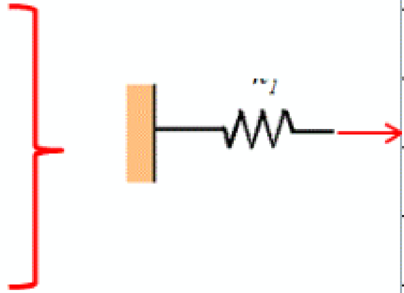
Combining springs



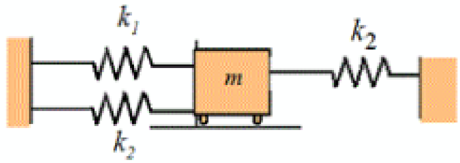
Parallel (forces add) $k_{eff}=k_1+k_2$



Series (lengths add) $\frac{1}{k_{eff}}=\frac{1}{k_1}+\frac{1}{k_2}$



These are all in parallel



$k_{eff}=k_1+2k_2$

Example of solving a vibration EOM

Solve $$\frac{d^2 y}{dt^2} + 4y = 0 \qquad y = 1 \quad \frac{dy}{dt} = 0 \quad t = 0$$

Use tabulated solutions

- This is "Case I"

Rearrange our equation

$$\frac{1}{4}\frac{d^2y}{dt^2} + y = 0$$

$\frac{1}{4} \to 1/\omega_n^2$, $y \to x$, $0 \to C$

$\Rightarrow \omega_n = 2 \qquad C = 0$

Solution to Case I (From handout)

Equation $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + x = C$ Initial Conditions $x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0$

Solution
$$x = C + X_0 \sin(\omega_n t + \phi)$$
$$X_0 = \sqrt{(x_0 - C)^2 + v_0^2 / \omega_n^2} \qquad \phi = \tan^{-1}\left(\frac{(x_0 - C)\omega_n}{v_0}\right)$$

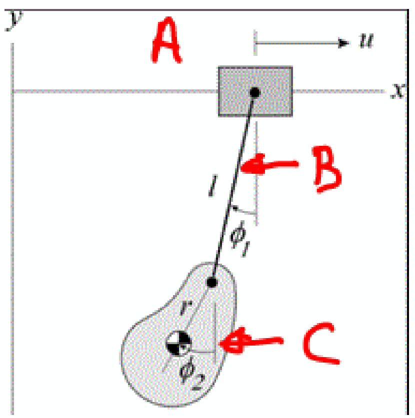
Or
$$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

Initial conditions $\Rightarrow x_0 = 1 \qquad V_0 = 0 \Rightarrow X_0 = 1 \qquad \phi = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

$$\Rightarrow y(t) = \sin(2t + \pi/2) = \cos(2t)$$

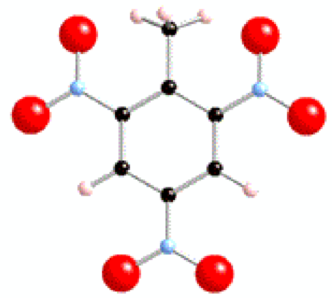
Example: Find the number of DOF and vibration modes of the systems shown

(a)



2D Idealization of an overhead crane

(b)



Molecule

(a) 2D system #DOF = 2p + 3r − c

B/C : rigid bodies A = particle (could also assume rigid body)

Constraints: 2 pin joints, each 2 constraints (x, y prevented)
Rail prevents A from moving vertically − 1 constraint

#DOF = 2 + 6 − 5 = 3

Check : Coord: $\{u, \phi_1, \phi_2\} = 3$

(b) : Formula for molecules : $\# DOF = 3p = 3 \times 20 = 60$ DOF

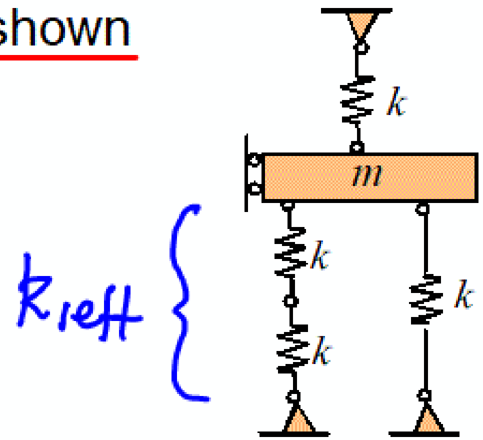
Vibration modes = # DOF − # "rigid body modes"

For a molecule, # rigid body modes = 3 translations
3 rotations

Vibration modes : $3p - 6 = 54$

Example: Find the natural frequency of the system shown

Formula $\omega_n = \sqrt{\frac{k_{eff}}{m}}$



Combine series springs : $\frac{1}{k_{1eff}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{1eff} = \frac{k}{2}$

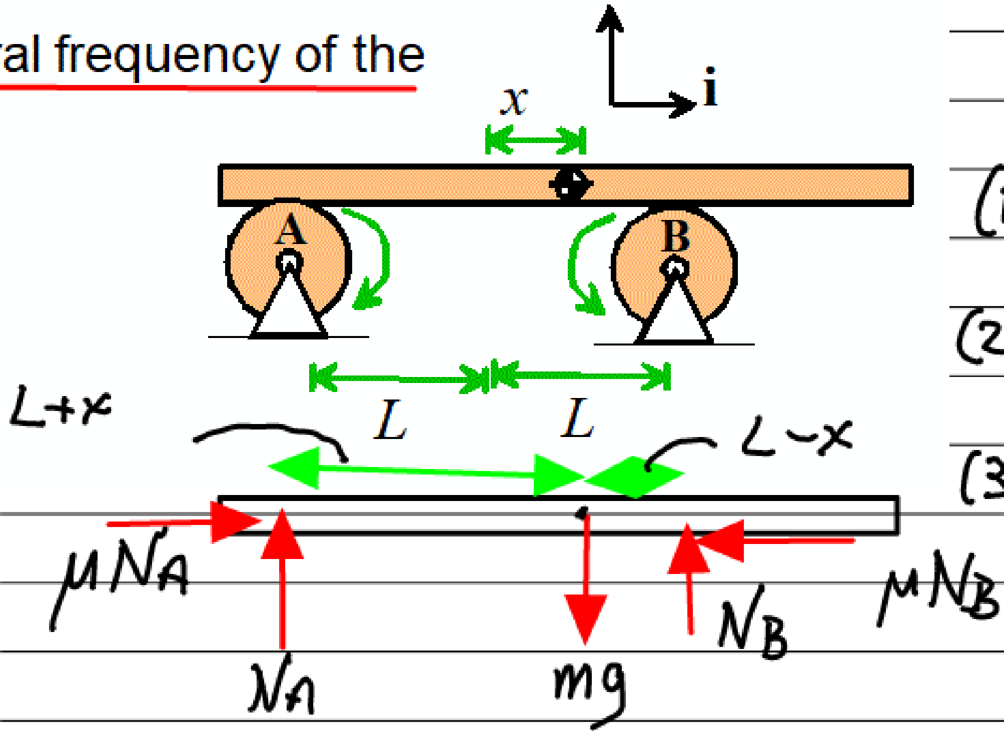
Series ⇒ Lengths add

Parallel ⇒ forces add - all others in parallel

$k_{eff} = \frac{k}{2} + k + k = \frac{5k}{2} \Rightarrow \omega_n = \sqrt{\frac{5k}{2m}}$

Example: Find the natural frequency of the 'friction oscillator'

Assume wheels spin fast enough to ensure slip



Approach

(1) $\underline{F} = m\underline{a}$, $\underline{M}_c = 0$

(2) EOM

(3) Read off ω_n

$$\underline{F} = m\underline{a} \Rightarrow \mu(N_A - N_B)\underline{i} + (N_A + N_B - mg)\underline{j} = m\frac{d^2x}{dt^2}\underline{i} \quad (1)$$

$$\underline{M}_c = 0 \Rightarrow [N_B(L-x) - N_A(L+x)]\underline{k} = \underline{0} \quad (2)$$

Solve!

$$N_A + N_B = mg \quad \underline{j} \text{ component of } (1)$$

$$(N_B - N_A)L - (N_A + N_B)x = 0 \quad (2)$$

$$\Rightarrow N_B - N_A = mg\, x/L$$

Hence $-\mu mg\frac{x}{L} = m\frac{d^2x}{dt^2}$ ($\underline{i}$ component of (1))

Rearrange: $\frac{L}{g\mu}\frac{d^2x}{dt^2} + x = 0$

$1/\omega_n^2$ $\Rightarrow$ $\omega_n = \sqrt{\frac{\mu g}{L}}$

Compare to "Case I" eom