Course Outline

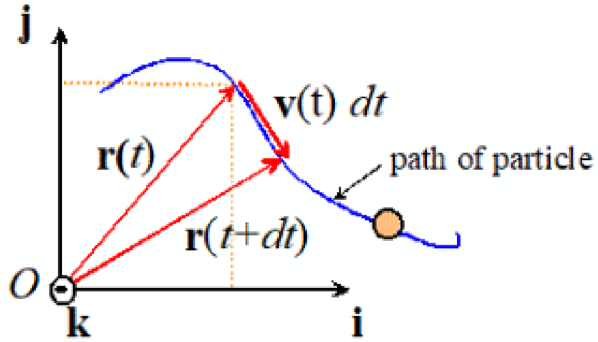
1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton's laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy;
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies

Particle Dynamics – concept checklist

- Understand the concept of an 'inertial frame'
- Be able to idealize an engineering design as a set of particles, and know when this idealization will give accurate results
- Describe the motion of a system of particles (eg components in a fixed coordinate system; components in a polar coordinate system, etc)
- Be able to differentiate position vectors (with proper use of the chain rule!) to determine velocity and acceleration; and be able to integrate acceleration or velocity to determine position vector.
- Be able to describe motion in normal-tangential and polar coordinates (eg be able to write down vector components of velocity and acceleration in terms of speed, radius of curvature of path, or coordinates in the cylindrical-polar system).
- Be able to convert between Cartesian to normal-tangential or polar coordinate descriptions of motion
- Be able to draw a correct free body diagram showing forces acting on system idealized as particles
- Be able to write down Newton's laws of motion in rectangular, normal-tangential, and polar coordinate systems
- Be able to obtain an additional moment balance equation for a rigid body moving without rotation or rotating about a fixed axis at constant rate.
- Be able to use Newton's laws of motion to solve for unknown accelerations or forces in a system of particles

Particle Kinematics

Inertial frame – non accelerating, non rotating reference frame
Particle – point mass at some position in space



Position Vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity Vector $\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$

$$= \frac{d}{dt}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\Rightarrow v_x(t) = \frac{dx}{dt} \qquad v_y(t) = \frac{dy}{dt} \qquad v_z(t) = \frac{dz}{dt}$$

- Direction of velocity vector is parallel to path
- Magnitude of velocity vector is distance traveled / time

Acceleration Vector

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} = \frac{d}{dt}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}$$

$$\Rightarrow a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Also $a_x(t) = \frac{dv_x}{dx}v_x \qquad a_y(t) = \frac{dv_y}{dy}v_y \qquad a_z(t) = \frac{dv_z}{dz}v_z$

Particle Kinematics

- Straight line motion with constant acceleration

$$\mathbf{r} = \left[X_0 + V_0 t + \frac{1}{2} a t^2 \right] \mathbf{i} \qquad \mathbf{v} = \left(V_0 + a t \right) \mathbf{i} \qquad \mathbf{a} = a \mathbf{i}$$

- Time/velocity/position dependent acceleration – use calculus

$$\mathbf{r} = \left(X_0 + \int_0^t v(t)\,dt \right) \mathbf{i} \qquad \mathbf{v} = \left(V_0 + \int_0^t a(t)\,dt \right) \mathbf{i}$$

$$a = \frac{dv}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{V_0}^{v} f(v)\,dv = \int_0^t g(t)\,dt$$

$$v = \frac{dx}{dt} = \frac{g(t)}{f(x)} \Rightarrow \int_{X_0}^{x(t)} f(x)\,dx = \int_0^t g(t)\,dt$$

$$\frac{dv}{dt} = a(x)$$

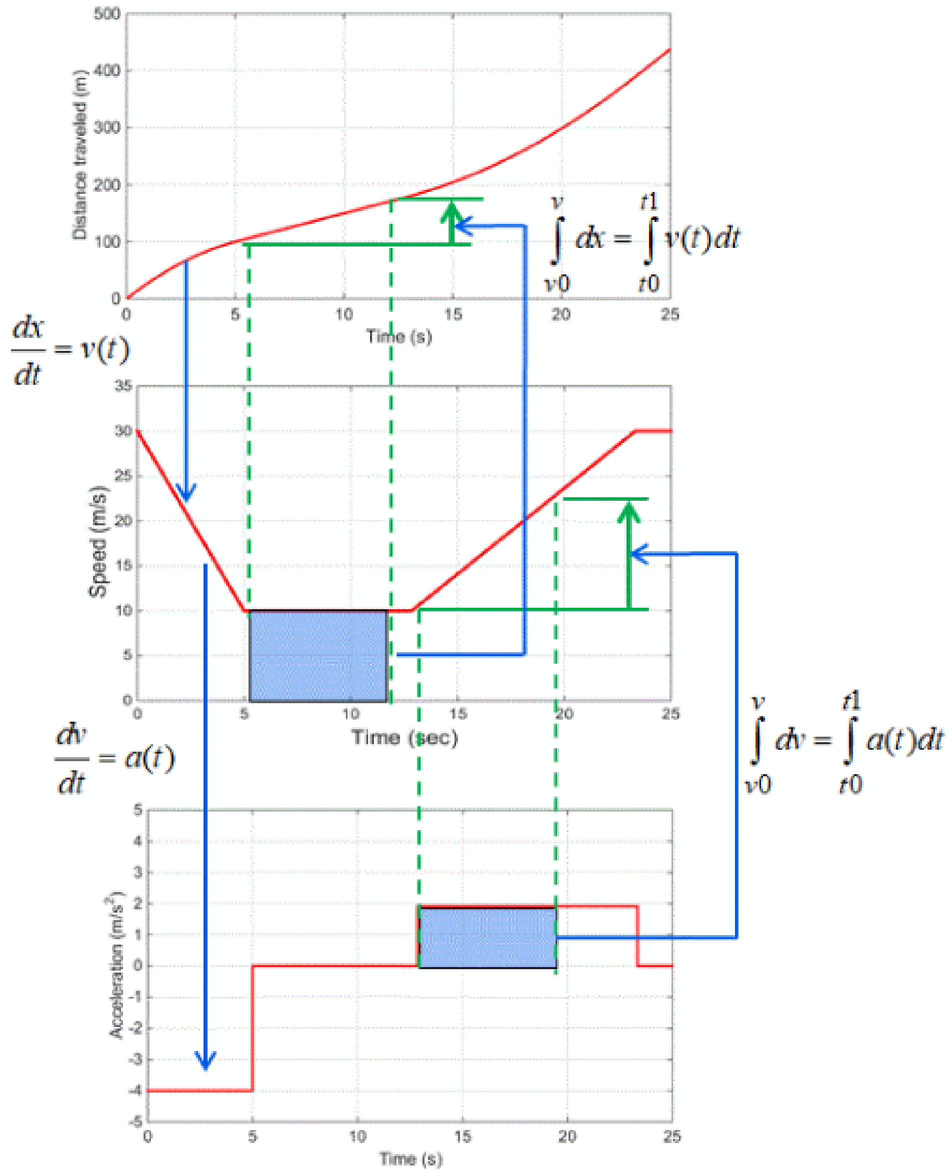
$$\Rightarrow \frac{dv}{dx}\frac{dx}{dt} = a(x) \Rightarrow v\frac{dv}{dx} = a(x)$$

$$\int_{V_0}^{v(t)} v\,dv = \int_0^{x(t)} a(x)\,dx$$

Graphical x-v-a relations

- Speed is the slope of the distance-v-time curve
- Distance is the area under the speed-v-time curve

- Acceleration is the slope of the speed-v-time curve
- Speed is the area under the acceleration-v-time curve



Particle Kinematics

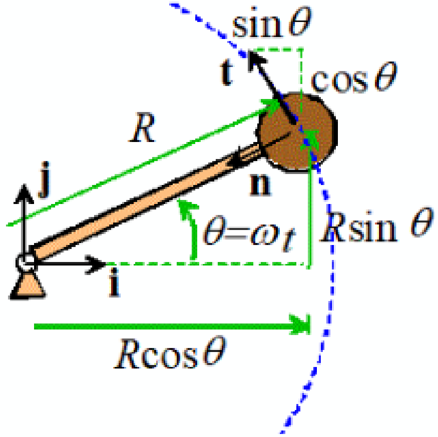
- Circular Motion at const speed

$$\theta = \omega t \quad s = R\theta \quad V = \omega R$$

$$\mathbf{r} = R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{v} = \omega R(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = V\mathbf{t}$$

$$\mathbf{a} = -\omega^2 R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \omega^2 R\mathbf{n} = \frac{V^2}{R}\mathbf{n}$$



- General circular motion

$$\omega = d\theta / dt \quad \alpha = d\omega / dt = d^2\theta / dt^2$$

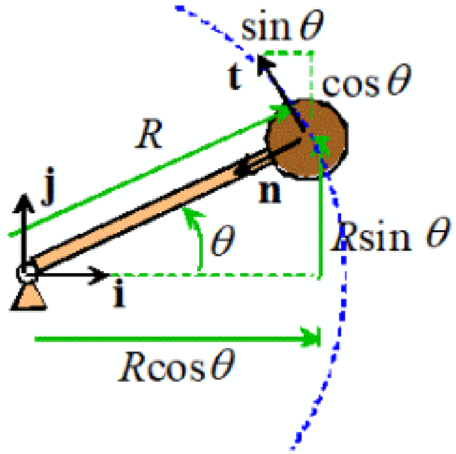
$$s = R\theta \quad V = ds / dt = R\omega$$

$$\mathbf{r} = R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{v} = \omega R(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = V\mathbf{t}$$

$$\mathbf{a} = R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \alpha R\mathbf{t} + \omega^2 R\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

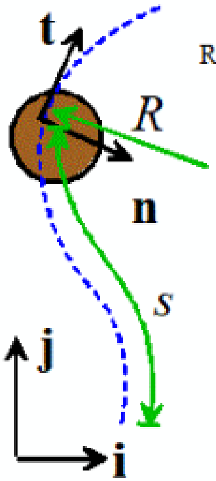


Particle Kinematics

- Motion along an arbitrary path

$$\mathbf{v} = V\mathbf{t}$$

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$



Radius of curvature $R = \dfrac{1}{\sqrt{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2}}$

$$\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$$

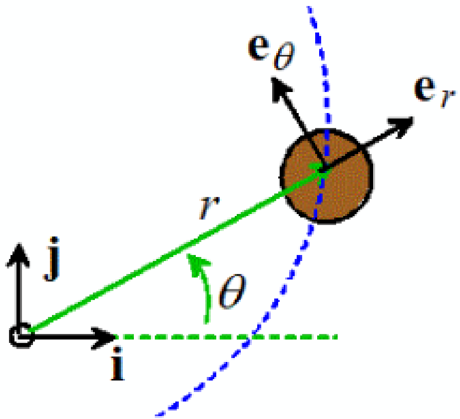
- Polar Coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}(y/x)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta$$

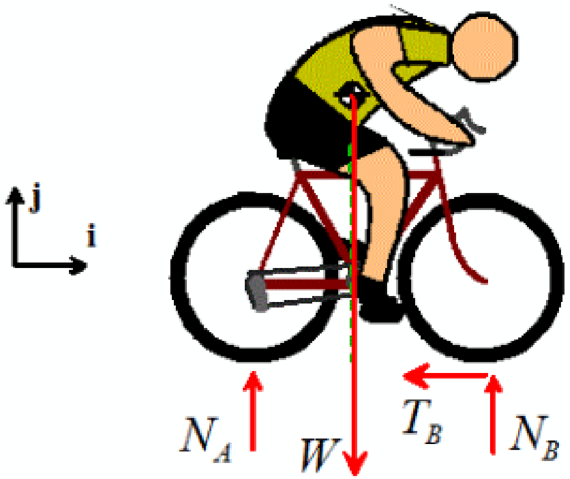


Summary of important concepts and equations

Newton's laws

- For a particle $\mathbf{F} = m\mathbf{a}$

- For a rigid body in motion without rotation, or a particle on a massless frame



$$\mathbf{M}_C = \mathbf{0}$$

You MUST take moments about center of mass

Using Newton's laws

Calculating forces required to cause prescribed motion

- Idealize system
- Free body diagram
- Kinematics (describe motion – usually goal is to find formula for acceleration)
- **F**=m**a** for each particle.
- $\mathbf{M}_G = \mathbf{0}$ (for steadily or non-rotating rigid bodies or frames only – this is a special case of the moment-angular momentum formula for rigid bodies)
- Solve for unknown forces or accelerations (just like statics)

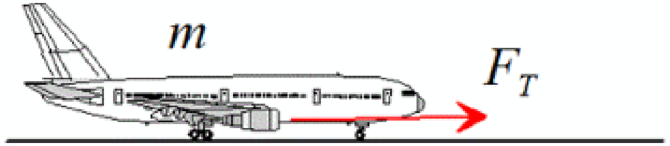
Topics for today's section

- Example of straight-line motion with variable acceleration
- Example using normal-tangential coordinates
- Example using polar coordinates

Example of straight-line motion with variable a

Aircraft starts from rest.

Engine thrust. $F_T = F_0\left(1-\frac{v}{v_0}\right)$



Must reach speed v_{TO} to take off

1. Find a formula for speed as a function of time (neglect drag).
2. Find a formula for distance traveled as a function of time
3. Find a formula for the minimum length of runway required to takeoff

① $\frac{dv}{dt} = a = \frac{F_0}{m}\left(1-\frac{V}{V_0}\right) \Rightarrow \int_0^V \frac{dV}{1-V/V_0} = \int_0^t \frac{F_0}{m}\,dt$

$\left[-V_0 \log\left(1-\frac{V}{V_0}\right)\right]_0^V = \left[\frac{F_0}{m}t\right]_0^t$

$\Rightarrow \boxed{V = V_0\left(1 - \exp\left(\frac{-F_0 t}{mV_0}\right)\right)}$

$$\frac{dx}{dt} = V \quad = \int_0^x dx = \int_0^t V_0\left(1 - \exp\left(-\frac{F_0 t}{V_0 m}\right)\right) dt$$

$$x = \left[V_0 t + \frac{V_0^2 m}{F_0} \exp\left(-\frac{F_0 t}{V_0 m}\right)\right]_0^t$$

$$\Rightarrow x = V_0 t + \frac{V_0^2 m}{F_0}\left\{\exp\left(-\frac{F_0 t}{V_0 m}\right) - 1\right\}$$

③ Solve (1) for t; subs in ② for x $(V = V_{T0}\ ;\ t\ t_{T0})$

From ① : $V_{T0} = V_0\left\{1 - \exp\left(-\frac{F_0 t}{mV_0}\right)\right\} \Rightarrow t_{T0} = -\frac{mV_0}{F_0}\log\left(1 - \frac{V}{V_{T0}}\right)$

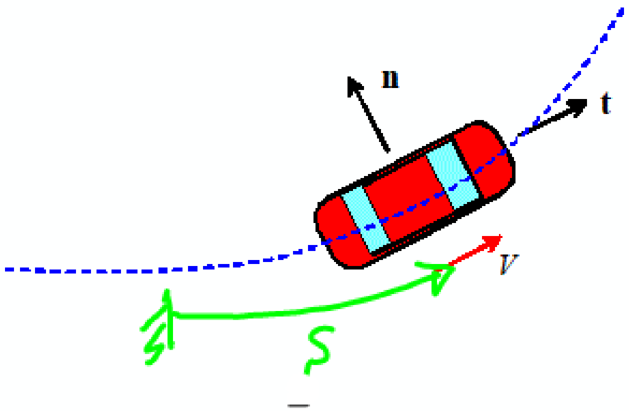
Also $\frac{V_0^2 m}{F_0}\left\{\exp\left(-\frac{F_0 t}{mV_0}\right) - 1\right\} = -\frac{V_{T0} V_0 m}{F_0}$

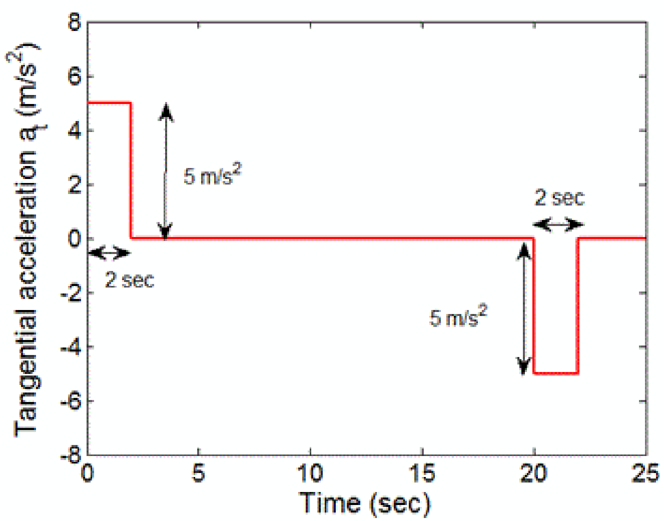
$$\Rightarrow X_{T0} = -\frac{mV_0^2}{F_0}\log\left(1 - \frac{V}{V_{T0}}\right) - \frac{V_{T0} V_0 m}{F_0}$$

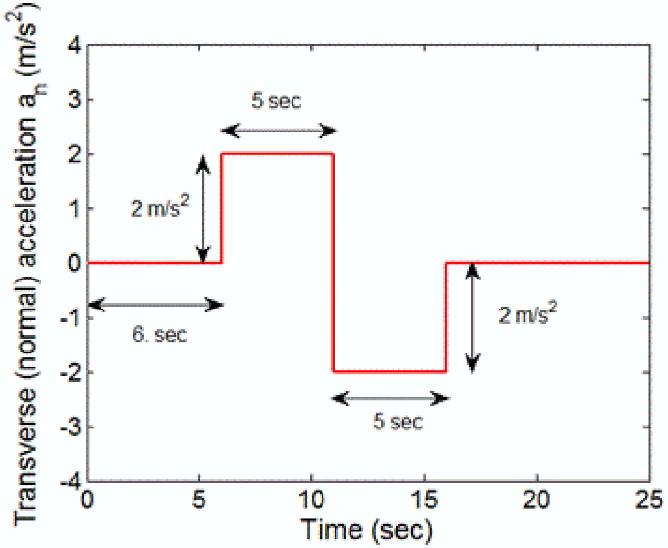
Example using n-t coordinates (Midterm 2012).

A vehicle is instrumented to measure acceleration components in directions parallel and perpendicular to the car's direction of motion. (A positive transverse accel means the car accelerates to the left). The Car is at rest at the origin at time *t=0* facing the *x* direction.

1. Sketch a graph showing the car's speed as a function of time.
2. Sketch the subsequent path of the vehicle





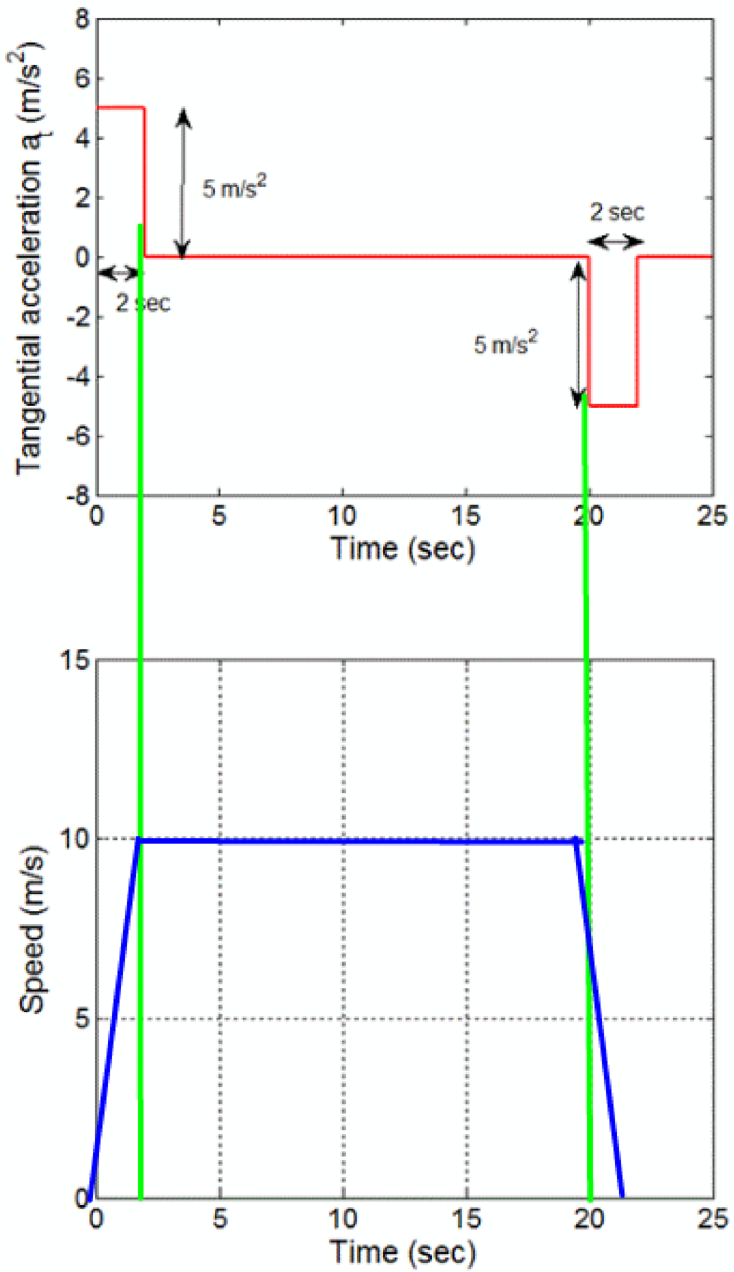


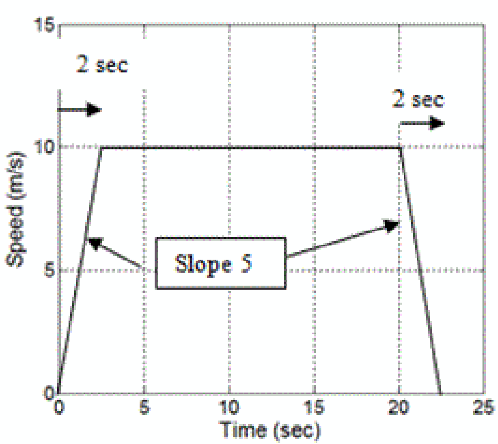
Formulas: $\frac{d\bar{V}}{dt} = a_t$ $\quad \bar{V} = \frac{ds}{dt}$ $\quad \underline{a} = \frac{d\bar{V}}{dt}\underline{t} + \frac{\bar{V}^2}{R}\underline{n}$

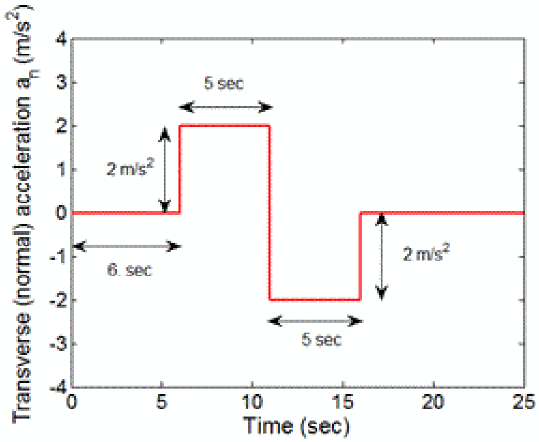
We know V = area under a_t graph

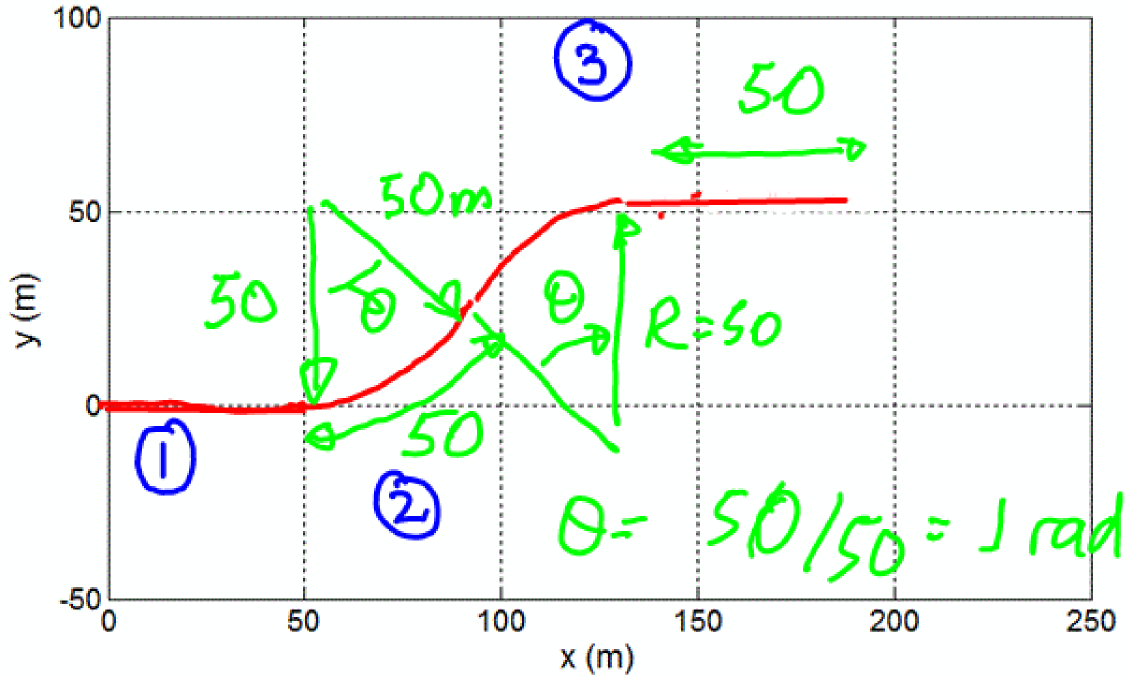
At time $t=2s$ speed = $2 \times 5 = 10 m/s$

When $a_t = 0$ speed = constant









① For $0 < t < 6$

(i) $a_n = 0 \Rightarrow \frac{V^2}{R} = 0$

$\Rightarrow R = V^2/0 \Rightarrow R \to \infty$

Straight line

Dist traveled = area under V curve $= 5 \times 2 + 40 = 50m$

② $6 < t < 11$ a_n = const V = const

$\Rightarrow R$ = const $V^2/R = a_n$

$\Rightarrow R = V^2/a_n = 50$

We travel $s = 10 \times 5 = 50m$ during this phase

③ Remaining path from symmetry

Example using polar coordinates

The motion of a particle is described using polar coordinates as:

$$r = 1 + t - (\pi / 4) \qquad \theta = t$$

At the instant when

$$\theta = \pi / 4$$

(1) The position vector in $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

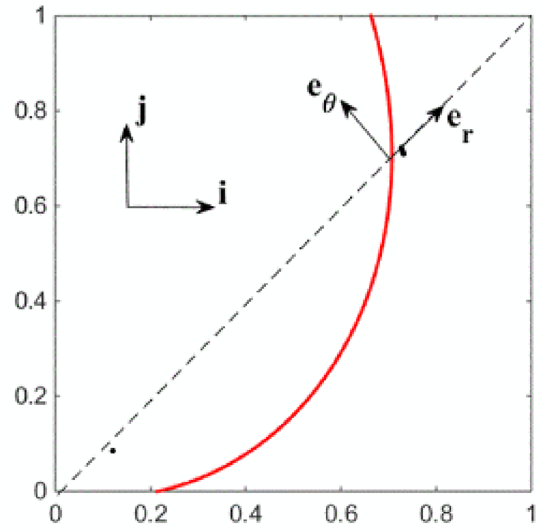
(2) The velocity vector in $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

(3) The acceleration vector in $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

(4) A unit vector parallel to path in $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

(5) The velocity in normal-tangential coordinates

(6) The acceleration in normal-tangential coordinates



Formulas :

$$\underline{r} = r\,\underline{e_r}$$

$$\underline{v} = \frac{dr}{dt}\underline{e_r} + r\frac{d\theta}{dt}\underline{e_\theta}$$

$$\underline{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\underline{e_r} + \left\{2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right\}\underline{e_\theta}$$

① when $\theta = \pi/4$ $t = \pi/4$ $\Rightarrow$ $r = 1$ $\boxed{\underline{r} = \underline{e_r} \ m}$

② $\frac{dr}{dt} = 1$ $\frac{d\theta}{dt} = 1$ $\Rightarrow$ $\boxed{\underline{V} = \underline{e_r} + \underline{e_\theta}}$ (r=1 also)

③ $\frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$ $\Rightarrow$ $\boxed{\underline{a} = -\underline{e_r} + 2\underline{e_\theta}}$

④ Recall $\underline{v} = \bar{V}\,\underline{t}$ $V = |\underline{v}|$

Hence $\underline{t} = \frac{\underline{v}}{\bar{V}}$ $|\underline{v}| = |\underline{e_r} + \underline{e_\theta}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\boxed{\underline{t} = (\underline{e_r} + \underline{e_\theta})/\sqrt{2}}$

⑤ formula $\boxed{\underline{v} = \bar{V}\,\underline{t} = \sqrt{2}\,\underline{t}}$

⑥ We know $\underline{a} = a_t \underline{t} + a_n \underline{n}$

Dot both sides with $\underline{t}$:

$$\underline{a}\cdot\underline{t} = a_t \underbrace{\underline{t}\cdot\underline{t}}_{=1} + a_n \underbrace{\underline{t}\cdot\underline{n}}_{=0}$$

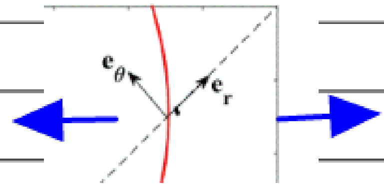
$$a_t = \underline{a}\cdot\underline{t} = (-\underline{e_r} + 2\underline{e_\theta})\cdot(\underline{e_r} + \underline{e_\theta}) / \sqrt{2}$$

$$= (-1 + 2) / \sqrt{2} = \frac{1}{\sqrt{2}}$$

To find a_n we first need to find $\underline{n}$ in $\underline{e_r}$, $\underline{e_\theta}$ coords

We know $\underline{n}$ is perpendicular to $\underline{k}$ and $\underline{t} \Rightarrow \underline{n} = \pm \underline{k} \times \underline{t}$

Choose $\underline{n} = +\underline{k} \times \underline{t}$ to point to center of curvature

$+\underline{k} \times \underline{t}$  $-\underline{k} \times \underline{t}$

Hence $\underline{n} = \underline{k} \times \underline{t} = \underline{k} \times (\underline{e_r} + \underline{e_\theta}) / \sqrt{2}$

Note $\underline{k} \times \underline{e_r} = \underline{e_\theta}$ $\quad \underline{k} \times \underline{e_\theta} = -\underline{e_r}$ (right hand rule)

Hence $\underline{n} = (-\underline{e_r} + \underline{e_\theta}) / \sqrt{2}$

Finally $a_n = \underline{a} \cdot \underline{n} = (-\underline{e_r} + 2\underline{e_\theta}) \cdot (-\underline{e_r} + \underline{e_\theta}) / \sqrt{2}$

$= 3/\sqrt{2}$

$$\boxed{\underline{a} = \frac{1}{\sqrt{2}} \underline{t} + \frac{3}{\sqrt{2}} \underline{n}}$$

Check $|\underline{a}| = \left| \frac{1}{\sqrt{2}} \underline{t} + \frac{3}{\sqrt{2}} \underline{n} \right| = \sqrt{5}$

or $|\underline{a}| = |-\underline{e_r} + 2\underline{e_\theta}| = \sqrt{5}$ ✓