Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion; Newton's laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy;
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion; kinematics formulas
   - Dynamics formulas for rigid bodies; calculating moments of inertia
   - Motion of systems of rigid bodies
   - Energy and momentum for rigid bodies

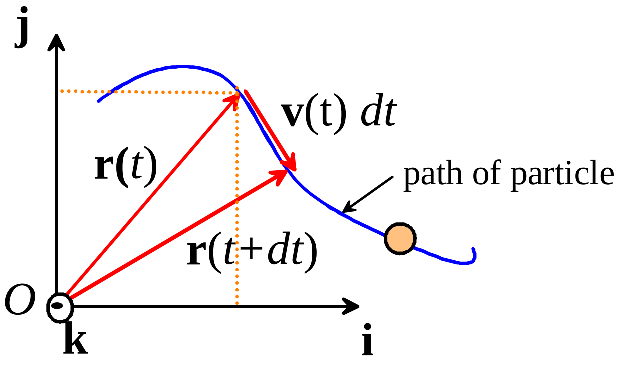
Exam topics

Particle Dynamics: Concept Checklist

- Understand the concept of an ‘inertial frame’
- Be able to idealize an engineering design as a set of particles, and know when this idealization will give accurate results
- Describe the motion of a system of particles (eg components in a fixed coordinate system; components in a polar coordinate system, etc)
- Be able to differentiate position vectors (with proper use of the chain rule!) to determine velocity and acceleration; and be able to integrate acceleration or velocity to determine position vector.
- Be able to describe motion in normal-tangential and polar coordinates (eg be able to write down vector components of velocity and acceleration in terms of speed, radius of curvature of path, or coordinates in the cylindrical-polar system).
- Be able to convert between Cartesian to normal-tangential or polar coordinate descriptions of motion
- Be able to draw a correct free body diagram showing forces acting on system idealized as particles
- Be able to write down Newton’s laws of motion in rectangular, normal-tangential, and polar coordinate systems
- Be able to obtain an additional moment balance equation for a rigid body moving without rotation or rotating about a fixed axis at constant rate.
- Be able to use Newton’s laws of motion to solve for unknown accelerations or forces in a system of particles
- Use Newton’s laws of motion to derive differential equations governing the motion of a system of particles
- Be able to re-write second order differential equations as a pair of first-order differential equations in a form that MATLAB can solve

Particle Kinematics

Inertial frame – non accelerating, non rotating reference frame
Particle – point mass at some position in space



Position Vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity Vector
$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$
$$= \frac{d}{dt}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$
$$\Rightarrow v_x(t) = \frac{dx}{dt} \qquad v_y(t) = \frac{dy}{dt} \qquad v_z(t) = \frac{dz}{dt}$$

- Direction of velocity vector is parallel to path
- Magnitude of velocity vector is distance traveled / time

Acceleration Vector

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} = \frac{d}{dt}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}$$
$$\Rightarrow a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Particle Kinematics

- Straight line motion with constant acceleration

$$\mathbf{r} = \left[X_0 + V_0 t + \frac{1}{2} a t^2 \right] \mathbf{i} \qquad \mathbf{v} = \left(V_0 + at \right) \mathbf{i} \qquad \mathbf{a} = a\mathbf{i}$$

- Time/velocity/position dependent acceleration – use calculus

$$\mathbf{r} = \left(X_0 + \int_0^t v(t) dt \right) \mathbf{i} \qquad \mathbf{v} = \left(V_0 + \int_0^t a(t) dt \right) \mathbf{i}$$

$$\int_{V_0}^{v(t)} v dv = \int_0^{x(t)} a(x) dx$$

$$a = \frac{dv}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{V_0}^{v} f(v) dv = \int_0^t g(t) dt$$

$$v = \frac{dx}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{X_0}^{x(t)} f(x) dv = \int_0^t v(t) dt$$

Particle Kinematics

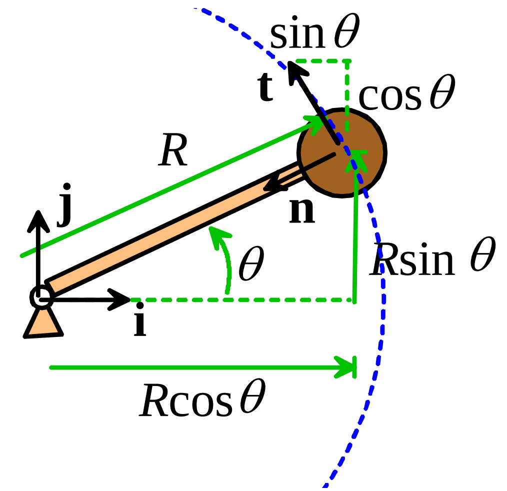
- Circular Motion at const speed

$$\theta = \omega t \quad s = R\theta \quad V = \omega R$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = -\omega^2 R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \omega^2 R\mathbf{n} = \frac{V^2}{R}\mathbf{n}$$



- General circular motion

$$\omega = d\theta / dt \quad \alpha = d\omega / dt = d^2\theta / dt^2$$

$$s = R\theta \quad V = ds / dt = R\omega$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \alpha R\mathbf{t} + \omega^2 R\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

Particle Kinematics

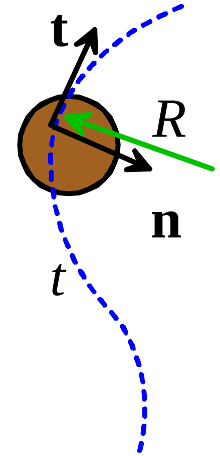
- Arbitrary path

$$\mathbf{r} = x(\lambda)\mathbf{i} + y(\lambda)\mathbf{j}$$

$$\mathbf{v} = V\mathbf{t}$$

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

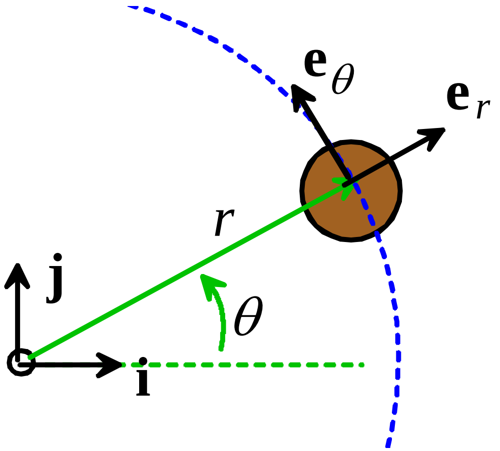
$$\frac{1}{R} = \frac{\left| \frac{dx}{d\lambda}\frac{d^2y}{d\lambda^2} - \frac{dy}{d\lambda}\frac{d^2x}{d\lambda^2} \right|}{\left\{ \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 \right\}^{3/2}}$$



- Polar Coordinates

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

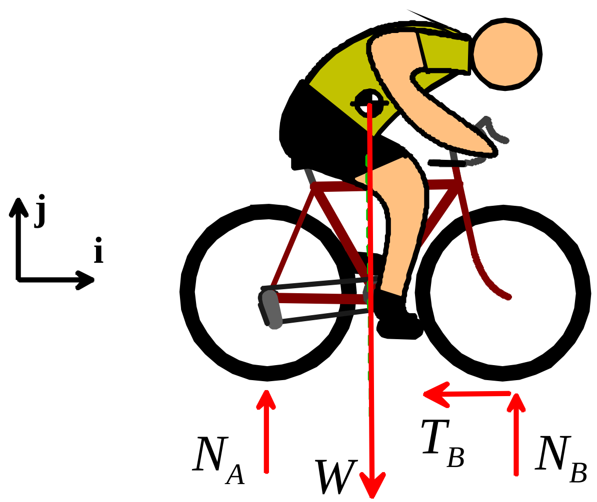
$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right)\mathbf{e}_\theta$$



Newton's laws

- For a particle $$\mathbf{F} = m\mathbf{a}$$

- For a rigid body in motion without rotation, or a particle on a massless frame



$$\mathbf{M}_C = \mathbf{0}$$

You MUST take moments about center of mass

Calculating forces required to cause prescribed motion of a particle

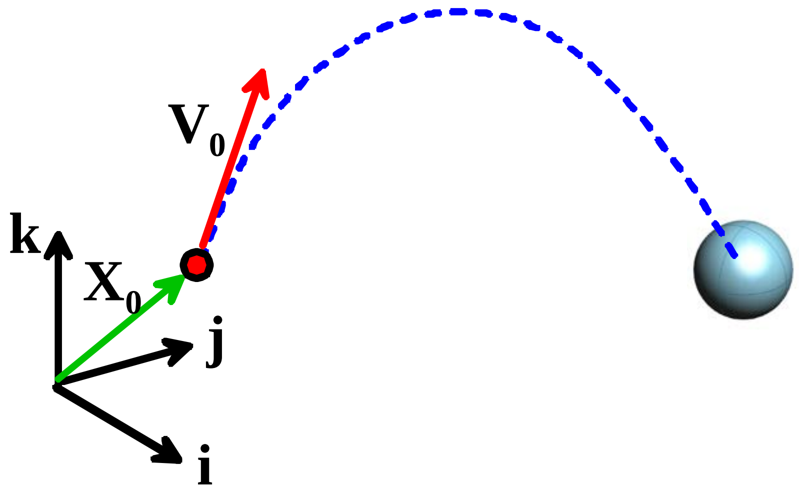
- Idealize system
- Free body diagram
- Kinematics
- **F**=m**a** for each particle.
- $\mathbf{M}_C = \mathbf{0}$ (for rigid bodies or frames only)
- Solve for unknown forces or accelerations

Deriving Equations of Motion for particles

1. Idealize system
2. Introduce variables to describe motion (often *x,y* coords, but we will see other examples)
3. Write down **r**, differentiate to get **a**
4. Draw FBD
5. $\mathbf{F} = m\mathbf{a}$
6. If necessary, eliminate reaction forces
7. Result will be differential equations for coords defined in (2), e.g. $m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = kY_0 \sin\omega t$
8. Identify initial conditions, and solve ODE

Motion of a projectile

$$\left.\begin{aligned}\mathbf{r} &= X_0\mathbf{i} + Y_0\mathbf{j} + Z_0\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= V_{x0}\mathbf{i} + V_{y0}\mathbf{j} + V_{z0}\mathbf{k}\end{aligned}\right\} t = 0$$



$$\mathbf{r} = \left(X_0 + V_{x0}t\right)\mathbf{i} + \left(Y_0 + V_{y0}t\right)\mathbf{j} + \left(Z_0 + V_{z0}t - \frac{1}{2}gt^2\right)\mathbf{k}$$

$$\mathbf{v} = \left(V_{x0}\right)\mathbf{i} + \left(V_{y0}\right)\mathbf{j} + \left(V_{z0} - gt\right)\mathbf{k}$$

$$\mathbf{a} = -g\mathbf{k}$$

Rearranging differential equations for MATLAB

- Example $$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$$
- Introduce $v = dy / dt$
- Then $$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} v \\ -2\zeta\omega_n v - \omega_n^2 y \end{bmatrix}$$
- This has form $$\frac{d\mathbf{w}}{dt} = f(t, \mathbf{w}) \qquad \mathbf{w} = \begin{bmatrix} y \\ v \end{bmatrix}$$

Conservation laws for particles: Concept Checklist

- Know the definitions of power (or rate of work) of a force, and work done by a force
- Know the definition of kinetic energy of a particle
- Understand power-work-kinetic energy relations for a particle
- Be able to use work/power/kinetic energy to solve problems involving particle motion
- Be able to distinguish between conservative and non-conservative forces
- Be able to calculate the potential energy of a conservative force
- Be able to calculate the force associated with a potential energy function
- Know the work-energy relation for a system of particles; (energy conservation for a closed system)
- Use energy conservation to analyze motion of conservative systems of particles

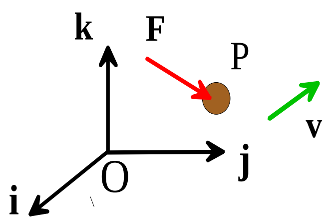
- Know the definition of the linear impulse of a force
- Know the definition of linear momentum of a particle
- Understand the impulse-momentum (and force-momentum) relations for a particle
- Understand impulse-momentum relations for a system of particles (momentum conservation for a closed system)
- Be able to use impulse-momentum to analyze motion of particles and systems of particles
- Know the definition of restitution coefficient for a collision
- Predict changes in velocity of two colliding particles in 2D and 3D using momentum and the restitution formula

- Know the definition of angular impulse of a force
- Know the definition of angular momentum of a particle
- Understand the angular impulse-momentum relation
- Be able to use angular momentum to solve central force problems

Work and Energy relations for particles

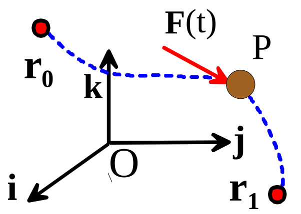
Rate of work done by a force (power developed by force)

$$P = \mathbf{F} \cdot \mathbf{v}$$



Total work done by a force

$$W = \int_0^{t_1} \mathbf{F} \cdot \mathbf{v} dt \qquad W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$



Kinetic energy

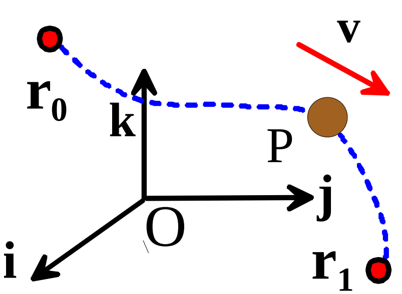
$$T = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

Power-kinetic energy relation

$$P = \frac{dT}{dt}$$

Work-kinetic energy relation

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = T - T_0$$

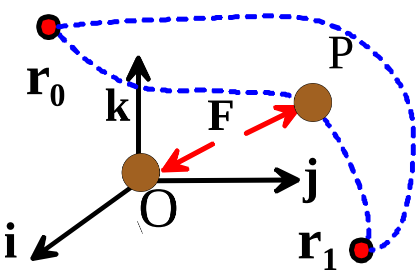


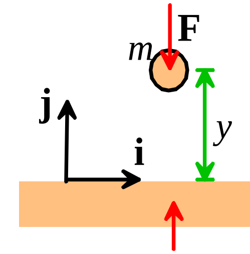
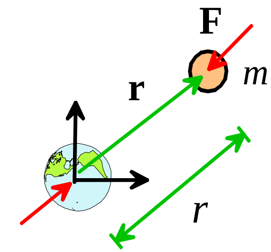
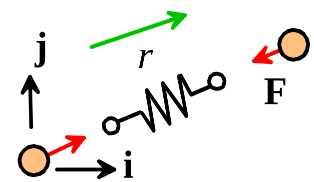
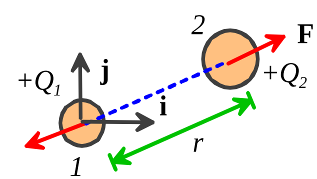
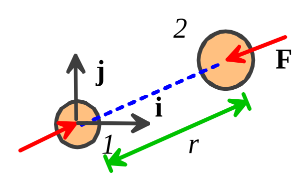
Potential energy

Potential energy of a conservative force (pair)

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} + \text{constant}$$

$$\mathbf{F} = -\text{grad}(V)$$



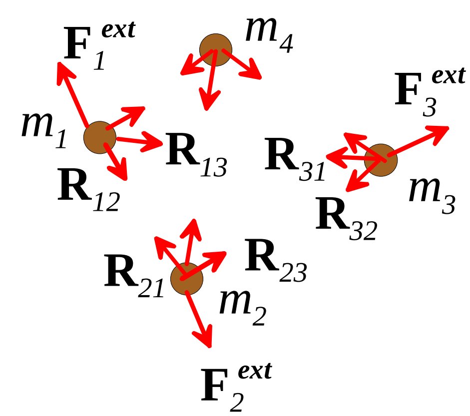
Type of force	Potential energy	
Gravity acting on a particle near earths surface	$V = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$V = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$V = \dfrac{1}{2}k(r - L_0)^2$	
Force acting between two charged particles	$V = \dfrac{Q_1 Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

Energy relations for conservative systems subjected to external forces

Internal Forces: (forces exerted by one part of the system on another) $\mathbf{R}_{ij}$

External Forces: (any other forces) $\mathbf{F}_i^{ext}$

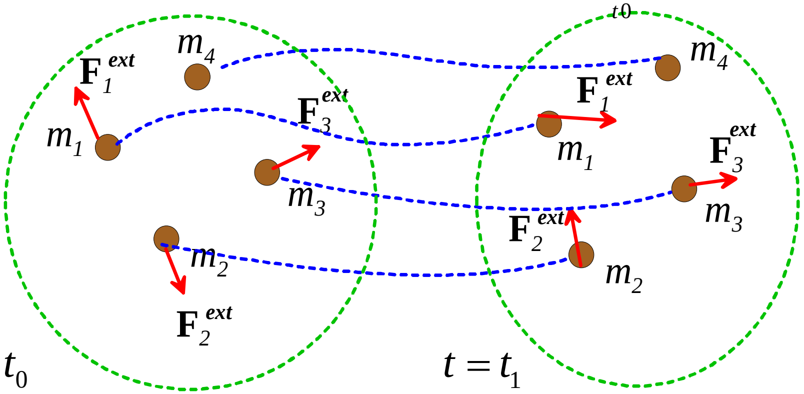
System is conservative if all internal forces are conservative forces (or constraint forces)



Energy relation for a conservative system

$$External\ \ Power\ \ P^{ext}(t)$$

$$External\ \ work\ \ \Delta W^{ext} = \int_{t0}^{t1} P(t)dt$$



$t = t_0$

$Total\ \ KE\ \ T_0^{TOT}$

$Total\ \ PE\ \ V_0^{TOT}$

$t = t_1$

$Total\ \ KE\ \ T_1^{TOT}$

$Total\ \ PE\ \ V_1^{TOT}$

$$\Delta W_{ext} = T_1^{TOT} + V_1^{TOT} - \left(T_0^{TOT} + V_0^{TOT} \right)$$

Special case – zero external work:

$$T_1^{TOT} + V_1^{TOT} = T_0^{TOT} + V_0^{TOT}$$

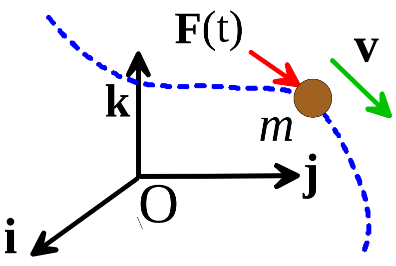
KE+PE = constant

Impulse-momentum for a single particle

Definitions

Linear Impulse of a force $\mathbf{I} = \int_{t_0}^{t_1} \mathbf{F}(t)dt$

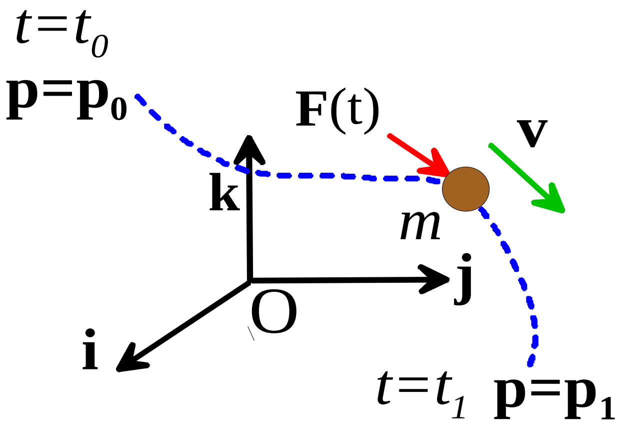
Linear momentum of a particle $\mathbf{p} = m\mathbf{v}$



Impulse-Momentum relations

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{I} = \mathbf{p}_1 - \mathbf{p}_0$$

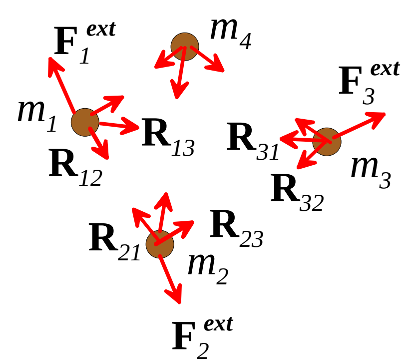


Impulse-momentum for a system of particles

$\mathbf{R}_{ij}$ Force exerted on particle i by particle j

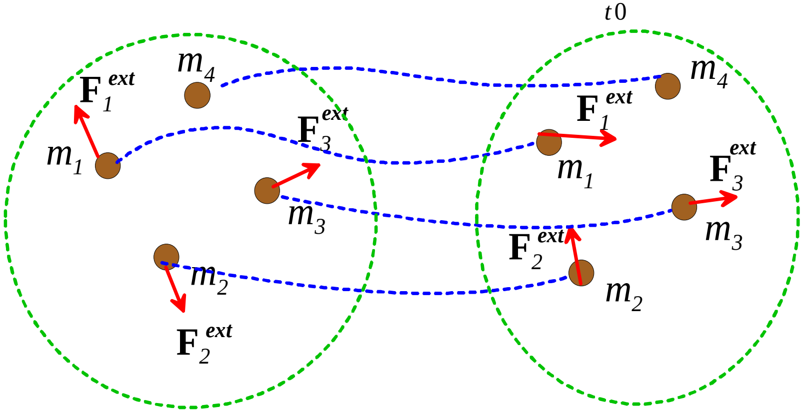
$\mathbf{F}_i^{ext}$ External force on particle i

$\mathbf{v}_i$ Velocity of particle i



$Total \;\; External \;\; Force \;\; \mathbf{F}^{TOT}(t)$

$$Total \; External \; Impulse \;\; \mathbf{I}^{TOT} = \int_{t0}^{t1} \mathbf{F}^{TOT}(t)dt$$



$t = t_0$

$Total \;\; momentum \;\; \mathbf{p}_0^{TOT}$

$t = t_1$

$Total \;\; momentum \;\; \mathbf{p}_1^{TOT}$

Impulse-momentum for the system:

$$\mathbf{F}^{TOT} = \frac{d\mathbf{p}^{TOT}}{dt}$$

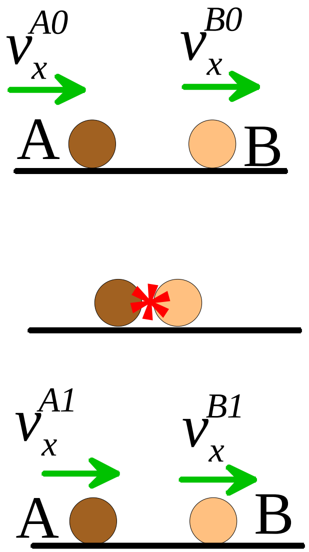
$$\mathbf{I}^{TOT} = \mathbf{p}_1^{TOT} - \mathbf{p}_0^{TOT}$$

Special case – zero external impulse:

$$\mathbf{p}_1^{TOT} = \mathbf{p}_0^{TOT}$$

(Linear momentum conserved)

Collisions



Momentum

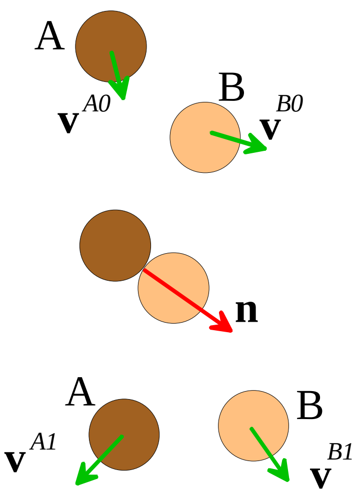
$$m_A v_x^{A1} + m_B v_x^{B1} = m_A v_x^{A0} + m_B v_x^{B0}$$

Restitution formula

$$v^{B1} - v^{A1} = -e\left(v^{B0} - v^{A0}\right)$$

$$v^{B1} = v^{B0} - \frac{m_A}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$

$$v^{A1} = v^{A0} + \frac{m_B}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$



Momentum

$$m_B\ \mathbf{v}^{B1} + m_A \mathbf{v}^{A1} = m_B\ \mathbf{v}^{B0} + m_A \mathbf{v}^{A0}$$

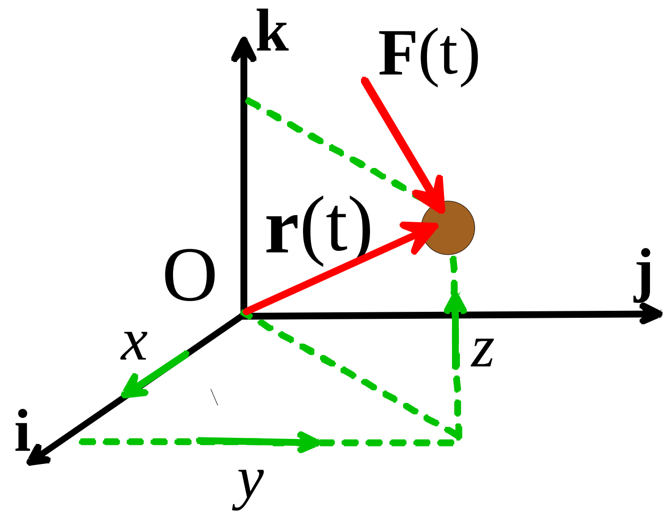
Restitution formula

$$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Angular Impulse-Momentum Equations for a Particle



Angular Impulse $$\mathbf{A} = \int_{t_0}^{t_1} \mathbf{r}(t) \times \mathbf{F}(t) dt$$

Angular Momentum $$\mathbf{h} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

Impulse-Momentum relations $$\mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt} \qquad \mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$$

Special Case $$\mathbf{A} = \mathbf{0} \Rightarrow \mathbf{h}_1 = \mathbf{h}_0$$ Angular momentum conserved

Useful for central force problems (when forces on a particle always act through a single point, eg planetary gravity)

Free Vibrations – concept checklist

You should be able to:

1. Understand simple harmonic motion (amplitude, period, frequency, phase)
2. Identify # DOF (and hence # vibration modes) for a system
3. Understand (qualitatively) meaning of ‘natural frequency’ and ‘Vibration mode’ of a system
4. Calculate natural frequency of a 1DOF system (linear and nonlinear)

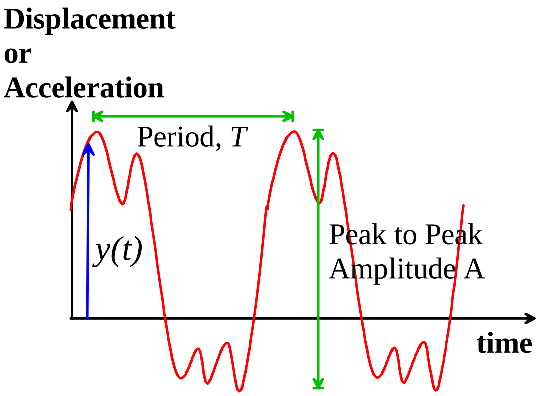
5. Write the EOM for simple spring-mass-damper systems by inspection

6. Understand natural frequency, damped natural frequency, and ‘Damping factor’ for a dissipative 1DOF vibrating system
7. Know formulas for nat freq, damped nat freq and ‘damping factor’ for spring-mass system in terms of k,m,c
8. Understand underdamped, critically damped, and overdamped motion of a dissipative 1DOF vibrating system
9. Be able to determine damping factor and natural frequency from a measured free vibration response
10. Be able to predict motion of a freely vibrating 1DOF system given its initial velocity and position, and apply this to design-type problems

Vibrations and simple harmonic motion

Typical vibration response

- Period, frequency, angular frequency amplitude



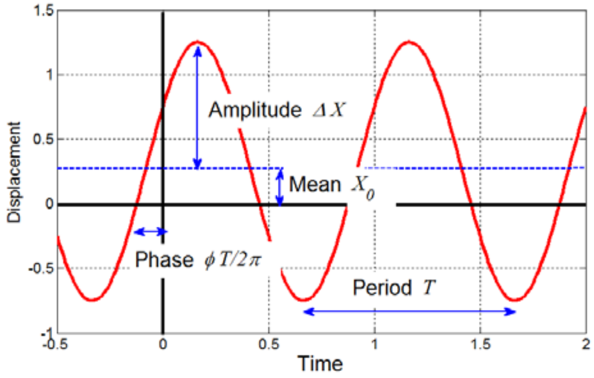
Simple Harmonic Motion

$$x(t) = X_0 + \Delta X \sin\left(\omega t + \phi\right)$$

$$v(t) = \Delta V \cos(\omega t + \phi)$$

$$a(t) = -\Delta A \sin(\omega t + \phi)$$

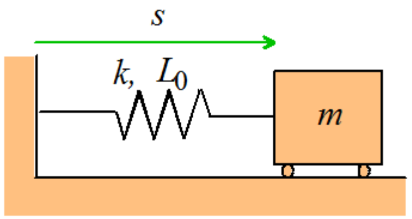
$$\Delta V = \omega \Delta X \qquad \Delta A = \omega \Delta V$$



Vibration of 1DOF conservative systems

Harmonic Oscillator

Canonical Vibration Problem: The spring mass system is released with velocity v_0 from position s_0 at time *t=0*. Find *s(t)*.



Derive EOM (F=ma) $$\frac{m}{k}\frac{d^2 s}{dt^2} + s = L_0$$

Compare with 'standard' differential equation

Equation $$\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$$ Initial Conditions $$x = x_0 \qquad \frac{dx}{dt} = v_0 \qquad t = 0$$

$$x = s \qquad C = L_0 \qquad x_0 = s_0$$

$$\frac{1}{\omega_n^2} = \frac{m}{k}$$

Solution $$x = C + X_0 \sin(\omega_n t + \phi)$$

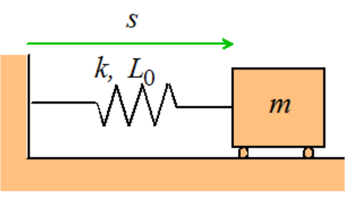
$$X_0 = \sqrt{(x_0 - C)^2 + v_0^2 / \omega_n^2} \qquad \phi = \tan^{-1}\left(\frac{(x_0 - C)\omega_n}{v_0}\right)$$

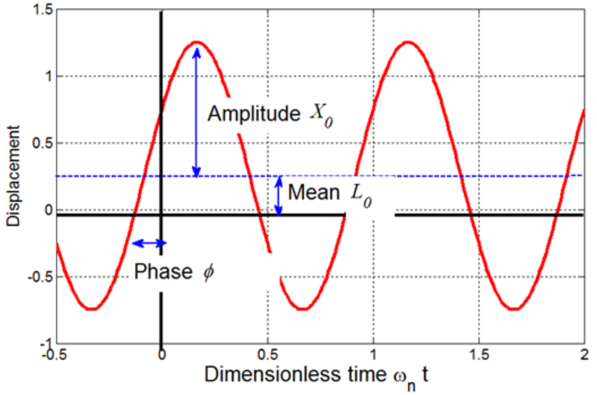
Or $$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

Solution

$$s(t) = L_0 + \sqrt{(s_0 - L_0)^2 + v_0^2 / \omega_n^2}\,\sin(\omega_n t + \phi)$$

$$\left.\begin{aligned} s &= s_0 \\ \frac{ds}{dt} &= v_0 \end{aligned}\right\} t = 0$$



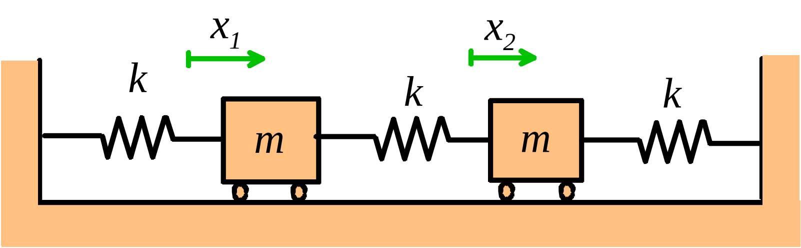


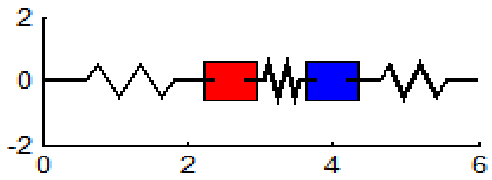
Natural Frequency $$\omega_n = \sqrt{\frac{k}{m}}$$

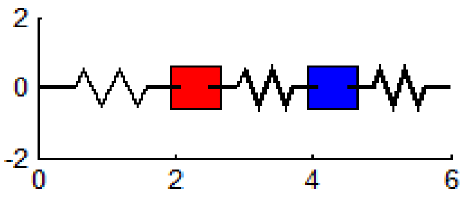
Vibration modes and natural frequencies

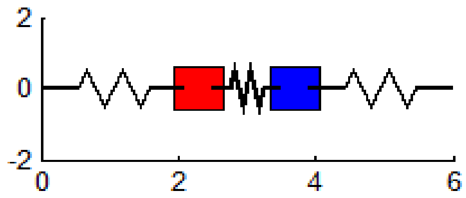
•Vibration modes: special initial deflections that cause entire system to vibrate harmonically

•Natural Frequencies are the corresponding vibration frequencies







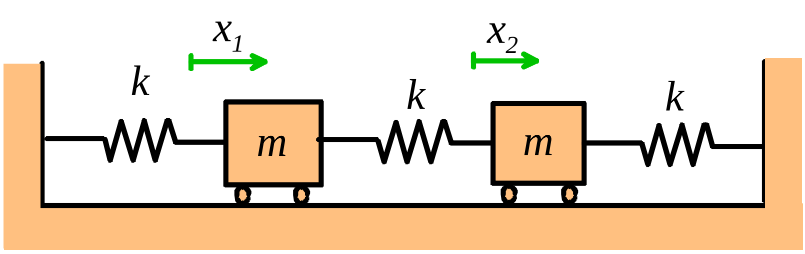


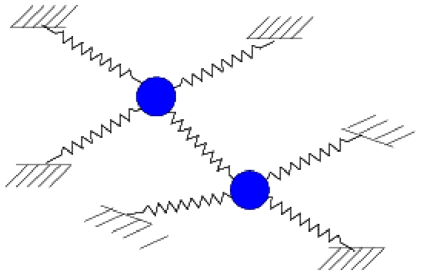
Number of DOF (and vibration modes)

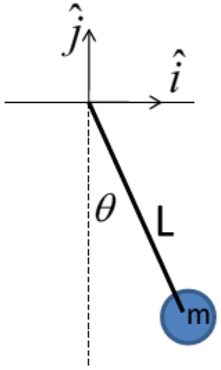
In 2D: # DOF = 2*# particles + 3*# rigid bodies - # constraints
In 3D: # DOF = 3*# particles + 6*# rigid bodies - # constraints

Expected # vibration modes = # DOF - # rigid body modes

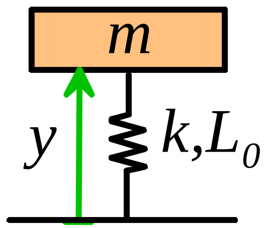
A ‘rigid body mode’ is steady rotation or translation of the entire system at constant speed. The maximum number of ‘rigid body’ modes (in 3D) is 6; in 2D it is 3. Usually only things like a vehicle or a molecule, which can move around freely, have rigid body modes.

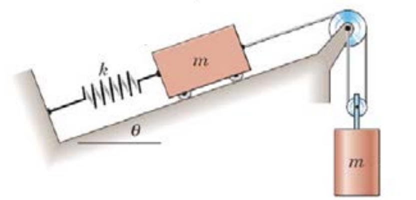






Calculating nat freqs for 1DOF systems – the basics

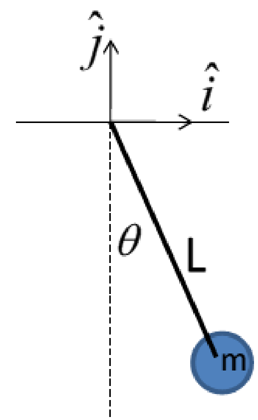




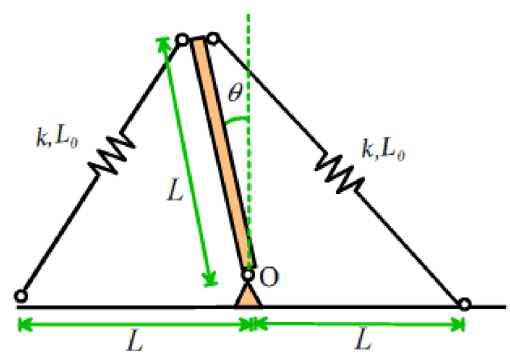
EOM for small vibration of any 1DOF undamped system has form

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+y=C$$

ω_n is the natural frequency



1. Get EOM (F=ma or energy)
2. Linearize (sometimes)
3. Arrange in standard form
4. Read off nat freq.



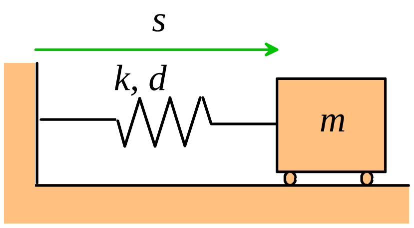
Tricks for calculating natural frequencies of 1DOF undamped systems

- Using energy conservation to find EOM

$$KE + PE = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 + \frac{1}{2} k (s - L_0)^2 = const$$

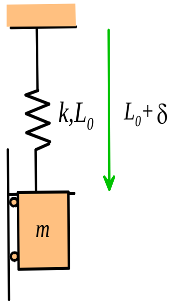
$$\Rightarrow \frac{d}{dt}(KE + PE) = m \left(\frac{ds}{dt} \right) \frac{d^2 s}{dt^2} + k(s - L_0) \frac{ds}{dt} = 0$$

$$\Rightarrow m \frac{d^2 s}{dt^2} + ks = kL_0$$



- Nat freq is related to static deflection

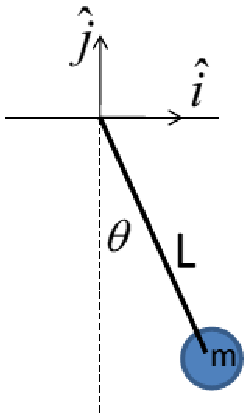
$$\omega_n = \sqrt{\frac{g}{\delta}}$$



Linearizing EOM

Sometimes EOM has form

$$\frac{d^2y}{dt^2} + f(y) = C$$



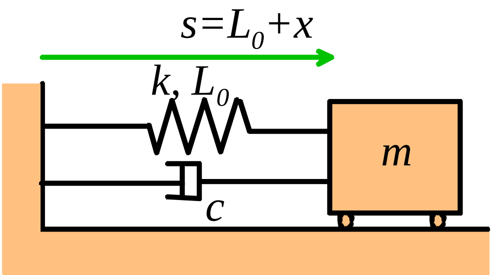
We cant solve this in general…
Instead, assume y is small

$$m\frac{d^2y}{dt^2} + f(0) + \left.\frac{df}{dy}\right|_{y=0} y + \ldots = C$$

$$m\frac{d^2y}{dt^2} + \left.\frac{df}{dy}\right|_{y=0} y = C - f(0)$$

There are short-cuts to doing the Taylor expansion

Writing down EOM for spring-mass-damper systems

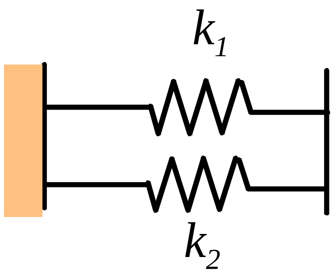


Commit this to memory! (or be able to derive it…)

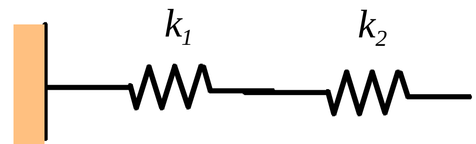
$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}}$$

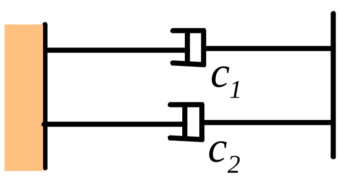
x(t) is the ‘dynamic variable’ (deflection from static equilibrium)



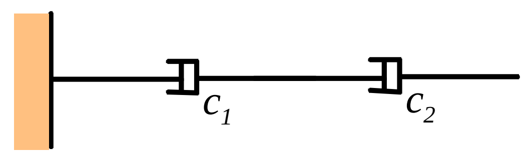
Parallel: stiffness $k = k_1 + k_2$



Series: stiffness $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

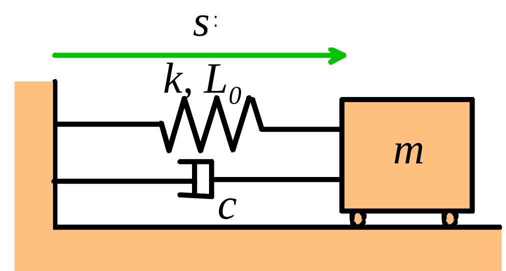


Parallel: coefficient $c = c_1 + c_2$



Parallel: coefficient $\frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2}$

Canonical damped vibration problem



EOM $m\dfrac{d^2s}{dt^2}+c\dfrac{ds}{dt}+ks=kL_0$ with $s=s_0 \quad \dfrac{ds}{dt}=v_0 \quad t=0$

Standard Form $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2}+\dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt}+x=C \quad x=x_0 \quad \dfrac{dx}{dt}=v_0 \quad t=0$

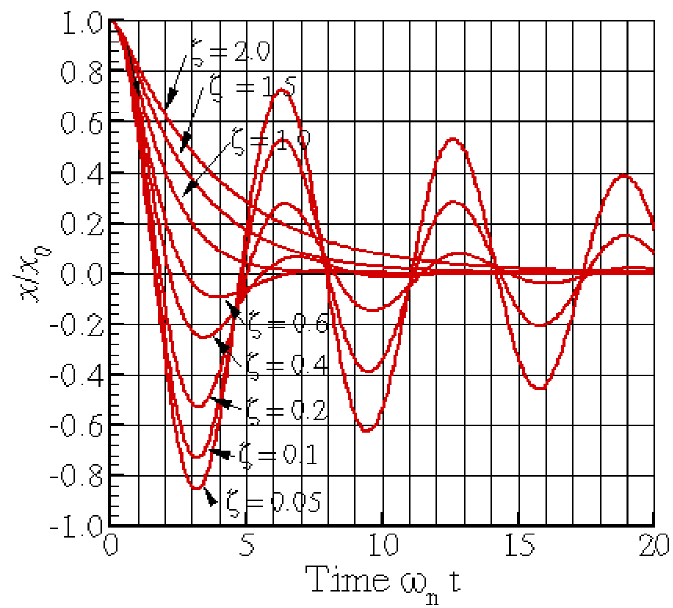
$s\equiv x \qquad \omega_n=\sqrt{\dfrac{k}{m}} \qquad \varsigma=\dfrac{c}{2\sqrt{km}} \qquad C=L_0 \qquad x_0\equiv s_0$

$$\omega_d=\omega_n\sqrt{\left|1-\varsigma^2\right|}$$

Overdamped $\varsigma>1$

Critically Damped $\varsigma=1$

Underdamped $\varsigma<1$

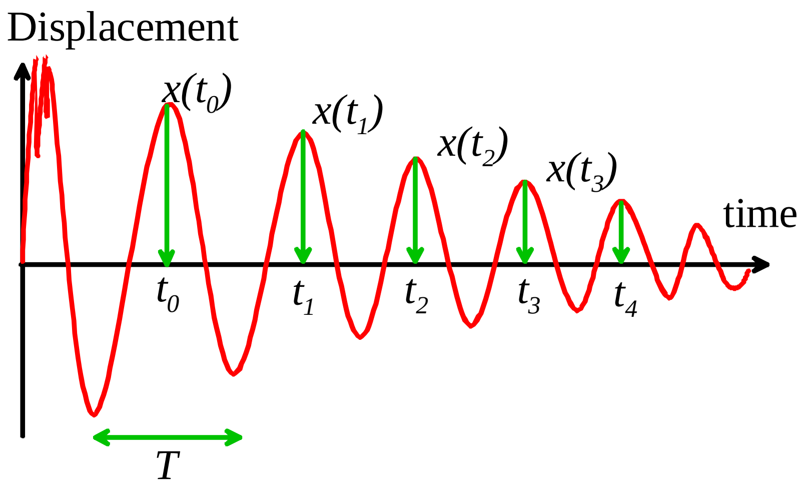


Overdamped $\varsigma>1$ $x(t)=C+\exp(-\varsigma\omega_n t)\left\{\dfrac{v_0+(\varsigma\omega_n+\omega_d)(x_0-C)}{2\omega_d}\exp(\omega_d t)-\dfrac{v_0+(\varsigma\omega_n-\omega_d)(x_0-C)}{2\omega_d}\exp(-\omega_d t)\right\}$

Critically Damped $\varsigma=1$ $x(t)=C+\left\{(x_0-C)+\left[v_0+\omega_n(x_0-C)\right]t\right\}\exp(-\omega_n t)$

Underdamped $\varsigma<1$ $x(t)=C+\exp(-\varsigma\omega_n t)\left\{(x_0-C)\cos\omega_d t+\dfrac{v_0+\varsigma\omega_n(x_0-C)}{\omega_d}\sin\omega_d t\right\}$

Calculating natural frequency and damping factor from a measured vibration response



Measure log decrement:

$$\delta = \frac{1}{n}\log\left(\frac{x(t_0)}{x(t_n)}\right)$$

Measure period: T

Then

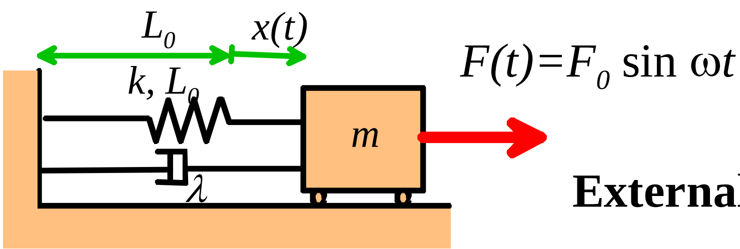
$$\varsigma = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \qquad \omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T}$$

Forced Vibrations – concept checklist

You should be able to:

1. Be able to derive equations of motion for spring-mass systems subjected to external forcing (several types) and solve EOM using complex vars, or by comparing to solution tables
2. Understand (qualitatively) meaning of ‘transient’ and ‘steady-state’ response of a forced vibration system (see Java simulation on web)
3. Understand the meaning of ‘Amplitude’ and ‘phase’ of steady-state response of a forced vibration system
4. Understand amplitude-v-frequency formulas (or graphs), resonance, high and low frequency response for 3 systems
5. Determine the amplitude of steady-state vibration of forced spring-mass systems.
6. Deduce damping coefficient and natural frequency from measured forced response of a vibrating system
7. Use forced vibration concepts to design engineering systems

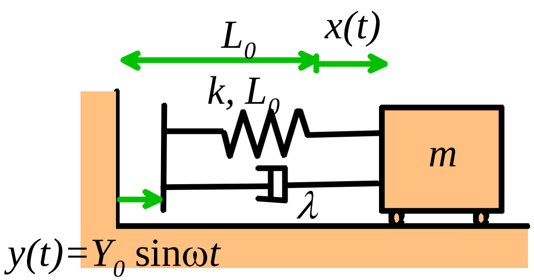
EOM for forced vibrating systems



External forcing

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=KF_0\sin\omega t$$

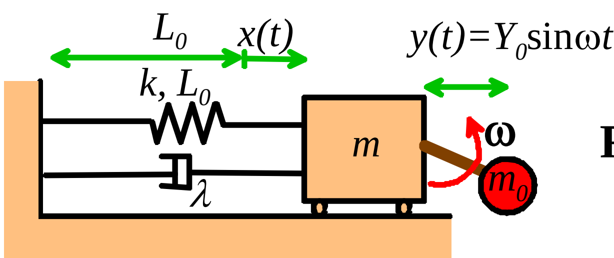
$$\omega_n=\sqrt{\frac{k}{m}},\qquad \varsigma=\frac{\lambda}{2\sqrt{km}},\qquad K=\frac{1}{k}$$



Base Excitation

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$$

$$\omega_n=\sqrt{\frac{k}{m}},\qquad \varsigma=\frac{\lambda}{2\sqrt{km}},\qquad K=1$$



Rotor Excitation

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}=K\frac{Y_0\omega^2}{\omega_n^2}\sin\omega t$$

$$\omega_n=\sqrt{\frac{k}{M}}\qquad \varsigma=\frac{\lambda}{2\sqrt{kM}}\qquad K=\frac{m_0}{M}\qquad M=m+m_0$$

Steady-state and Transient solution to EOM

Equation $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF_0\sin(\omega t)$ Initial Conditions $x=x_0 \quad \frac{dx}{dt}=v_0 \quad t=0$

Full Solution $x(t)=C+x_h(t)+x_p(t)$

Steady state part (particular integral) $x_p(t)=X_0\sin\left(\omega t+\phi\right)$

$$X_0=\frac{KF_0}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

Transient part (complementary integral)

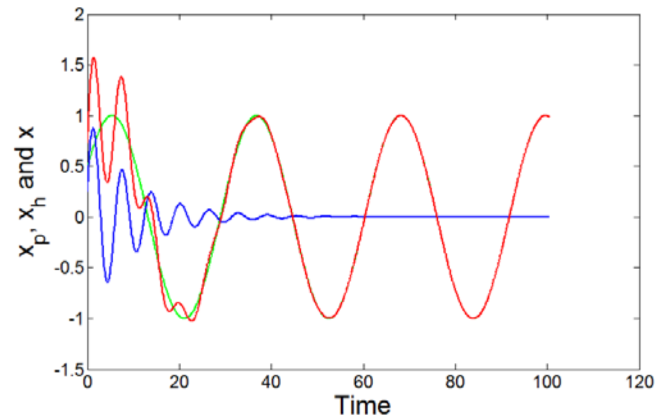
Overdamped $\varsigma>1$ $$x_h(t)=C+\exp(-\varsigma\omega_n t)\left\{\frac{v_0^h+(\varsigma\omega_n+\omega_d)x_0^h}{2\omega_d}\exp(\omega_d t)-\frac{v_0^h+(\varsigma\omega_n-\omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$$

Critically Damped $\varsigma=1$ $$x_h(t)=C+\left\{x_0^h+\left[v_0^h+\omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

Underdamped $\varsigma<1$ $$x_h(t)=C+\exp(-\varsigma\omega_n t)\left\{x_0^h\cos\omega_d t+\frac{v_0^h+\varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

$$\omega_d=\omega_n\sqrt{\left|1-\varsigma^2\right|}$$

$$x_0^h=x_0-C-x_p(0)=x_0-C-X_0\sin\phi \qquad v_0^h=v_0-\left.\frac{dx_p}{dt}\right|_{t=0}=v_0-X_0\omega\cos\phi$$



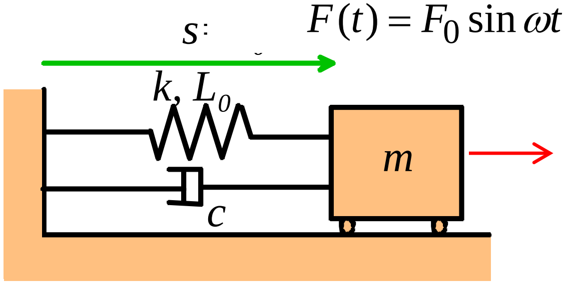
Canonical externally forced system (steady state solution)

Steady state solution to $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF_0\sin(\omega t)$ $\quad \omega = 2\pi / T$

$$\omega_n=\sqrt{\frac{k}{m}} \quad \zeta=\frac{c}{2\sqrt{km}} \quad K=\frac{1}{k}$$

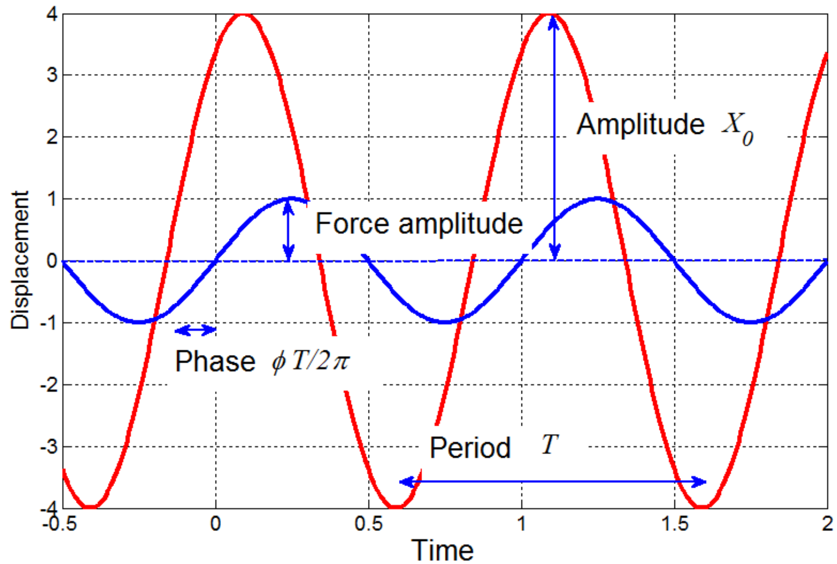
$$x_p(t)=X_0\sin(\omega t+\phi)$$

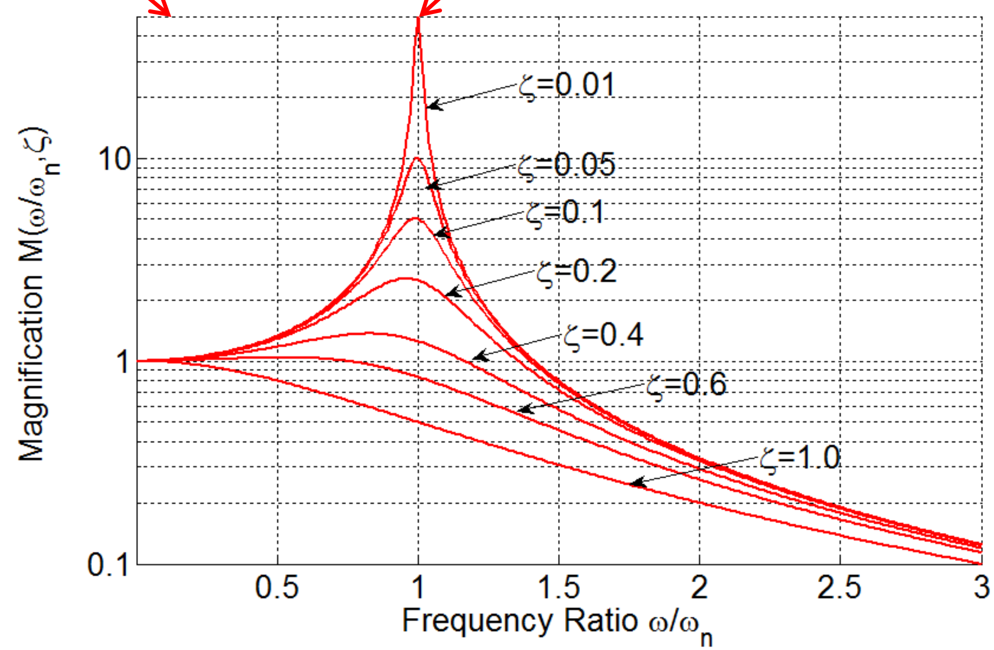
$F(t)=F_0\sin\omega t$



$$X_0=KF_0M(\omega/\omega_n,\zeta) \qquad M=\frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

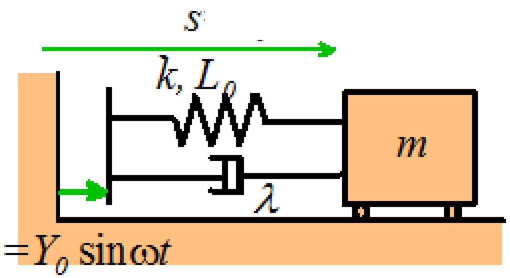
$$M_{\max}\approx\frac{1}{2\zeta}$$





Canonical base excited system (steady state solution)

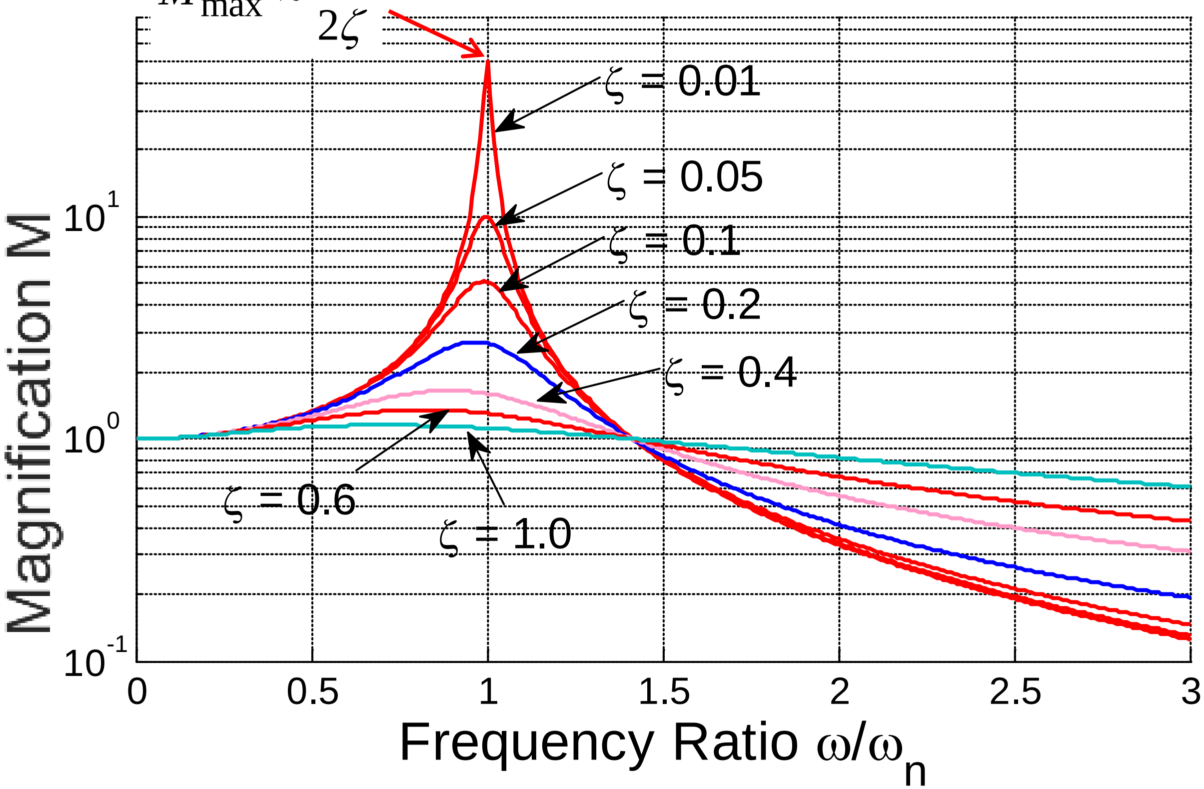
Steady state solution to $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$$



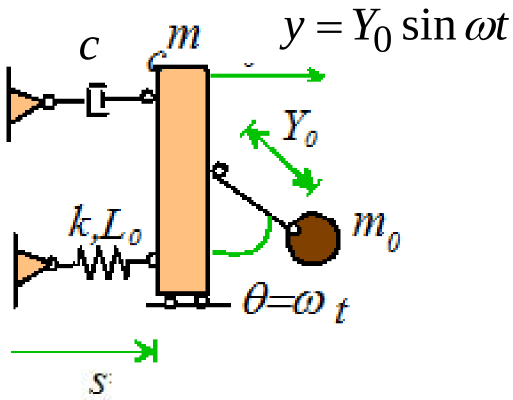
$$\omega_n=\sqrt{\frac{k}{m}}\quad \zeta=\frac{c}{2\sqrt{km}}\quad K=1 \qquad x_p(t)=X_0\sin(\omega t+\phi)$$

$$X_0=KY_0M(\omega,\omega_n,\zeta)\qquad M=\frac{\left\{1+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}}\qquad \phi=\tan^{-1}\frac{-2\varsigma\omega^3/\omega_n^3}{1-(1-4\varsigma^2)\omega^2/\omega_n^2}$$

$$M_{\max}\approx\frac{1}{2\zeta}$$



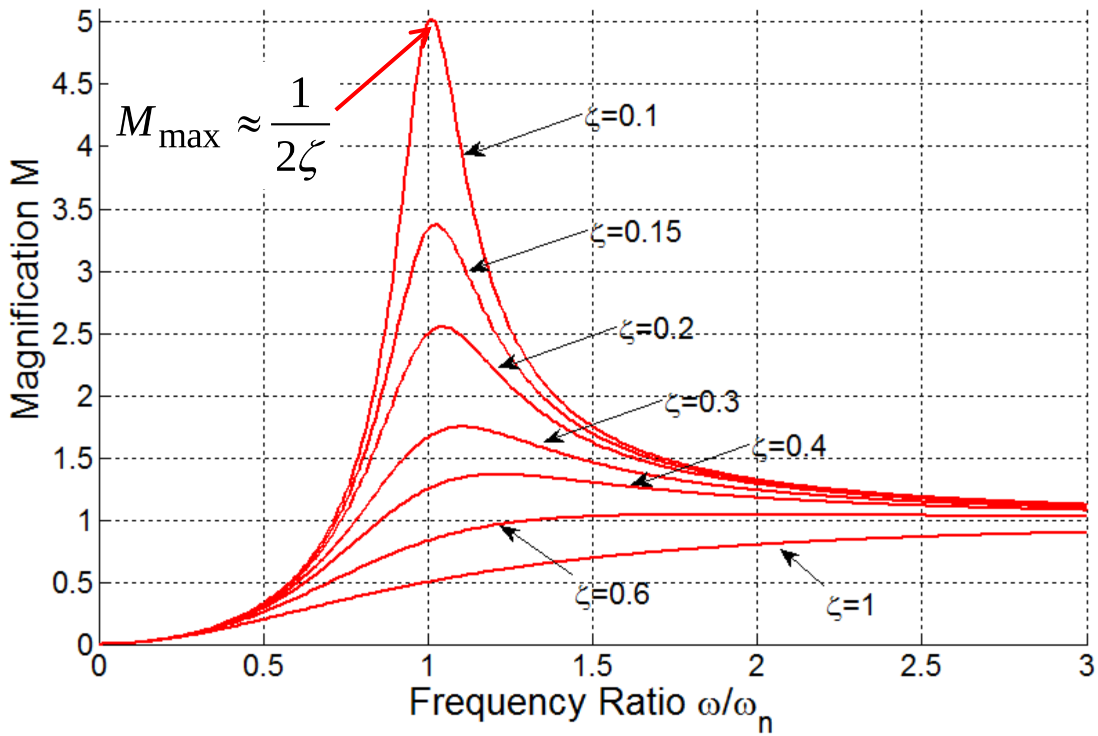
Canonical rotor excited system (steady state solution)



Steady state solution to $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C - \frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

$$\omega_n = \sqrt{\frac{k}{m+m_0}} \quad \zeta = \frac{c}{2\sqrt{k(m+m_0)}} \quad K = \frac{m_0}{m+m_0} \qquad x_p(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = KY_0 M(\omega, \omega_n, \zeta) \qquad M = \frac{\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$



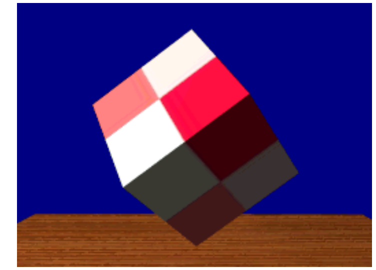
Dynamics of Rigid Bodies– concept checklist

1. Understand angular velocity and acceleration vectors; be able to integrate / differentiate angular velocities / accelerations for planar motion.
2. Understand formulas relating velocity/acceleration of two points on a rigid body
3. Understand constraints at joints and contacts between rigid bodies
4. Be able to relate velocities, accelerations, or angular velocities/accelerations of two members in a system of links or rigid bodies
5. Be able to analyze motion in systems of gears
6. Understand formulas relating velocity/angular velocity and acceleration/angular acceleration of a rolling wheel

7. Be able to calculate mass moments of inertia of simple shapes; use parallel axis theorem to shift axis of inertia or calculate mass moments of inertia for a set of rigid bodies connected together
8. Understand $\sum \mathbf{M}_G = I_G \boldsymbol{\alpha}$ for planar motion of a rigid body
9. Understand and know when you can use $\sum \mathbf{M}_0 = I_0 \boldsymbol{\alpha}$
10. Be able to calculate accelerations / forces in a system of planar rigid bodies subjected to forces using dynamics equations and kinematics equations

11. Understand power/work/potential energy of a rigid body; use energy methods to analyze motion in a system of rigid bodies
12. Understand angular momentum of a rigid body; use angular momentum to analyze motion of rigid bodies

Describing rotational motion of a rigid body

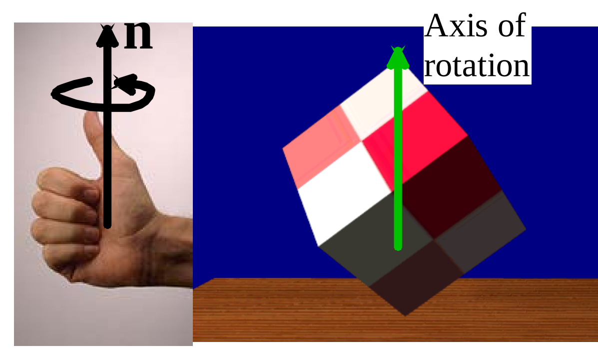


Angular velocity vector:

1. Direction – parallel to rotation axis (RH screw rule)
2. Magnitude – angle (radians) turned per sec

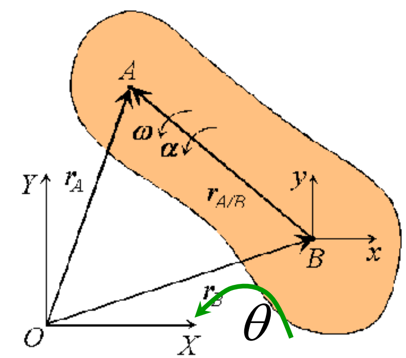
$$\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{n} = \omega\mathbf{n}$$

Angular acceleration vector: $\boldsymbol{\alpha} = \dfrac{d\omega}{dt}\mathbf{n}$



For planar motion: $\omega = \dfrac{d\theta}{dt}$ $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

$$\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{k} \qquad \boldsymbol{\alpha} = \frac{d^2\theta}{dt^2}\mathbf{k}$$



Pure Moments (torques): $\mathbf{M} = M\mathbf{n}$

A pure moment is a generalized force that induces rotational motion without translation of center of mass

A motor shaft is an example of an object that exerts a moment – the shaft is parallel to the direction of the moment **n**

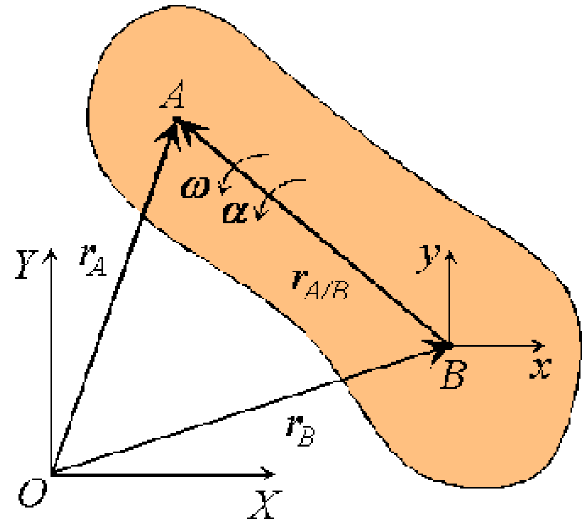
Rigid body kinematics

Velocities of two points on a rigid body are related by

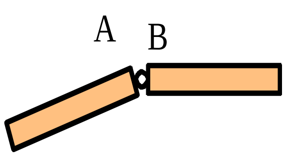
$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

Accelerations of two points on a rigid body are related by

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})$$

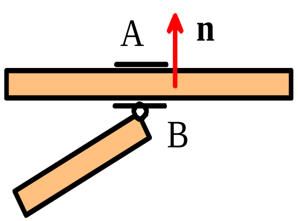


Continuity conditions



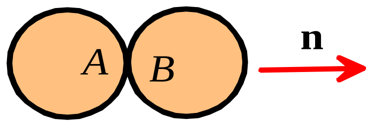
$$\mathbf{v}_A = \mathbf{v}_B$$

$$\mathbf{a}_A = \mathbf{a}_B$$



$$\mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n}$$

$$\mathbf{a}_A \cdot \mathbf{n} = \mathbf{a}_B \cdot \mathbf{n}$$



No slip $\mathbf{v}_A = \mathbf{v}_B$

Tangential accels equal

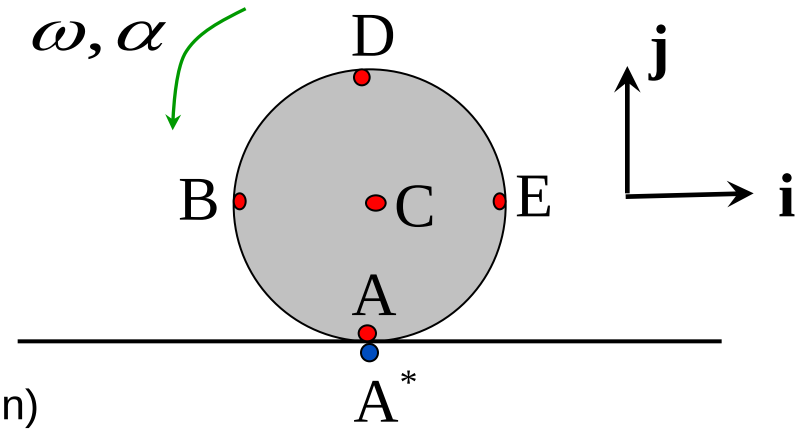
Slip $\mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n}$

Accels arbitrary

Kinematics of a Rolling Wheel

Wheel has angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$

Wheel has angular acceleration $\boldsymbol{\alpha} = \alpha \mathbf{k}$



Wheel rolls without slip
This means that velocity of A is zero
(wheel has same velocity as the ground, see animation)

Point A also has zero acceleration in the **i** direction
(tangential accelerations are equal at the contact
A has a nonzero upwards acceleration, however)

The rigid body formula tells us that

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{C/A}$$

$$= \mathbf{0} + \omega \mathbf{k} \times R\mathbf{j}$$

$$\mathbf{v}_C = -\omega R \mathbf{i}$$

Then differentiate wrt time to see $\mathbf{a}_C = -\alpha R \mathbf{i}$

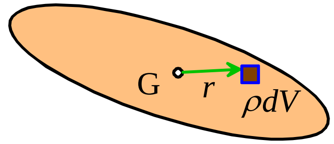
To find velocity or accel at A, B, D, E use the standard rigid body formulas….

Dynamics of rigid bodies

Preliminary definitions: mass moments of inertia used in planar motion (general 3D more complex)

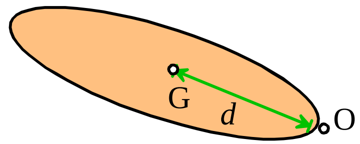
$$\textit{Mass density } \rho$$

$$\textit{Total Mass}: M = \int_V \rho \, dV \qquad \textit{COM position}: \mathbf{r}_G = \frac{1}{M}\int_V \rho \mathbf{r} \, dV$$



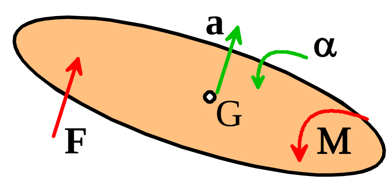
$$\textit{Inertia about an axis through origin}: I = \int_V \rho r^2 \, dV$$

$$\textit{Parallel Axis Theorem}: I_O = I_G + md^2$$



Equations of Motion

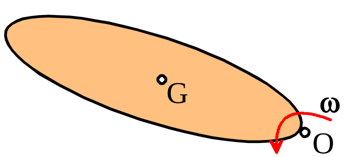
Translational motion $\mathbf{F} = m\mathbf{a}_G$ (must use acceleration of COM)



Rotational motion $$\sum \mathbf{M}_G = \sum_{Forces} \mathbf{r}_{F/G} \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} = I_G \boldsymbol{\alpha}$$

This rotational motion equation is valid **ONLY** for planar motion – 3D motion has another term

For rotation about a fixed axis only $$\sum \mathbf{M}_O = \sum_{Forces} \mathbf{r}_{F/O} \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} = I_O \boldsymbol{\alpha}$$



Free body diagrams with friction

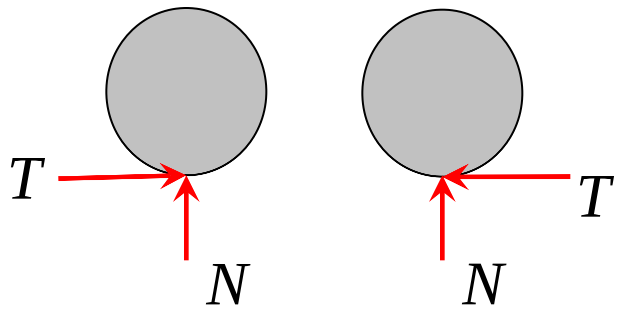
Rolling without slip

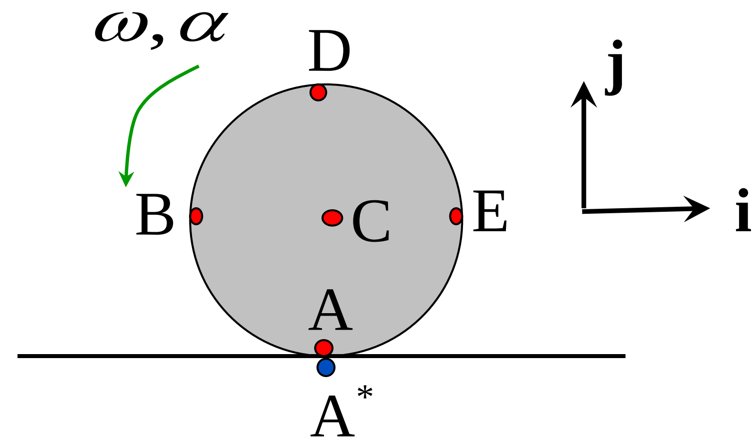
$$\mathbf{v}_C = -\omega R\mathbf{i}$$

$$\mathbf{a}_C = -\alpha R\mathbf{i}$$

$$|T| < \mu N$$

Both FBDs below are correct



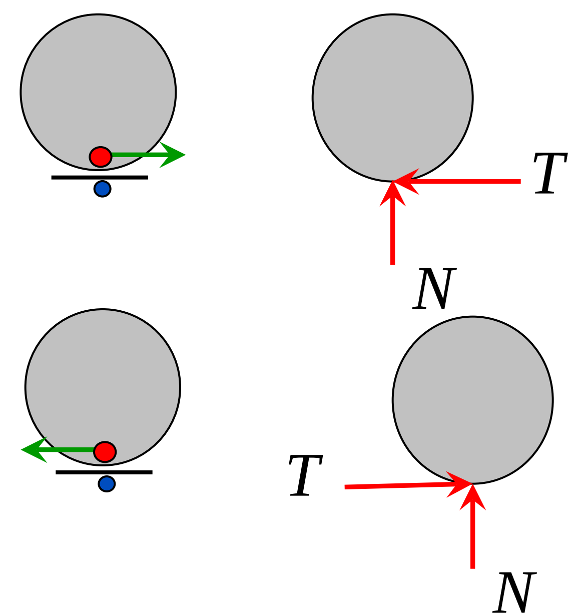


Rolling with sliding: Friction force must oppose sliding

$v_{Cx} + \omega R > 0 \Rightarrow$ A moves to right wrt A*

$v_{Cx} + \omega R < 0 \Rightarrow$ A moves to left wrt A*

$$T = \mu N$$



Analyzing motion of systems of rigid bodies

1. Identify each particle/rigid body in the system

2. Draw a FBD for each particle / rigid body separately

3. Write down $\mathbf{F} = m\mathbf{a}$ for each rigid body and particle

4. Write down $\sum \mathbf{M}_G = I_G \boldsymbol{\alpha}$ for each rigid body (for rotation about a fixed point can also use $\sum \mathbf{M}_O = I_O \boldsymbol{\alpha}$

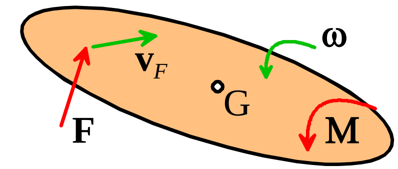
5. Look for points in system where acceleration is known or related (eg contacts, joints, etc)

6. Use $\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/A})$ to relate accelerations and angular accelerations of rigid bodies

7. Solve system of equations from 3, 4, 6 to calculate unknown reactions and accelerations / angular accelerations

Energy methods for rigid bodies

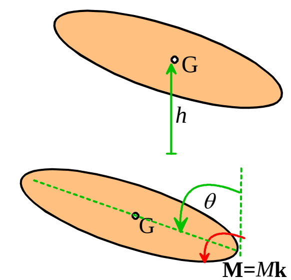
Power (rate of work done) by forces and moments acting on a rigid body



$$P = \sum_{Forces} \mathbf{F} \cdot \mathbf{v}_F + \sum_{Pure\ Moments} \mathbf{M} \cdot \boldsymbol{\omega}$$

Total work done $$W = \int_{t_0}^{t_1} P(t)dt = \int_{t_0}^{t_1} \left(\sum_{Forces} \mathbf{F} \cdot \mathbf{v}_F + \sum_{Pure\ Moments} \mathbf{M} \cdot \boldsymbol{\omega} \right) dt$$

Gravitational potential energy of a rigid body – use position of COM $V = mgh$

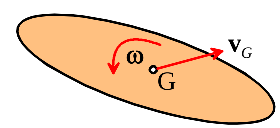


Potential energy of a constant moment (planar motion only) $V = -M\theta$

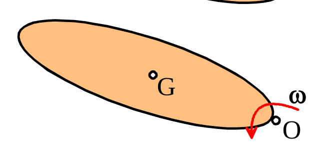
Potential energy of a torsional spring $V = \frac{1}{2}\kappa\theta^2$

Kinetic energy of a rigid body

$T = \frac{1}{2} m |\mathbf{v}_G|^2 + \frac{1}{2} I_G |\boldsymbol{\omega}|^2$ General – can always use this



$T = \frac{1}{2} I_O |\boldsymbol{\omega}|^2$ Rotation about a **fixed axis** only
(use parallel axis theorem to find Io)

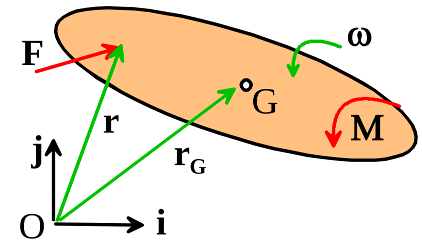


Power-KE relation $P = \frac{dT}{dt}$ **Work-KE relation** $W = T_1 - T_0$

Work- energy relation for a conservative system $W^{ext} = T_1 + V_1 - (T_0 + V_0)$

If no external work is done on a conservative system $T_1 + V_1 = (T_0 + V_0)$

Angular momentum for rigid bodies



Angular impulse about COM

(note that COM need not be fixed)

$$\mathbf{A}_G = \int_{t_0}^{t_1} \left(\sum_{Forces} (\mathbf{r} - \mathbf{r}_G) \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} \right) dt$$

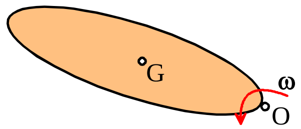
Angular impulse about a fixed point

$$\mathbf{A}_0 = \int_{t_0}^{t_1} \left(\sum_{Forces} \mathbf{r} \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} \right) dt$$

Angular momentum about COM $\mathbf{h}_G = I_G \boldsymbol{\omega}$

Angular momentum about a fixed point $\mathbf{h}_O = \mathbf{r}_G \times m\mathbf{v}_G + I_G \boldsymbol{\omega}$

Special case: rotation about a fixed point $\mathbf{h}_O = I_O \boldsymbol{\omega}$



Impulse-momentum relations (COM)

$$\sum_{Forces} (\mathbf{r} - \mathbf{r}_G) \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} = \frac{d\mathbf{h}_G}{dt} \qquad \mathbf{A}_G = \mathbf{h}_{G1} - \mathbf{h}_{G0}$$

Momentum is conserved if $\mathbf{A}_G = \mathbf{0}$

Impulse-momentum relations (Fixed point)

$$\sum_{Forces} \mathbf{r} \times \mathbf{F} + \sum_{Pure\ Moments} \mathbf{M} = \frac{d\mathbf{h}_O}{dt} \qquad \mathbf{A}_O = \mathbf{h}_{O1} - \mathbf{h}_{O0}$$

Momentum is conserved if $\mathbf{A}_O = \mathbf{0}$