Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton's laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy;
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies

Exam topics

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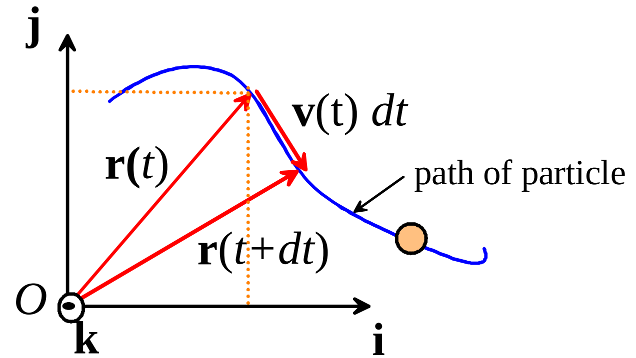
Particle Dynamics – concept checklist

- Understand the concept of an ‘inertial frame’
- Be able to idealize an engineering design as a set of particles, and know when this idealization will give accurate results
- Describe the motion of a system of particles (eg components in a fixed coordinate system; components in a polar coordinate system, etc)
- Be able to differentiate position vectors (with proper use of the chain rule!) to determine velocity and acceleration; and be able to integrate acceleration or velocity to determine position vector.
- Be able to describe motion in normal-tangential and polar coordinates (eg be able to write down vector components of velocity and acceleration in terms of speed, radius of curvature of path, or coordinates in the cylindrical-polar system).
- Be able to convert between Cartesian to normal-tangential or polar coordinate descriptions of motion
- Be able to draw a correct free body diagram showing forces acting on system idealized as particles
- Be able to write down Newton’s laws of motion in rectangular, normal-tangential, and polar coordinate systems
- Be able to obtain an additional moment balance equation for a rigid body moving without rotation or rotating about a fixed axis at constant rate.
- Be able to use Newton’s laws of motion to solve for unknown accelerations or forces in a system of particles
- Use Newton’s laws of motion to derive differential equations governing the motion of a system of particles
- Be able to re-write second order differential equations as a pair of first-order differential equations in a form that MATLAB can solve

Particle Kinematics

Inertial frame – non accelerating, non rotating reference frame
Particle – point mass at some position in space



Position Vector $$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Velocity Vector $$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$

$$= \frac{d}{dt}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\Rightarrow v_x(t) = \frac{dx}{dt} \qquad v_y(t) = \frac{dy}{dt} \qquad v_z(t) = \frac{dz}{dt}$$

- Direction of velocity vector is parallel to path
- Magnitude of velocity vector is distance traveled / time

Acceleration Vector

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} = \frac{d}{dt}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}$$

$$\Rightarrow a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Also $$a_x(t) = \frac{dv_x}{dx}v_x \qquad a_y(t) = \frac{dv_y}{dy}v_y \qquad a_z(t) = \frac{dv_z}{dz}v_z$$

Particle Kinematics

- Straight line motion with constant acceleration

$$\mathbf{r} = \left[X_0 + V_0 t + \frac{1}{2} a t^2 \right] \mathbf{i} \qquad \mathbf{v} = (V_0 + at)\mathbf{i} \qquad \mathbf{a} = a\mathbf{i}$$

- Time/velocity/position dependent acceleration – use calculus

$$\mathbf{r} = \left(X_0 + \int_0^t v(t)dt \right) \mathbf{i} \qquad \mathbf{v} = \left(V_0 + \int_0^t a(t)dt \right) \mathbf{i}$$

$$a = \frac{dv}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{V_0}^{v} f(v)dv = \int_0^t g(t)dt$$

$$v = \frac{dx}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{X_0}^{x(t)} f(x)dv = \int_0^t v(t)dt$$

$$\frac{dv}{dt} = a(x)$$

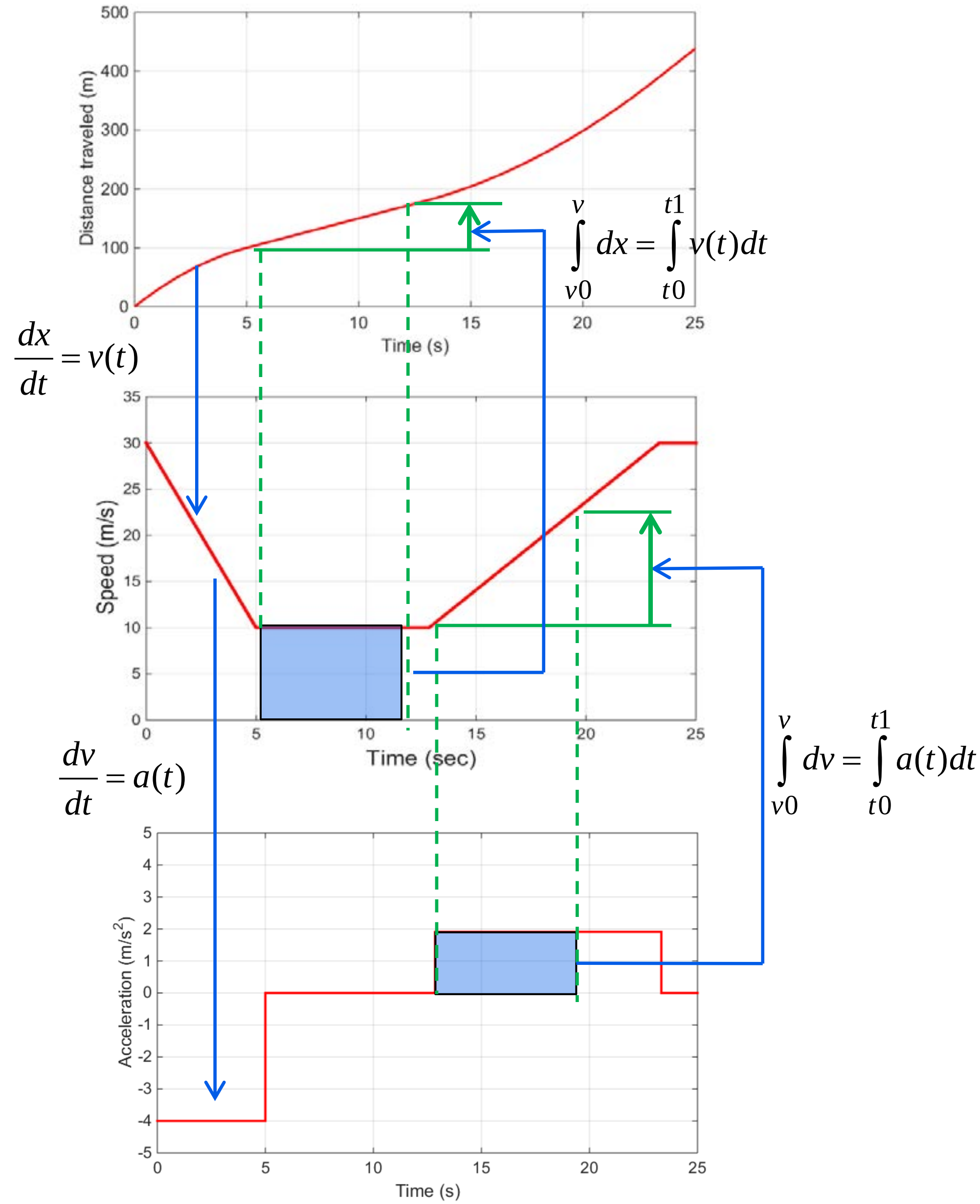
$$\Rightarrow \frac{dv}{dx}\frac{dx}{dt} = a(x) \Rightarrow v\frac{dv}{dx} = a(x)$$

$$\int_{V_0}^{v(t)} v dv = \int_0^{x(t)} a(x)dx$$

Graphical x-v-a relations

- Speed is the slope of the distance-v-time curve
- Distance is the area under the speed-v-time curve

- Acceleration is the slope of the speed-v-time curve
- Speed is the area under the acceleration-v-time curve



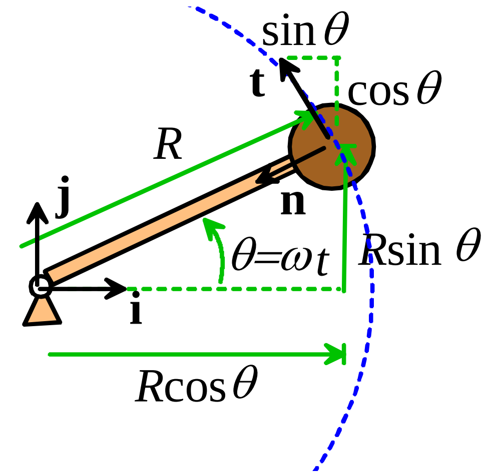
- Circular Motion at const speed

$$\theta = \omega t \quad s = R\theta \quad V = \omega R$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = -\omega^2 R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \omega^2 R\mathbf{n} = \frac{V^2}{R}\mathbf{n}$$



- General circular motion

$$\omega = d\theta / dt \quad \alpha = d\omega / dt = d^2\theta / dt^2$$

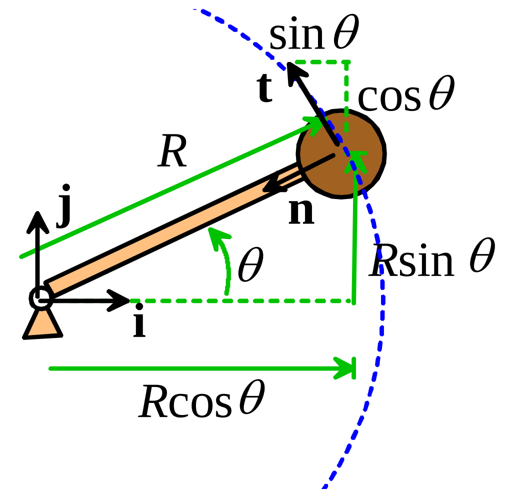
$$s = R\theta \quad V = ds / dt = R\omega$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

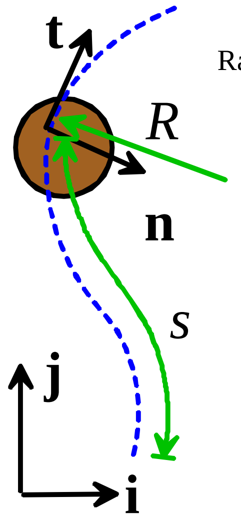
$$= \alpha R\mathbf{t} + \omega^2 R\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$



- Motion along an arbitrary path

$$\mathbf{v} = V\mathbf{t}$$

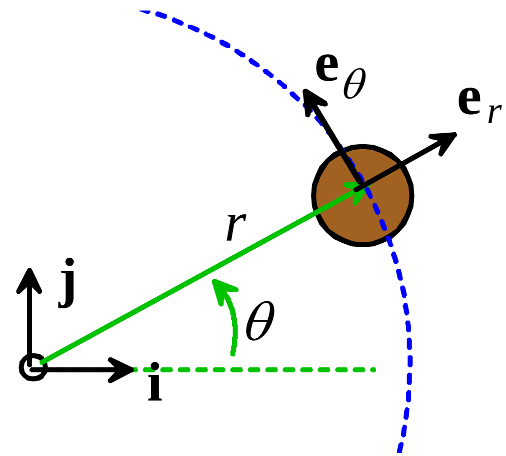
$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$



Radius of curvature $R = \dfrac{1}{\sqrt{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2}}$

$$\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$$

- Polar Coordinates



$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta$$

Calculating forces required to cause prescribed motion

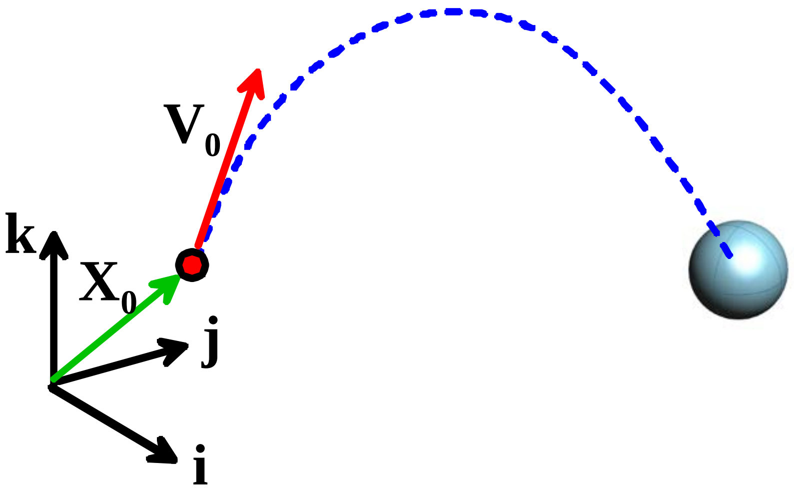
- Idealize system
- Free body diagram
- Kinematics (describe motion – usually goal is to find formula for acceleration)
- **F**=m**a** for each particle.
- $\mathbf{M}_G = \mathbf{0}$ (for steadily or non-rotating rigid bodies or frames only – this is a special case of the moment-angular momentum formula for rigid bodies)
- Solve for unknown forces or accelerations (just like statics)

Using Newton's laws to derive equations of motion

1. Idealize system
2. Introduce variables to describe motion (often *x,y* coords, but we will see other examples)
3. Write down **r**, differentiate to get **a**
4. Draw FBD
5. $\mathbf{F} = m\mathbf{a}$
6. If necessary, eliminate reaction forces
7. Result will be differential equations for coords defined in (2), e.g. $m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = kY_0 \sin \omega t$
8. Identify initial conditions, and solve ODE

Motion of a projectile in earths gravity

$$\left.\begin{aligned}\mathbf{r} &= X_0\mathbf{i} + Y_0\mathbf{j} + Z_0\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= V_{x0}\mathbf{i} + V_{y0}\mathbf{j} + V_{z0}\mathbf{k}\end{aligned}\right\} t = 0$$



$$\mathbf{r} = \left(X_0 + V_{x0}t\right)\mathbf{i} + \left(Y_0 + V_{y0}t\right)\mathbf{j} + \left(Z_0 + V_{z0}t - \frac{1}{2}gt^2\right)\mathbf{k}$$

$$\mathbf{v} = \left(V_{x0}\right)\mathbf{i} + \left(V_{y0}\right)\mathbf{j} + \left(V_{z0} - gt\right)\mathbf{k}$$

$$\mathbf{a} = -g\mathbf{k}$$

Rearranging differential equations for MATLAB

- Example $$\frac{d^2 y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$$
- Introduce $v = dy / dt$
- Then $$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} v \\ -2\zeta\omega_n v - \omega_n^2 y \end{bmatrix}$$
- This has form $$\frac{d\mathbf{w}}{dt} = f(t, \mathbf{w}) \qquad \mathbf{w} = \begin{bmatrix} y \\ v \end{bmatrix}$$

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Conservation Laws – concept checklist

- Know the definitions of power (or rate of work) of a force, and work done by a force
- Know the definition of kinetic energy of a particle
- Understand power-work-kinetic energy relations for a particle
- Be able to use work/power/kinetic energy to solve problems involving particle motion
- Be able to distinguish between conservative and non-conservative forces
- Be able to calculate the potential energy of a conservative force
- Be able to calculate the force associated with a potential energy function
- Know the work-energy relation for a system of particles; (energy conservation for a closed system)
- Use energy conservation to analyze motion of conservative systems of particles

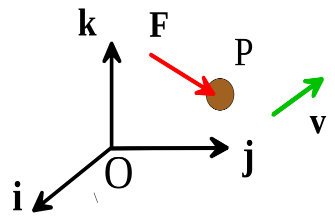
- Know the definition of the linear impulse of a force
- Know the definition of linear momentum of a particle
- Understand the impulse-momentum (and force-momentum) relations for a particle
- Understand impulse-momentum relations for a system of particles (momentum conservation for a closed system)
- Be able to use impulse-momentum to analyze motion of particles and systems of particles
- Know the definition of restitution coefficient for a collision
- Predict changes in velocity of two colliding particles in 2D and 3D using momentum and the restitution formula

- Know the definition of angular impulse of a force
- Know the definition of angular momentum of a particle
- Understand the angular impulse-momentum relation
- Be able to use angular momentum to solve central force problems/impact problems

Work-Energy relations for a single particle

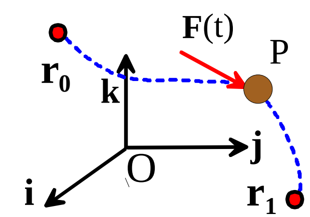
Rate of work done by a force (power developed by force)

$$P = \mathbf{F} \cdot \mathbf{v}$$



Total work done by a force

$$W = \int_0^{t_1} \mathbf{F} \cdot \mathbf{v} dt \qquad W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$



Kinetic energy

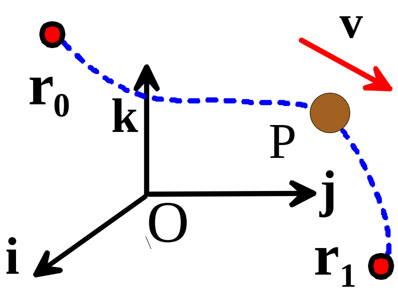
$$T = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

Power-kinetic energy relation

$$P = \frac{dT}{dt}$$

Work-kinetic energy relation

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = T - T_0$$

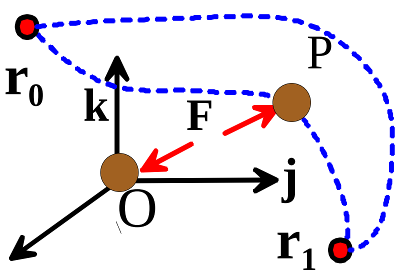


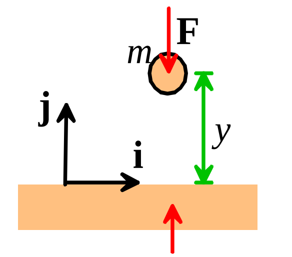
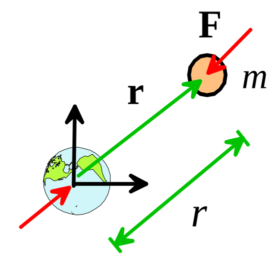
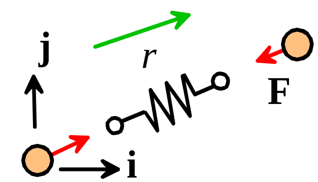
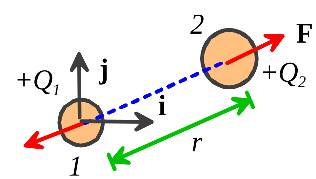
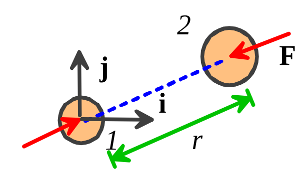
Potential Energy

Potential energy of a conservative force (pair)

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} + \text{constant}$$

$$\mathbf{F} = -\text{grad}(V)$$



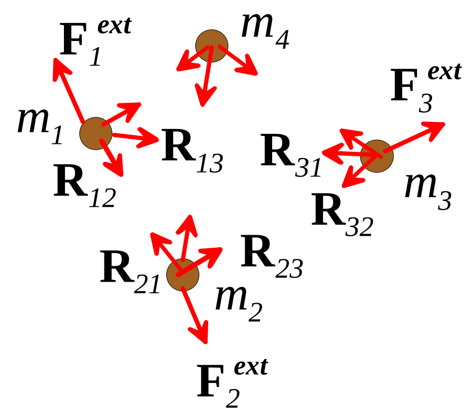
Type of force	Potential energy	
Gravity acting on a particle near earths surface	$V = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$V = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$V = \dfrac{1}{2}k\left(r - L_0\right)^2$	
Force acting between two charged particles	$V = \dfrac{Q_1 Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

Energy Relation for a Conservative System

Internal Forces: (forces exerted by one part of the system on another) $\mathbf{R}_{ij}$

External Forces: (any other forces) $\mathbf{F}_i^{ext}$

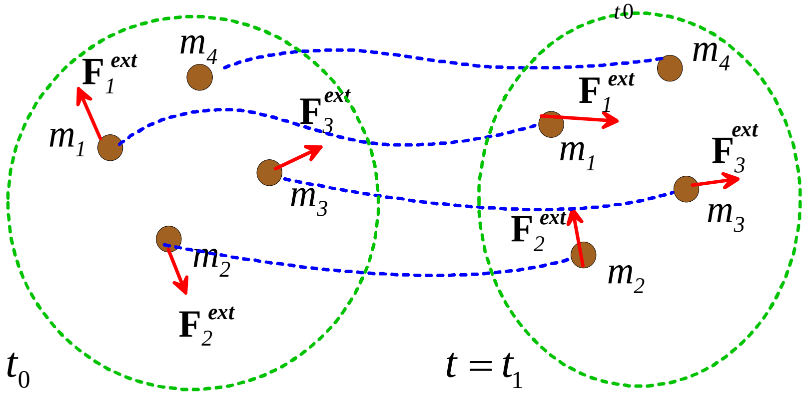
System is conservative if all internal forces are conservative forces (or constraint forces)



Energy relation for a conservative system

$$External\ \ Power\ \ P^{ext}(t)$$

$$External\ \ work\ \ \Delta W^{ext} = \int_{t0}^{t1} P(t)dt$$



$t = t_0$

$Total\ \ KE\ \ T_0^{TOT}$

$Total\ \ PE\ \ V_0^{TOT}$

$t = t_1$

$Total\ \ KE\ \ T_1^{TOT}$

$Total\ \ PE\ \ V_1^{TOT}$

$$\Delta W_{ext} = T_1^{TOT} + V_1^{TOT} - \left(T_0^{TOT} + V_0^{TOT}\right)$$

Special case – zero external work:

$$T_1^{TOT} + V_1^{TOT} = T_0^{TOT} + V_0^{TOT}$$

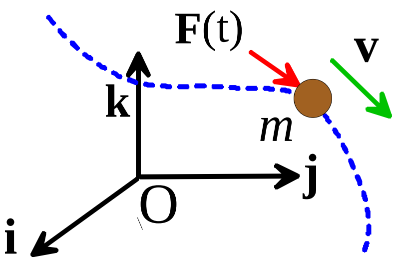
KE+PE = constant

Impulse-Momentum for a single particle

Definitions

Linear Impulse of a force $\mathbf{I} = \int_{t_0}^{t_1} \mathbf{F}(t)\,dt$

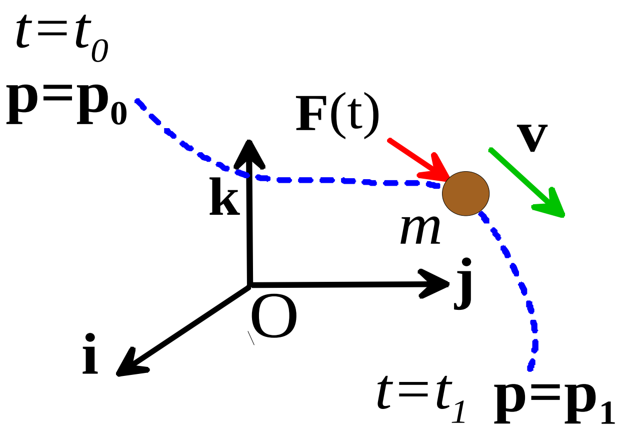
Linear momentum of a particle $\mathbf{p} = m\mathbf{v}$



Impulse-Momentum relations

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{I} = \mathbf{p}_1 - \mathbf{p}_0$$

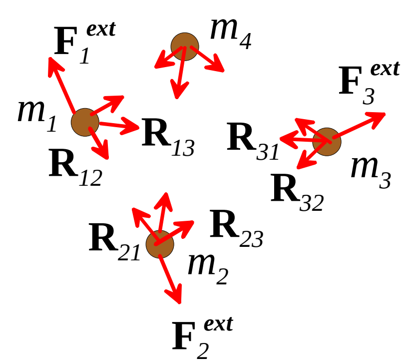


Impulse-Momentum for a system of particles

$\mathbf{R}_{ij}$ Force exerted on particle i by particle j

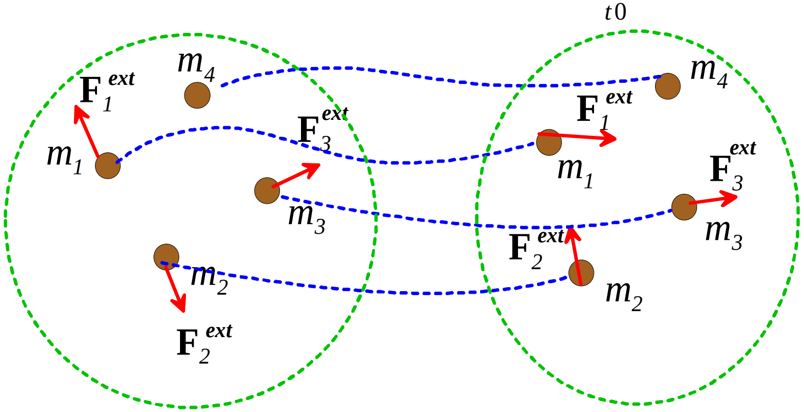
$\mathbf{F}_i^{ext}$ External force on particle i

$\mathbf{v}_i$ Velocity of particle i



$$Total \;\; External \;\; Force \;\; \mathbf{F}^{TOT}(t)$$

$$Total \; External \; Impulse \;\; \mathbf{I}^{TOT} = \int_{t0}^{t1} \mathbf{F}^{TOT}(t)dt$$



$t = t_0$

$Total \;\; momentum \;\; \mathbf{p}_0^{TOT}$

$t = t_1$

$Total \;\; momentum \;\; \mathbf{p}_1^{TOT}$

Impulse-momentum for the system:

$$\mathbf{F}^{TOT} = \frac{d\mathbf{p}^{TOT}}{dt}$$

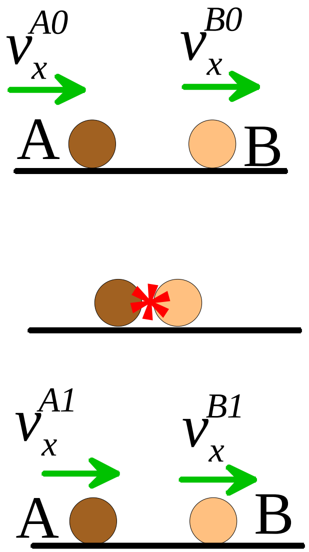
$$\mathbf{I}^{TOT} = \mathbf{p}_1^{TOT} - \mathbf{p}_0^{TOT}$$

Special case – zero external impulse:

$$\mathbf{p}_1^{TOT} = \mathbf{p}_0^{TOT}$$

(Linear momentum conserved)

Collisions

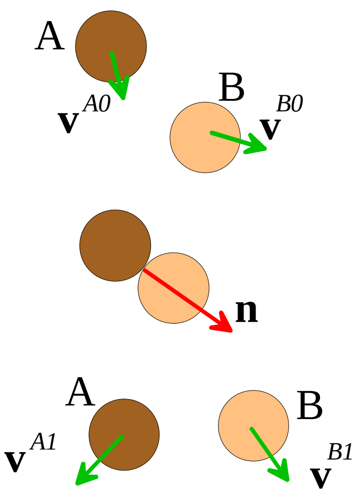


Momentum $$m_A v_x^{A1} + m_B v_x^{B1} = m_A v_x^{A0} + m_B v_x^{B0}$$

Restitution formula $$v^{B1} - v^{A1} = -e\left(v^{B0} - v^{A0}\right)$$

$$v^{B1} = v^{B0} - \frac{m_A}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$

$$v^{A1} = v^{A0} + \frac{m_B}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$



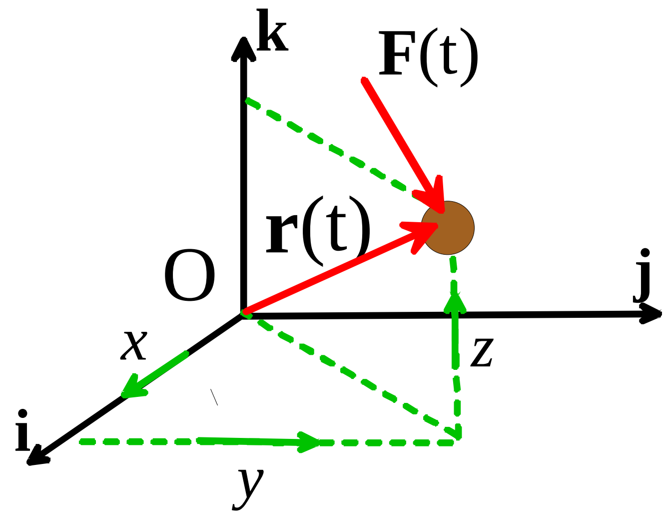
Momentum $$m_B\ \mathbf{v}^{B1} + m_A \mathbf{v}^{A1} = m_B\ \mathbf{v}^{B0} + m_A \mathbf{v}^{A0}$$

Restitution formula $$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Angular Impulse-Momentum Equations for a particle



Angular Impulse $\mathbf{A} = \int_{t_0}^{t_1} \mathbf{r}(t) \times \mathbf{F}(t) dt$

Angular Momentum $\mathbf{h} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$

Impulse-Momentum relations $\mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt}$ $\mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$

Special Case $\mathbf{A} = \mathbf{0} \Rightarrow \mathbf{h}_1 = \mathbf{h}_0$ Angular momentum conserved

Useful for central force problems (when forces on a particle always act through a single point, eg planetary gravity)

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   - Description of motion, coordinate systems; Newton's laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy;
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies

Free vibrations – concept checklist

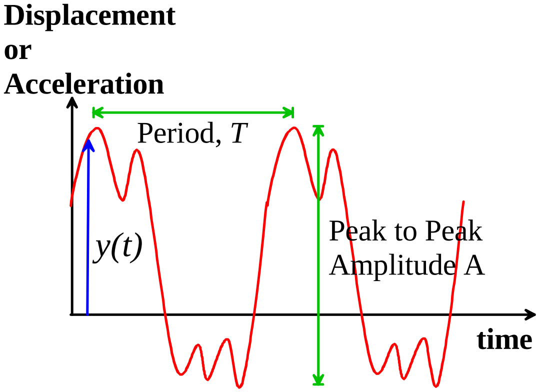
You should be able to:

1. Understand simple harmonic motion (amplitude, period, frequency, phase)
2. Understand the motion of a vibrating spring-mass system (and how the motion is predicted)
3. Calculate natural frequency of a 1 degree of freedom linear system (Derive EOM and use the solutions given on the handout)
4. Calculate the amplitude and phase of an undamped 1 DOF linear system from the initial conditions
5. Understand the concept of natural frequencies and mode shapes for vibration of a general undamped linear system
6. Combine series and parallel springs to simplify a system
7. Use energy to derive an equation of motion for a 1 DOF conservative system
8. Analyze small amplitude vibration of a nonlinear system (eg pendulum) by linearizing EOM with Taylor series
9. Understand natural frequency, damped natural frequency, and ‘Damping factor’ for a dissipative 1DOF vibrating system
10. Know formulas for nat freq, damped nat freq and ‘damping factor’ for spring-mass system in terms of k,m,c
11. Understand underdamped, critically damped, and overdamped motion of a dissipative 1DOF vibrating system
12. Be able to determine damping factor from a measured free vibration response (will be covered next lecture)
13. Be able to predict motion of a freely vibrating 1DOF system given its initial velocity and position, and apply this to design-type problems

Free vibrations

Typical vibration response

- Period, frequency, angular frequency amplitude



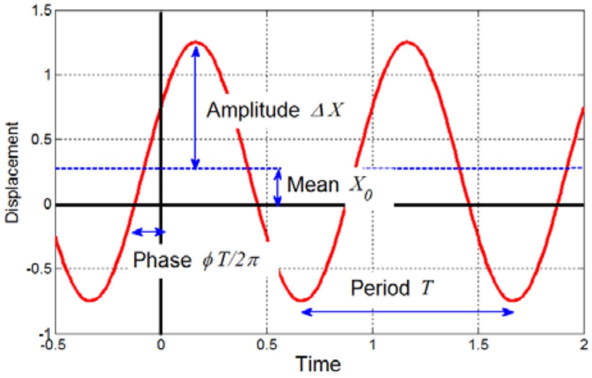
Simple Harmonic Motion

$$x(t) = X_0 + \Delta X \sin(\omega t + \phi)$$

$$v(t) = \Delta V \cos(\omega t + \phi)$$

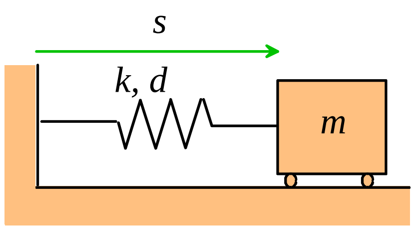
$$a(t) = -\Delta A \sin(\omega t + \phi)$$

$$\Delta V = \omega \Delta X \qquad \Delta A = \omega \Delta V$$



Free Vibration of Undamped 1DOF systems

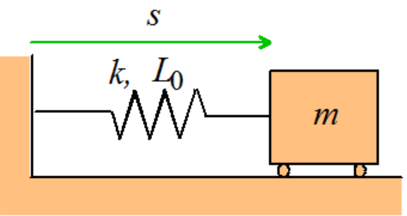
- Free -> No time dependent external forces
- Undamped -> No energy loss
- 1 DOF -> one variable describes system



Free vibrations

Harmonic Oscillator

Canonical Vibration Problem: The spring mass system is released with velocity v_0 from position s_0 at time *t=0* . Find *s(t)* .



Derive EOM (F=ma) $$\frac{m}{k}\frac{d^2 s}{dt^2} + s = L_0$$

Compare with 'standard' differential equation

Equation $\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$ Initial Conditions $x = x_0$ $\frac{dx}{dt} = v_0$ $t = 0$

$$x = s \qquad C = L_0 \qquad x_0 = s_0$$

$$\frac{1}{\omega_n^2} = \frac{m}{k}$$

Solution $$x = C + X_0 \sin(\omega_n t + \phi)$$

$$X_0 = \sqrt{(x_0 - C)^2 + v_0^2 / \omega_n^2} \qquad \phi = \tan^{-1}\left(\frac{(x_0 - C)\omega_n}{v_0}\right)$$

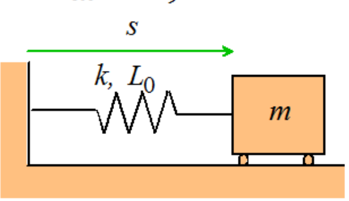
Or $$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

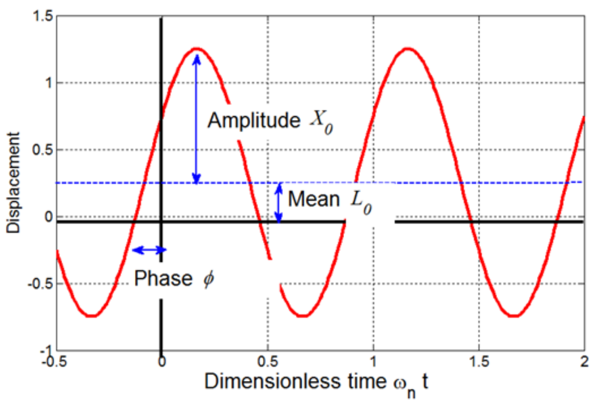
Solution

$$s(t) = L_0 + \sqrt{(s_0 - L_0)^2 + v_0^2 / \omega_n^2}\,\sin(\omega_n t + \phi)$$

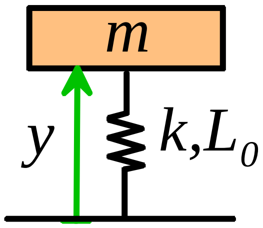
$$\left.\begin{array}{l} s = s_0 \\ \frac{ds}{dt} = v_0 \end{array}\right\} t = 0$$

Natural Frequency $$\omega_n = \sqrt{\frac{k}{m}}$$





Calculating natural frequencies for 1DOF systems



- Use F=ma (or energy) to find the equation of motion
- For an undamped system the equation will look like

$$A\frac{d^2y}{dt^2} + By = D$$

- Handout online gives solution to

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + x = C$$

- Rearrange your equation to look like this

$$\frac{A}{B}\frac{d^2y}{dt^2} + y = \frac{D}{B}$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + x = C$$

$$\frac{1}{\omega_n^2} = \frac{A}{B} \Rightarrow \omega_n = \sqrt{\frac{B}{A}}$$

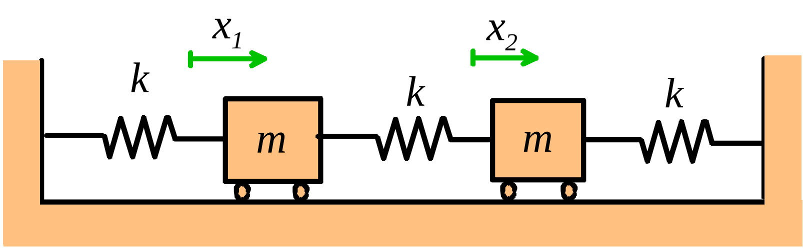
$$C = D/B$$

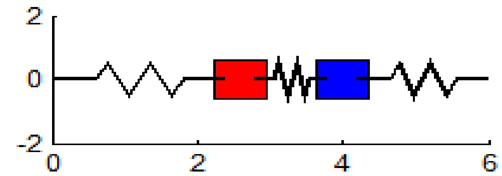
Natural Frequencies and Mode Shapes

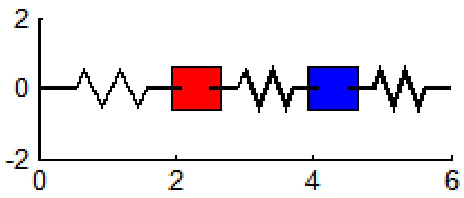
General system does not always vibrate harmonically

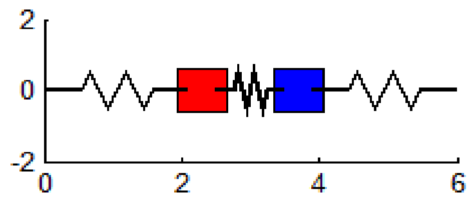
All unforced undamped systems vibrate harmonically at special frequencies, called **Natural Frequencies** of the system

The system will vibrate harmonically if it is released from rest with a special set of initial displacements, called **Mode Shapes** or **Vibration Modes**.







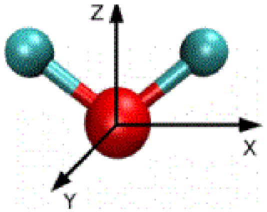


Counting degrees of freedom and vibration modes

DOF = no. coordinates required to describe motion

2D system # DOF = 2*p + 3*r-c

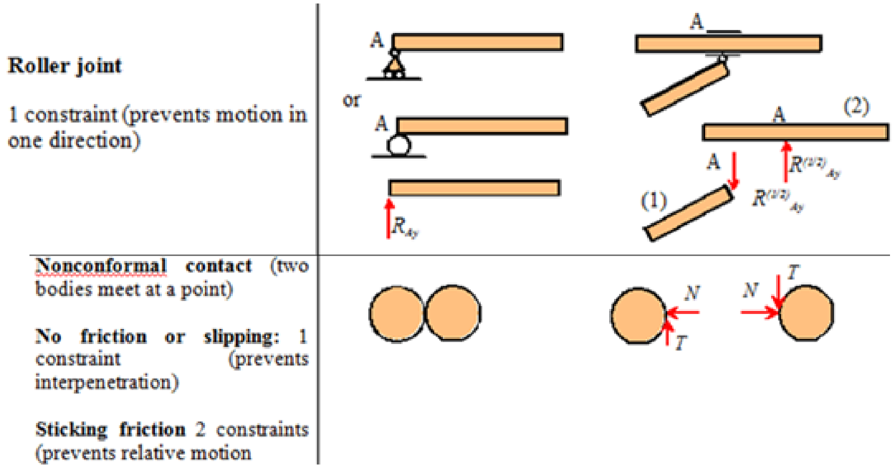
3D system # DOF = 3*p+6*r-c

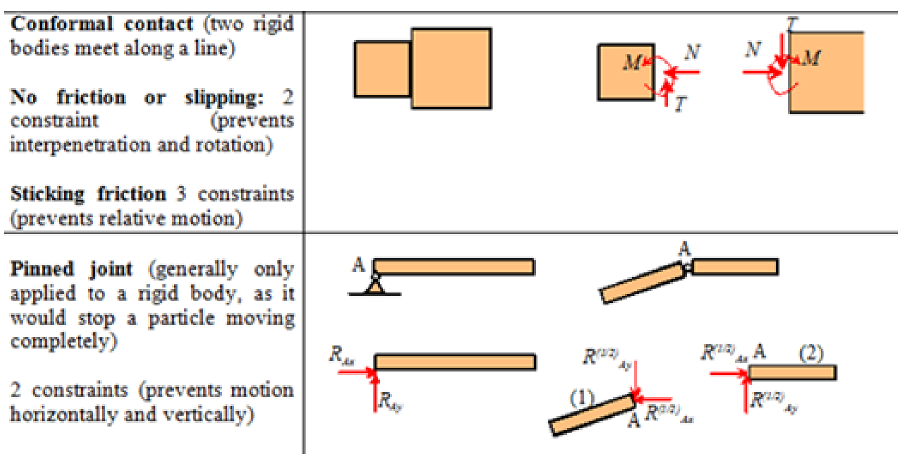


(d) Water molecule

Vibration modes = # DOF - # translation/rotation rigid body modes

Examples of 2D constraints



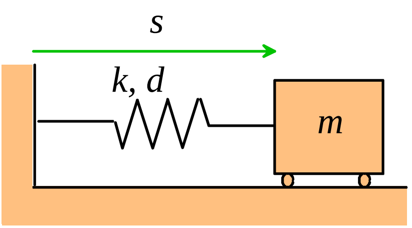


Using energy conservation to find EOM

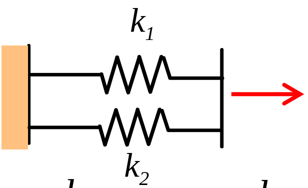
$$KE + PE = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 + \frac{1}{2} k (s - L_0)^2 = const$$

$$\Rightarrow \frac{d}{dt}(KE + PE) = m \left(\frac{ds}{dt} \right) \frac{d^2 s}{dt^2} + k(s - L_0) \frac{ds}{dt} = 0$$

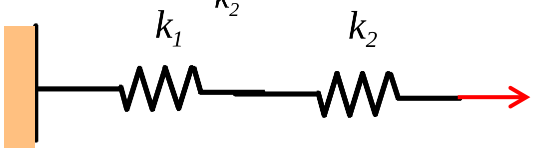
$$\Rightarrow m \frac{d^2 s}{dt^2} + ks = kL_0$$



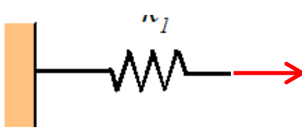
Combining springs



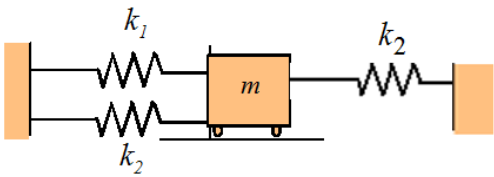
Parallel (forces add) $k_{eff} = k_1 + k_2$



Series (lengths add) $\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$



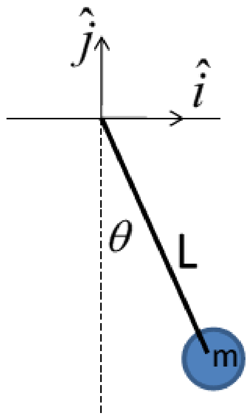
These are all in parallel



$k_{eff} = k_1 + 2k_2$

Calculating the natural frequency of a nonlinear system

Nonlinear systems



Sometimes EOM has form $$m\frac{d^2 y}{dt^2} + f(y) = 0$$

We cant solve this in general…
Instead, assume y is small, and note $f(0)=0$
(because acceleration must be zero for *y=0* for vibrations to be possible

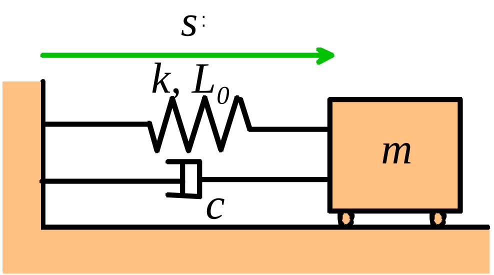
Simplify using Taylor expansion of *f*:

$$m\frac{d^2 y}{dt^2} + f(0) + \left.\frac{df}{dy}\right|_{y=0} y + \ldots = 0$$

$$m\frac{d^2 y}{dt^2} + \left.\frac{df}{dy}\right|_{y=0} y = 0$$

There are short-cuts to doing the Taylor expansion

Canonical damped vibration problem



EOM $$m\frac{d^2 s}{dt^2}+c\frac{ds}{dt}+ks=kL_0$$ with $s=s_0$ $\frac{ds}{dt}=v_0$ $t=0$

Standard Form $$\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$$ $x=x_0$ $\frac{dx}{dt}=v_0$ $t=0$

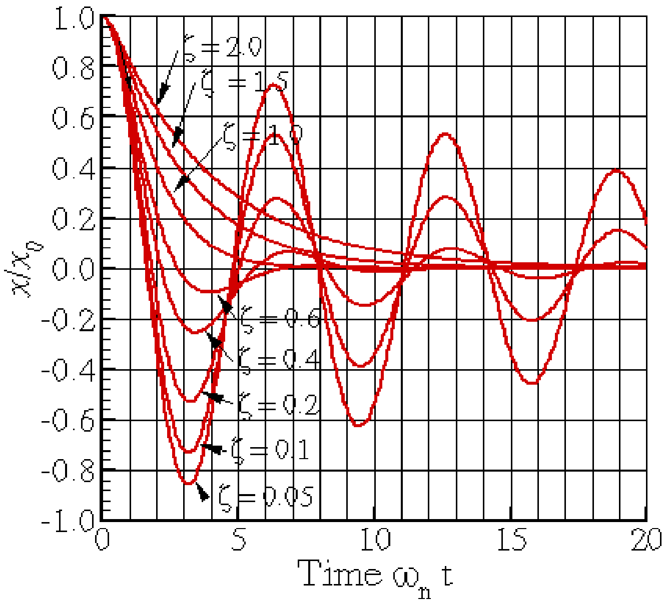
$$\omega_n=\sqrt{\frac{k}{m}} \qquad \varsigma=\frac{c}{2\sqrt{km}} \qquad C=L_0 \qquad x_0\equiv s_0$$

$$\omega_d=\omega_n\sqrt{\left|1-\varsigma^2\right|}$$

Overdamped $\varsigma>1$

Critically Damped $\varsigma=1$

Underdamped $\varsigma<1$



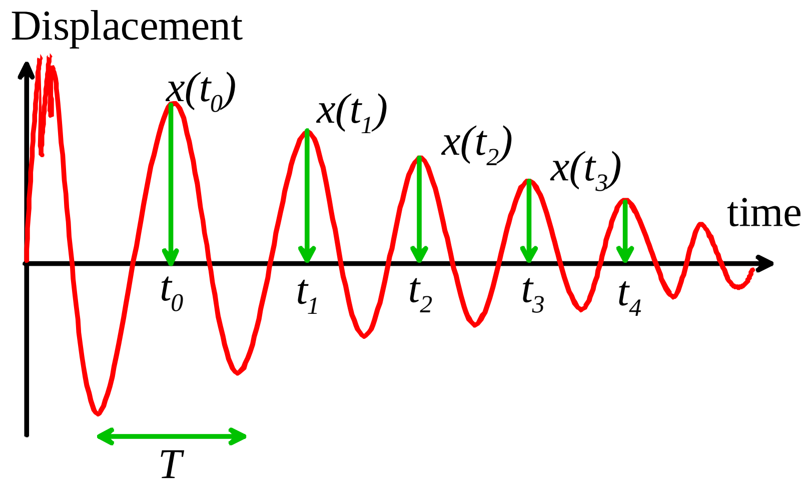
Overdamped $\varsigma>1$ $$x(t)=C+\exp(-\varsigma\omega_n t)\left\{\frac{v_0+(\varsigma\omega_n+\omega_d)(x_0-C)}{2\omega_d}\exp(\omega_d t)-\frac{v_0+(\varsigma\omega_n-\omega_d)(x_0-C)}{2\omega_d}\exp(-\omega_d t)\right\}$$

Critically Damped $\varsigma=1$ $$x(t)=C+\left\{(x_0-C)+\left[v_0+\omega_n(x_0-C)\right]t\right\}\exp(-\omega_n t)$$

Underdamped $\varsigma<1$ $$x(t)=C+\exp(-\varsigma\omega_n t)\left\{(x_0-C)\cos\omega_d t+\frac{v_0+\varsigma\omega_n(x_0-C)}{\omega_d}\sin\omega_d t\right\}$$

Application of damped vibrations

Calculating natural frequency and damping factor from a vibration measurement



Measure log decrement: $\delta = \frac{1}{n} \log \left(\frac{x(t_0)}{x(t_n)} \right)$

Measure period: T

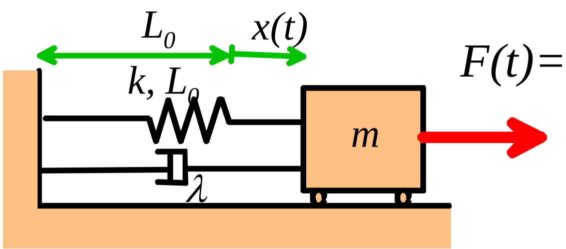
Then $\varsigma = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$ $\quad \omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T}$

Forced Vibrations – concept checklist

You should be able to:

1. Be able to derive equations of motion for spring-mass systems subjected to external forcing (several types) and solve EOM by comparing to solution tables
2. Understand (qualitatively) meaning of ‘transient’ and ‘steady-state’ response of a forced vibration system
3. Understand the meaning of ‘Amplitude’ and ‘phase’ of steady-state response of a forced vibration system
4. Understand amplitude-v-frequency formulas (or graphs), resonance, high and low frequency response for 3 systems
5. Determine the amplitude of steady-state vibration of forced spring-mass systems.
6. Use forced vibration concepts to design engineering systems

EOM for forced spring-mass systems

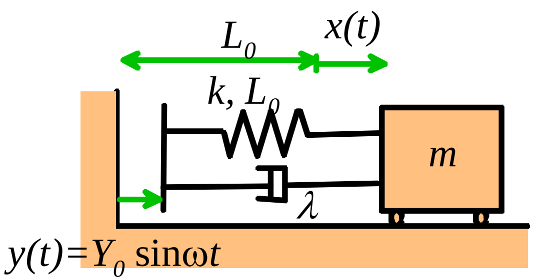


$F(t)=F_0 \sin \omega t$

External forcing

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=KF_0\sin\omega t$$

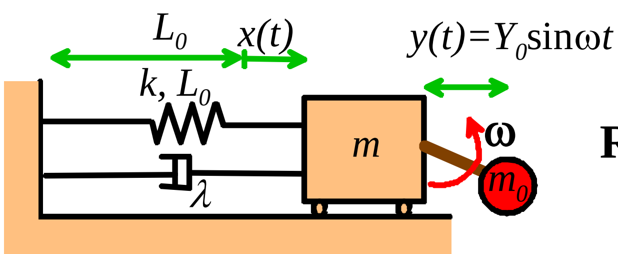
$$\omega_n=\sqrt{\frac{k}{m}},\qquad \varsigma=\frac{\lambda}{2\sqrt{km}},\qquad K=\frac{1}{k}$$



Base Excitation

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$$

$$\omega_n=\sqrt{\frac{k}{m}},\qquad \varsigma=\frac{\lambda}{2\sqrt{km}},\qquad K=1$$

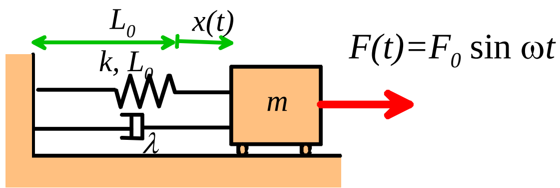


Rotor Excitation

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}=K\frac{Y_0\omega^2}{\omega_n^2}\sin\omega t$$

$$\omega_n=\sqrt{\frac{k}{m+m_0}}\qquad \varsigma=\frac{\lambda}{2\sqrt{k\left(m+m_0\right)}}\qquad K=\frac{m_0}{m+m_0}$$

Steady-state solution for external forcing



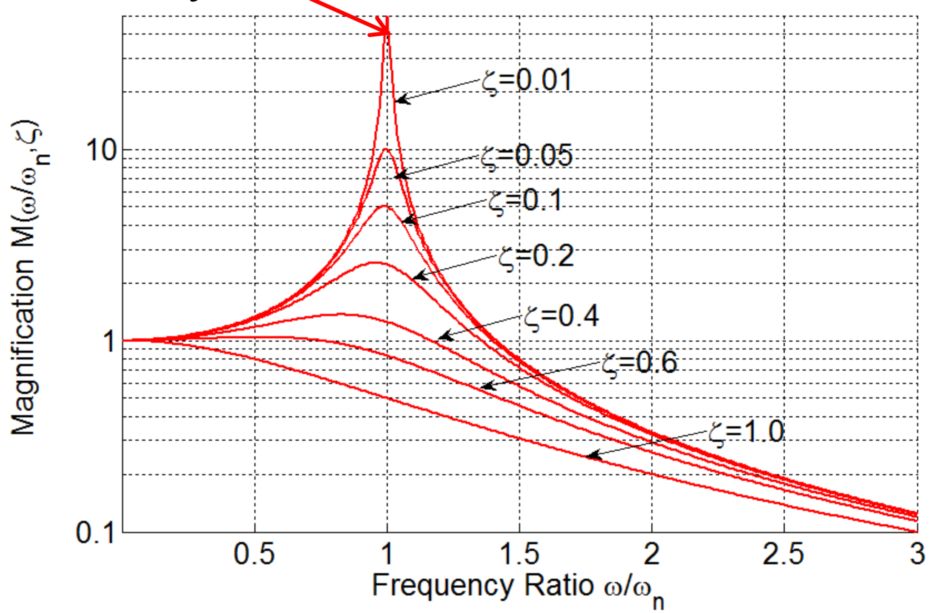
$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=KF(t)$$

$$\omega_n=\sqrt{\frac{k}{m}},\qquad \varsigma=\frac{\lambda}{2\sqrt{km}},\qquad K=\frac{1}{k}$$

$$x(t)=X_0\sin(\omega t+\phi)$$

$$X_0=KM(\omega,\omega_n,\zeta)F_0 \qquad M=\frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

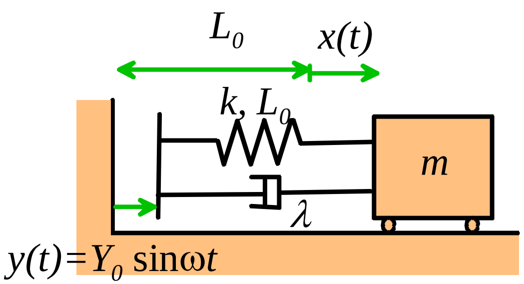
$$M_{\max}\approx\frac{1}{2\zeta}$$



System vibrates at same frequency as force

Amplitude depends on forcing frequency, nat frequency, and damping coeft.

Steady-state solution for base excitation

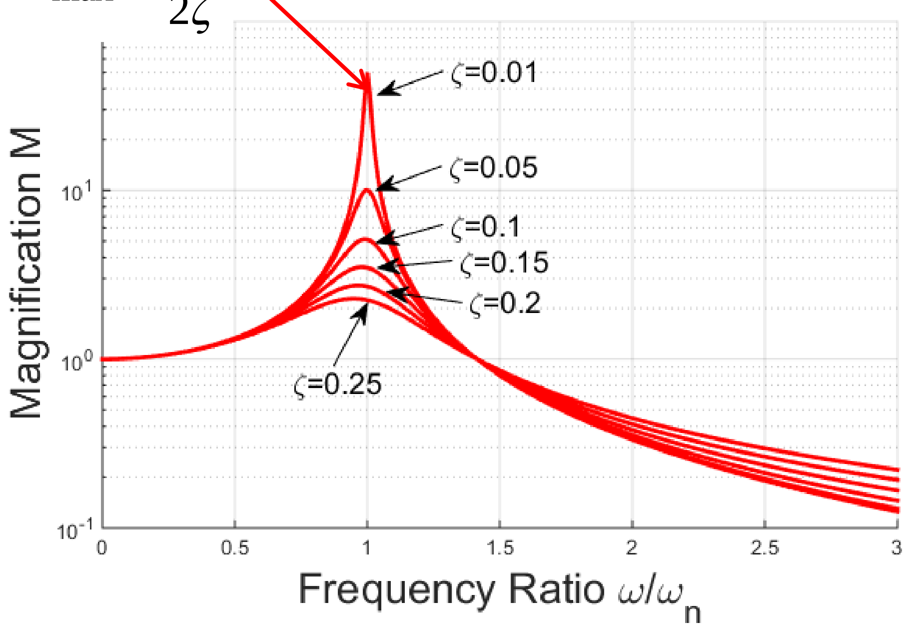


$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right) \qquad \omega_n=\sqrt{\frac{k}{m}}, \quad \varsigma=\frac{\lambda}{2\sqrt{km}}, \quad K=1$$

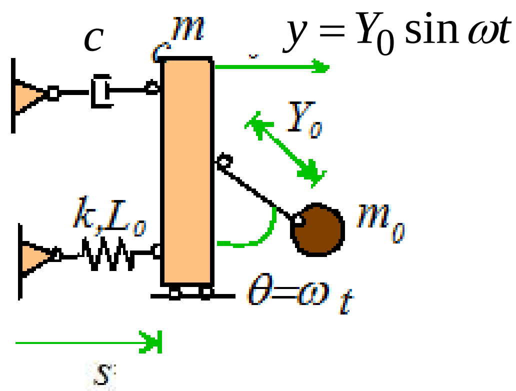
$$x(t)=X_0\sin(\omega t+\phi)$$

$$X_0=KM(\omega,\omega_n,\zeta)Y_0 \qquad M=\frac{\left\{1+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega^3/\omega_n^3}{1-(1-4\varsigma^2)\omega^2/\omega_n^2}$$

$$M_{\max}\approx\frac{1}{2\zeta}$$



Steady-state solution for rotor excitation

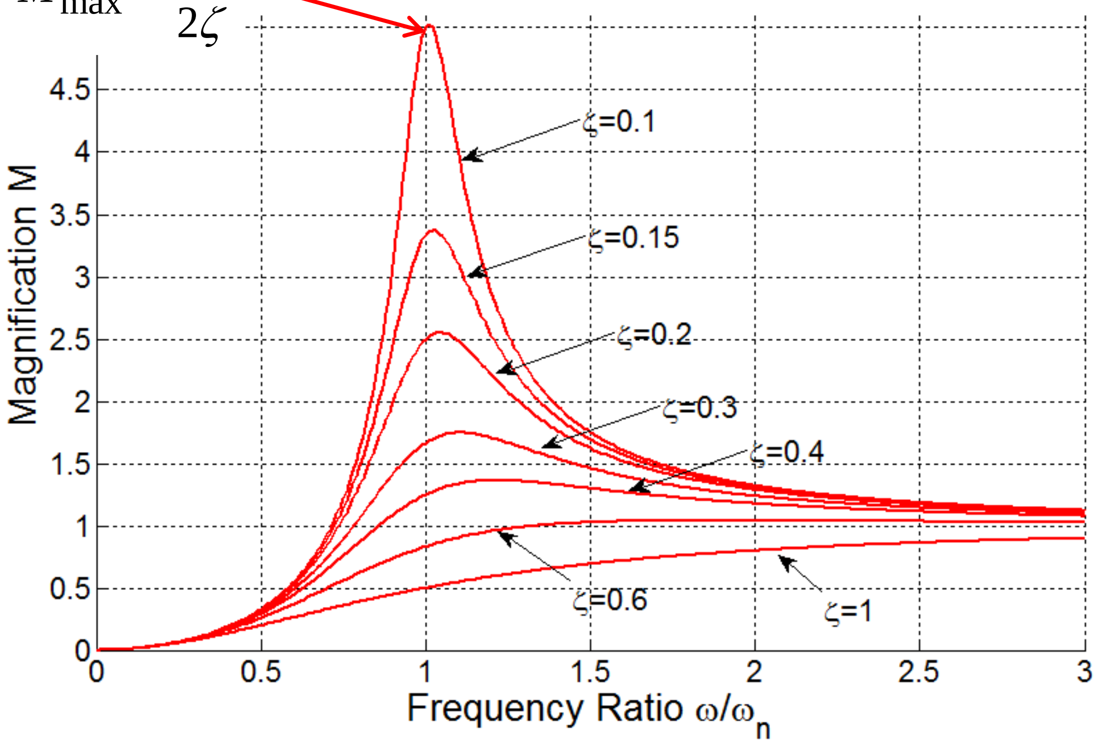


Steady state solution to $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C - \frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$

$\omega_n = \sqrt{\frac{k}{m+m_0}}$ $\zeta = \frac{c}{2\sqrt{k(m+m_0)}}$ $K = \frac{m_0}{m+m_0}$ $x_p(t) = X_0 \sin(\omega t + \phi)$

$X_0 = KY_0 M(\omega, \omega_n, \zeta)$ $M = \frac{\omega^2/\omega_n^2}{\left\{(1-\omega^2/\omega_n^2)^2 + (2\varsigma\omega/\omega_n)^2\right\}^{1/2}}$ $\phi = \tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$

$M_{\max} \approx \frac{1}{2\zeta}$



Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton's laws;
   - Calculating forces required to induce prescribed motion;
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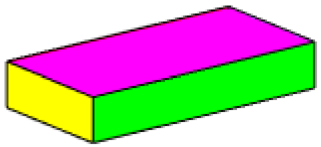
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies

Rigid Body Dynamics - Roadmap

1. Describing motion of a rigid body



- Rotation tensor (matrix)
- Angular Velocity Vector
- Spin tensor (matrix)

2. Analyzing motion in systems of rigid bodies

- Relating velocity/acceleration of two points on a rigid body
- Mechanisms
- Gears, pulleys and rolling wheels

3. Linear/Angular Momentum and Kinetic Energy of a rigid body

- Rigid body as an infinite number of particles
- Calculating inertia tensors
- Momentum and energy of a rotating body

4. Dynamics of rigid bodies

- Torques
- Force – linear momentum and moment – angular momentum relations
- Examples
- Using energy and momentum for rigid bodies

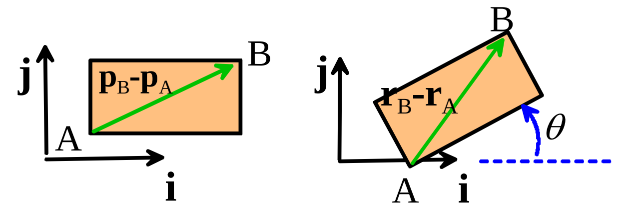
Rigid Body Dynamics – Concept checklist

1. Understand and manipulate rotation tensors in 2D and 3D
2. Understand angular velocity and acceleration vectors; be able to integrate / differentiate angular velocities / accelerations for planar motion.
3. Understand formulas relating velocity/acceleration of two points on a rigid body
4. Understand constraints at joints and contacts between rigid bodies
5. Be able to relate velocities, accelerations, or angular velocities/accelerations of two members in a system of links or rigid bodies
6. Be able to analyze motion in systems of gears
7. Understand formulas relating velocity/angular velocity and acceleration/angular acceleration of a rolling wheel
8. Be able to calculate the center of mass and mass moments of inertia of simple shapes; use parallel axis theorem to shift axis of inertia or calculate mass moments of inertia for a set of rigid bodies connected together
9. Understand how to calculate the angular momentum and kinetic energy of a rigid body
10. Understand the meaning of a ‘force couple’ or ‘pure moment/torque’
11. Understand the force-linear momentum and moment-angular momentum formulas

$$\sum \mathbf{F} = M\mathbf{a}_G \qquad \sum \mathbf{r}\times\mathbf{F} + \sum Q_z\mathbf{k} = \mathbf{r}_G \times M\mathbf{a}_G + I_{Gzz}\alpha_z\mathbf{k}$$

12. Understand the special case of these equations for fixed axis rotation
13. Be able to use dynamics equations and kinematics equations to calculate accelerations / forces in a system of planar rigid bodies subjected to forces
14. Understand power/work/potential energy of a rigid body; use energy methods to analyze motion in a system of rigid bodies
15. Use angular momentum to analyze motion of rigid bodies

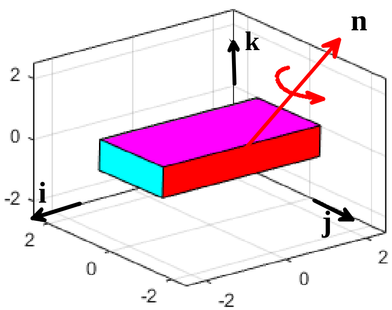
Rotation tensor (matrix) $\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$



2D rotations $\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3D rotation through θ about axis parallel to unit vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$

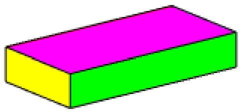
$$\mathbf{R} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$



$$1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz}$$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy} - R_{yz}\right)\mathbf{i} + \left(R_{xz} - R_{zx}\right)\mathbf{j} + \left(R_{yx} - R_{xy}\right)\mathbf{k}\right]$$

Sequence of rotations $\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)}$



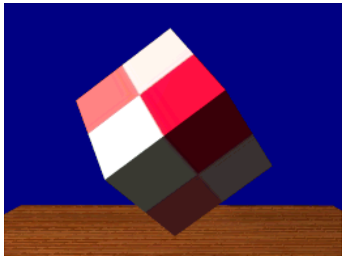
Orthogonality $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$

$\mathbf{R}^T$ and $\mathbf{R}$ represent opposite rotations

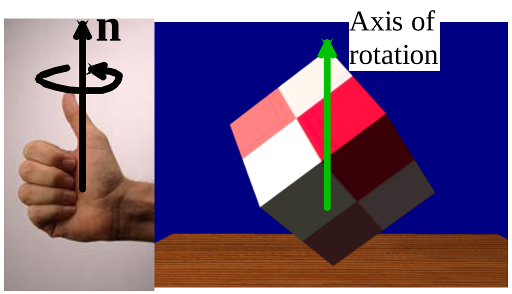
Rotational Motion

Angular velocity vector:

1. Direction – parallel to rotation axis (RH screw rule)
2. Magnitude – angle (radians) turned per sec



$$\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{n} = \omega\mathbf{n} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$



Angular acceleration vector: $\boldsymbol{\alpha} = \frac{d\omega}{dt}\mathbf{n}$

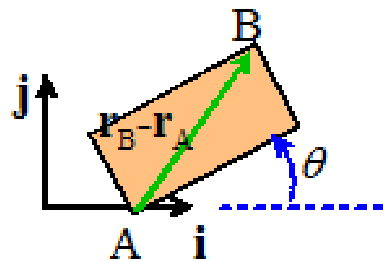
Spin Tensor

$$\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T \qquad \frac{d\mathbf{R}}{dt} = \mathbf{W}\mathbf{R} \qquad \mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$ for all vectors $\mathbf{u}$

For planar motion:

$$\omega_z = \frac{d\theta}{dt} \qquad \alpha_z = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \qquad \boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{k} \qquad \boldsymbol{\alpha} = \frac{d^2\theta}{dt^2}\mathbf{k}$$

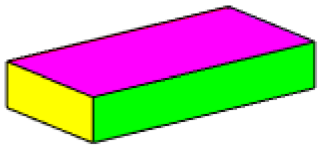


$$\mathbf{W} = \begin{bmatrix} 0 & -d\theta / dt \\ d\theta / dt & 0 \end{bmatrix}$$

Rigid Body Dynamics - Roadmap

1. Describing motion of a rigid body

- Rotation tensor (matrix)
- Angular Velocity Vector
- Spin tensor (matrix)



2. Analyzing motion in systems of rigid bodies

- Relating velocity/acceleration of two points on a rigid body
- Mechanisms
- Gears, pulleys and rolling wheels

3. Linear/Angular Momentum and Kinetic Energy of a rigid body

- Rigid body as an infinite number of particles
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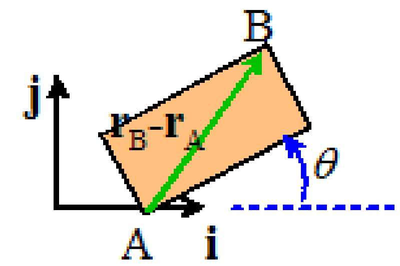
Rigid body kinematics formulas

Velocities of two points on a rigid body related by

$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$$



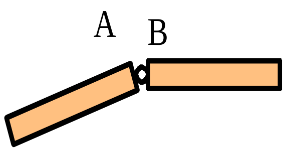
Accelerations of two points on a rigid body related by

$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times \{\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)\}$$

For 2D problems $\mathbf{v}_B - \mathbf{v}_A = \omega_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A)$

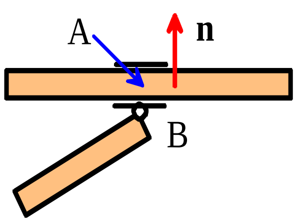
$$\mathbf{a}_B - \mathbf{a}_A = \alpha_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega_z^2 (\mathbf{r}_B - \mathbf{r}_A)$$

Constraints at connections



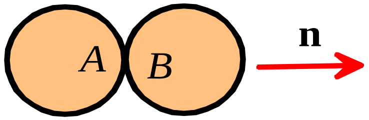
$$\mathbf{v}_A = \mathbf{v}_B$$

$$\mathbf{a}_A = \mathbf{a}_B$$



$$\mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n}$$

$$\mathbf{a}_A \cdot \mathbf{n} = \mathbf{a}_B \cdot \mathbf{n}$$



No slip $\mathbf{v}_A = \mathbf{v}_B$

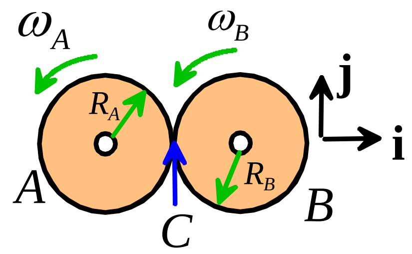
Tangential accels equal

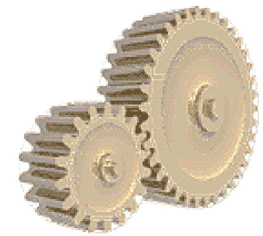
Slip $\mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n}$

Accels arbitrary

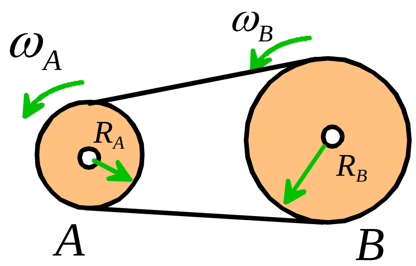
Gears, Belts and the rolling wheel

Velocities at C are equal $\Rightarrow \dfrac{\omega_B}{\omega_A} = -\dfrac{R_A}{R_B}$



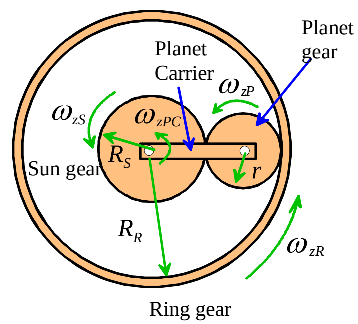


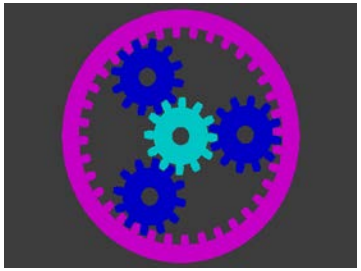
Belt speed is constant $\Rightarrow \dfrac{\omega_B}{\omega_A} = \dfrac{R_A}{R_B}$



Planetary gears
(solve with rotating frame)

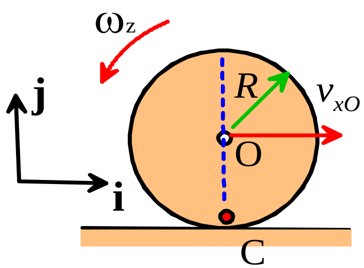
$$\frac{\omega_{zR} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{R_S}{R_R}$$



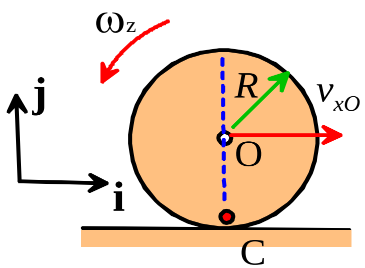


Wheel rolling without slip

C is stationary so $\mathbf{v}_O - \mathbf{v}_C = \omega_z \mathbf{k} \times (\mathbf{r}_O - \mathbf{r}_C) \Rightarrow v_{xO} = -\omega_z R \Rightarrow a_{xO} = -\alpha_z R$



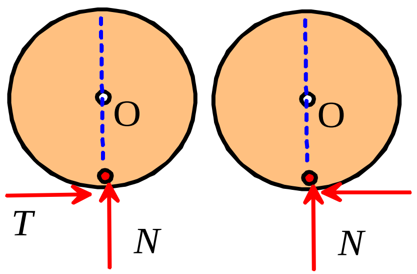
Wheels rolling and sliding on a stationary surface



$$\mathbf{v}_C - \mathbf{v}_O = \omega_z \mathbf{k} \times (\mathbf{r}_C - \mathbf{r}_O) \Rightarrow v_{xC} = v_{xO} + \omega_z R$$

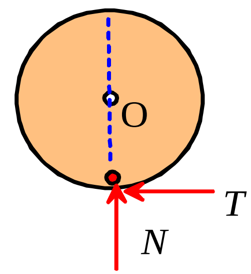
Wheel rolling without slip $\mathbf{v}_C = 0$ $\quad v_{xO} + \omega_z R = 0$

Both FBDs correct



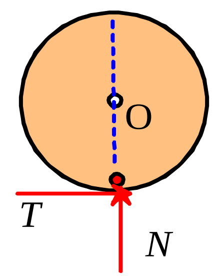
$|T| < \mu N$

Backspin $v_{xC} > 0$ $\quad v_{xO} + \omega_z R > 0$



$T = \mu N$

Topspin $v_{xC} < 0$ $\quad v_{xO} + \omega_z R < 0$



$T = \mu N$

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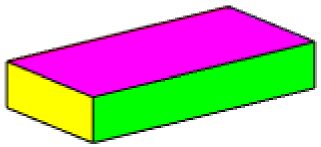

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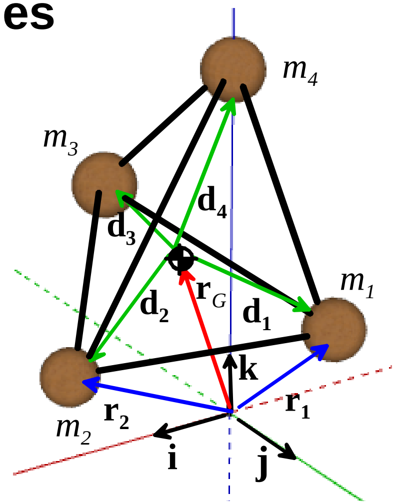
Calculating the momentum and energy of a rigid body

Preliminary: Momentum and Energy for a System of Particles



Total mass $M = \sum_i m_i$

Center of mass $\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i \qquad \mathbf{v}_G = \frac{d\mathbf{r}_G}{dt}$

Mass moment of inertia matrix $\mathbf{I}_G = \sum_i m_i \begin{bmatrix} d_{iy}^2 + d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz} \\ -d_{ix}d_{iy} & d_{ix}^2 + d_{iz}^2 & -d_{iy}d_{iz} \\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2 + d_{iy}^2 \end{bmatrix}$

Linear Momentum $\mathbf{p} = \sum_{particles} m_i \mathbf{v}_i = M\mathbf{v}_G$

Angular Momentum $\mathbf{h} = \sum_{particles} \mathbf{r}_i \times m_i \mathbf{v}_i = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G \boldsymbol{\omega}$

Kinetic Energy $T = \frac{1}{2}\sum_{particles} m_i \left|\mathbf{v}_i\right|^2 = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\left(\mathbf{I}_G\boldsymbol{\omega}\right)$

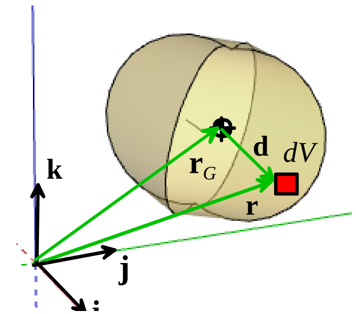
We use the same idea to calculate the momentum and energy of a rigid body.
The sums become integrals over an infinite number of infinitesimal particles

Inertial Properties

Inertial Properties of Rigid Bodies

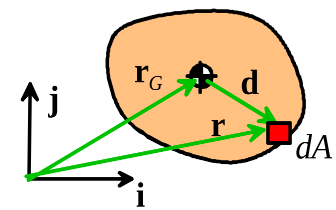
Total mass $M = \int_V \rho dV$

Center of mass $\mathbf{r}_G = \frac{1}{M}\int_V \mathbf{r}\rho dV \qquad \mathbf{v}_G = \frac{d\mathbf{r}_G}{dt}$



Mass moment of inertia

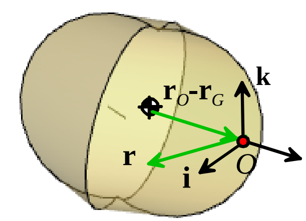
$$\mathbf{d} = \mathbf{r} - \mathbf{r}_G = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$$



3D: $$\mathbf{I}_G = \int_V \begin{bmatrix} d_{iy}^2 + d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz} \\ -d_{ix}d_{iy} & d_{ix}^2 + d_{iz}^2 & -d_{iy}d_{iz} \\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2 + d_{iy}^2 \end{bmatrix} \rho dV$$

2D: $$I_{Gzz} = \int_A (d_x^2 + d_y^2)\mu dA$$

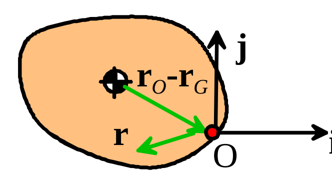
Parallel Axis Theorem



$$\mathbf{d} = \mathbf{r}_O - \mathbf{r}_G = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$$

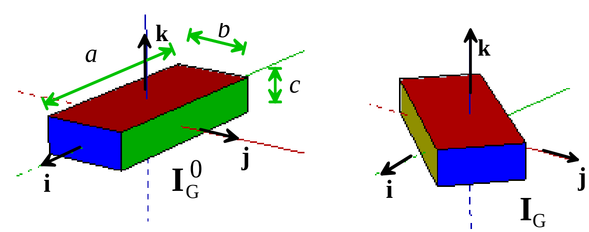
3D: $$\mathbf{I}_O = \mathbf{I}_G + M\begin{bmatrix} d_{iy}^2 + d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz} \\ -d_{ix}d_{iy} & d_{ix}^2 + d_{iz}^2 & -d_{iy}d_{iz} \\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2 + d_{iy}^2 \end{bmatrix}$$

2D: $$I_{Ozz} = I_{Gzz} + M(d_x^2 + d_y^2)$$

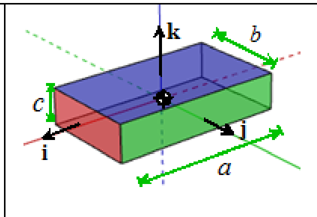
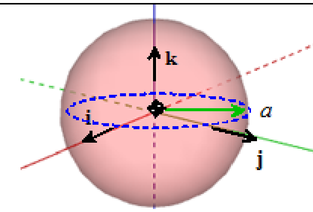
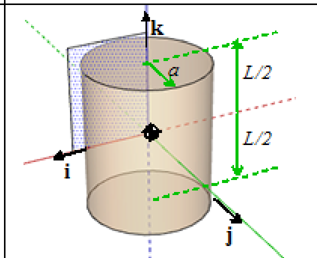
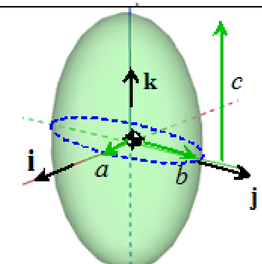
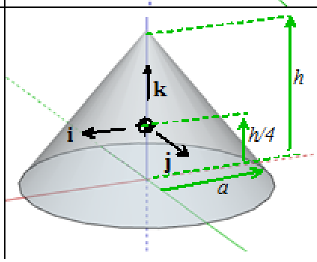
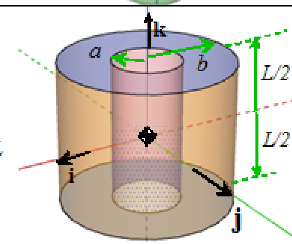


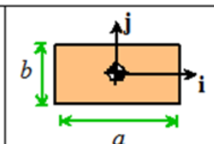
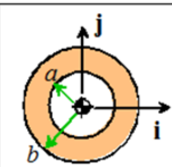
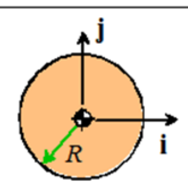
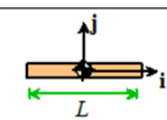
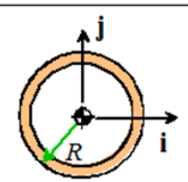
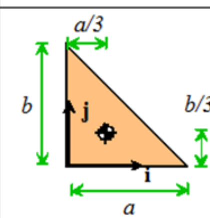
Rotation formula for inertia matrix

$$\mathbf{I}_G = \mathbf{R}\mathbf{I}_G^0\mathbf{R}^T \qquad \frac{d\mathbf{I}_G}{dt} = \mathbf{W}\mathbf{I}_G - \mathbf{I}_G\mathbf{W}$$



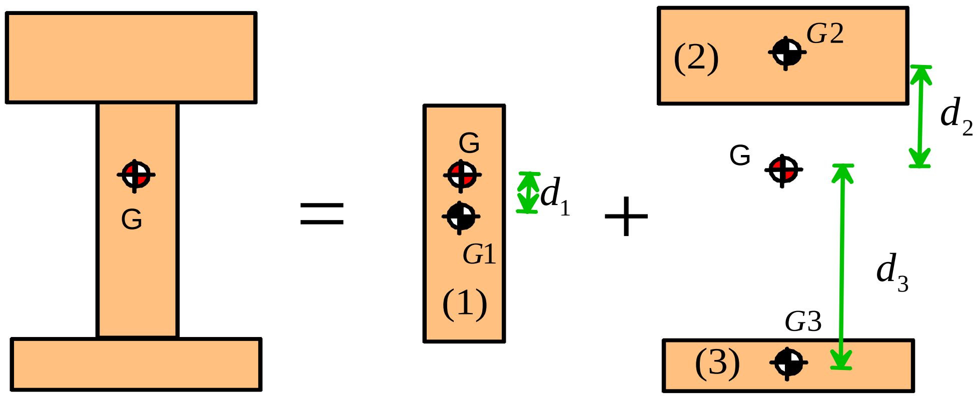
Mass Moments of Inertia

Prism $M = \rho abc$		$\frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$	**Solid Sphere** $M = \frac{4}{3}\pi\rho a^3$		$\frac{2Ma^2}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Solid Cylinder $M = \pi\rho a^2 L$		$\frac{ML^2}{12}\begin{bmatrix} 1+3a^2/L^2 & 0 & 0 \\ 0 & 1+3a^2/L^2 & 0 \\ 0 & 0 & 6a^2/L^2 \end{bmatrix}$	**Solid Ellipsoid** $M = \frac{4}{3}\pi\rho abc$		$\frac{M}{5}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$
Solid Cone $M = \frac{\pi}{3}\rho a^2 h$		$\frac{3Ma^2}{20}\begin{bmatrix} 1+h^2/(4a^2) & 0 & 0 \\ 0 & 1+h^2/(4a^2) & 0 \\ 0 & 0 & 2 \end{bmatrix}$	**Hollow Cylinder** $M = \pi\rho(b^2-a^2)L$		$\frac{M}{12}\begin{bmatrix} L^2+3(a^2+b^2) & 0 & 0 \\ 0 & L^2+3(a^2+b^2) & 0 \\ 0 & 0 & 6(a^2+b^2) \end{bmatrix}$

Square		$I_{Gzz} = \frac{M}{12}(a^2+b^2)$	**Hollow disk**		$I_{Gzz} = \frac{M}{2}\left(a^2+b^2\right)$
Disk		$I_{Gzz} = \frac{M}{2}R^2$	**Slender rod**		$I_{Gzz} = \frac{M}{12}L^2$
Thin ring		$I_{Gzz} = MR^2$	**Triangular Plate**		$\frac{M}{18}(a^2+b^2)$

Calculating mass moments of inertia by summation

(Illustrated with 2D example, same idea works in 3D)



To find position of COM and inertia of a complex shape, use:

Total mass $M = m_1 + m_2 + m_3$

Center of mass $\mathbf{r}_G = \frac{1}{M}(m_1\mathbf{r}_{G1} + m_2\mathbf{r}_{G2} + m_3\mathbf{r}_{G3})$

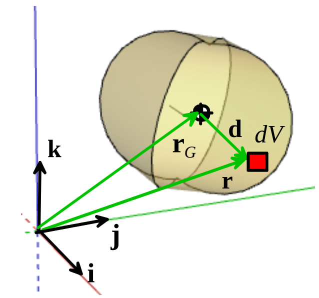
Mass moment of inertia (use parallel axis theorem and add all sections)

$$I_{Gzz} = I_{G1zz} + m_1 d_1^2 + I_{G2zz} + m_2 d_2^2 + I_{G3zz} + m_3 d_3^2$$

d_i is the distance of the COM of the ith section from the combined COM at G

Momentum and Energy Equations

Momentum and Energy of a rigid body

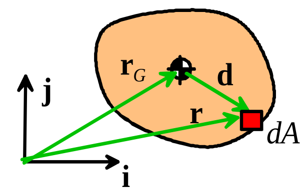


3D:

Linear Momentum $\mathbf{p} = M\mathbf{v}_G$

Angular Momentum $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}$

Kinetic Energy $T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\left(\mathbf{I}_G\boldsymbol{\omega}\right)$



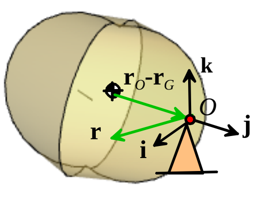
2D:

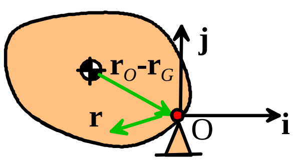
Linear Momentum $\mathbf{p} = M\mathbf{v}_G$

Angular Momentum $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + I_{Gzz}\omega_z\mathbf{k}$

Kinetic Energy $T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}I_{Gzz}\omega_z^2$

Special Case: Rotation about a fixed point





Angular Momentum $\mathbf{h} = \mathbf{I}_O\boldsymbol{\omega}$

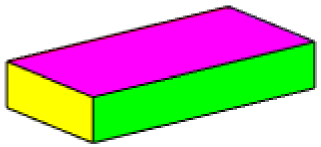
Kinetic Energy $T = \frac{1}{2}\boldsymbol{\omega}\cdot\left(\mathbf{I}_G\boldsymbol{\omega}\right)$

Angular Momentum $\mathbf{h} = I_{Ozz}\omega_z\mathbf{k}$

Kinetic Energy $T = \frac{1}{2}I_{Ozz}\omega_z^2$

Rigid Body Dynamics - Roadmap

1. Describing motion of a rigid body



- Rotation tensor (matrix)
- Angular Velocity Vector
- Spin tensor (matrix)

2. Analyzing motion in systems of rigid bodies

- Relating velocity/acceleration of two points on a rigid body
- Mechanisms
- Gears, pulleys and rolling wheels

3. Linear/Angular Momentum and Kinetic Energy of a rigid body

- Rigid body as an infinite number of particles
- Calculating inertia tensors
- Momentum and energy of a rotating body

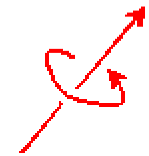
4. Dynamics of rigid bodies

- Torques
- Force – linear momentum and moment – angular momentum relations
- Examples of solutions to 2D problems
- Using energy and momentum for rigid bodies

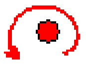
Torques (Couples, or 'pure moments')

Torque

A torque is a rotational force:
Causes rotation without translation



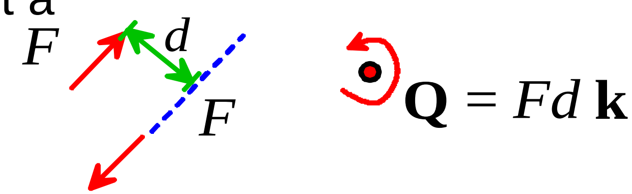
3D Torque



2D Torque

Torque is a vector: $\mathbf{Q} = Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}$
Torque has units of Nm

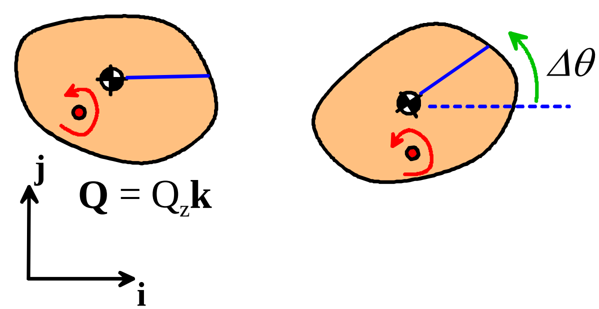
Two non-collinear equal and opposite forces exert a torque



Power of a torque $P = \mathbf{Q} \cdot \boldsymbol{\omega}$

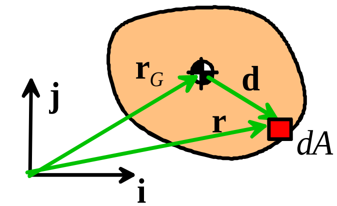
Work done by a torque $W = \int_0^t \mathbf{Q} \cdot \boldsymbol{\omega}\, dt$

For 2D: $W = \int_0^\theta Q_z\, d\theta$



2D inertia, parallel axis theorem

Inertial Properties

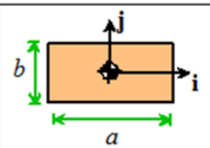
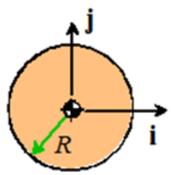
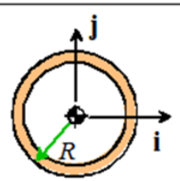


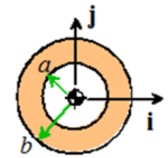
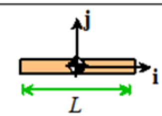
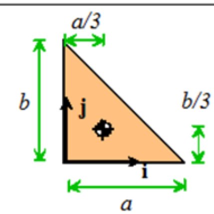
μ : Mass/unit area

Total mass $M = \int_A \mu dA$

Center of mass $\mathbf{r}_G = \frac{1}{M}\int_A \mathbf{r}\mu dA \qquad \mathbf{v}_G = \frac{d\mathbf{r}_G}{dt}$

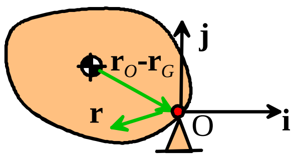
Mass moment of inertia $I_{Gzz} = \int_A (d_x^2 + d_y^2)\mu dA \qquad \mathbf{d} = \mathbf{r} - \mathbf{r}_G = d_x\mathbf{i} + d_y\mathbf{j}$

Square		$I_{Gzz} = \frac{M}{12}(a^2+b^2)$
Disk		$I_{Gzz} = \frac{M}{2}R^2$
Thin ring		$I_{Gzz} = MR^2$

Hollow disk		$I_{Gzz} = \frac{M}{2}\left(a^2+b^2\right)$
Slender rod		$I_{Gzz} = \frac{M}{12}L^2$
Triangular Plate		$\frac{M}{18}(a^2+b^2)$

Parallel Axis Theorem

$$I_{Ozz} = I_{Gzz} + M(d_x^2 + d_y^2)$$



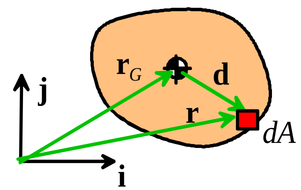
2D Momentum and energy

Momentum and Energy of a rigid body

Linear Momentum $\mathbf{p} = M\mathbf{v}_G$

Angular Momentum $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + I_{Gzz}\omega_z\mathbf{k}$

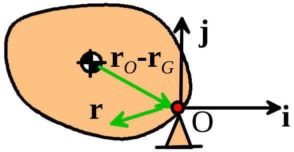
Kinetic Energy $T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}I_{Gzz}\omega_z^2$



Special Case: Rotation about a fixed point

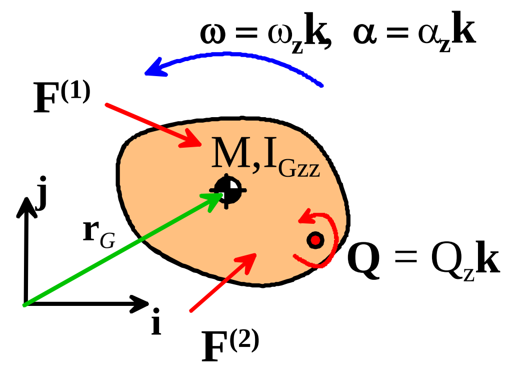
Angular Momentum $\mathbf{h} = I_{Ozz}\omega_z\mathbf{k}$

Kinetic Energy $T = \frac{1}{2}I_{Ozz}\omega_z^2$



2D equations of motion for rigid bodies

Analyzing 2D motion of a rigid body



Linear Momentum $$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt}$$

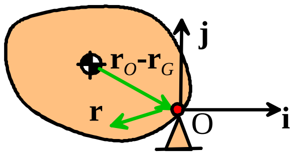
$$\Rightarrow \sum \mathbf{F} = M\mathbf{a}_G$$

Angular Momentum $$\sum \mathbf{r} \times \mathbf{F} + \sum Q_z \mathbf{k} = \frac{d\mathbf{h}}{dt}$$ (about origin)

$$\Rightarrow \sum \mathbf{r} \times \mathbf{F} + \sum Q_z \mathbf{k} = \mathbf{r}_G \times M\mathbf{a}_G + I_{Gzz}\alpha_z \mathbf{k}$$

Special Case: Rotation about a fixed point

$$\sum \mathbf{r} \times \mathbf{F} + \sum Q_z \mathbf{k} = I_{Ozz}\alpha_z \mathbf{k}$$

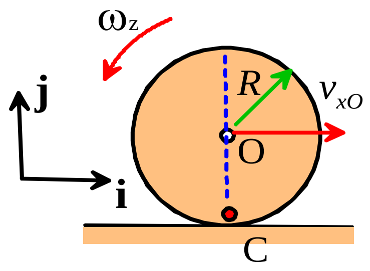


2D Kinematics formulas

Kinematics Formulas

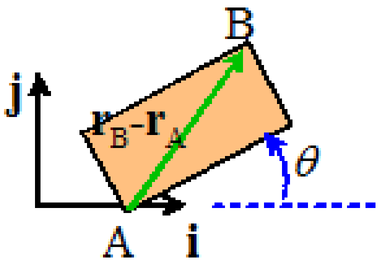
Wheel rolling without slip on stationary surface

$$\mathbf{v}_O - \mathbf{v}_C = \omega_z \mathbf{k} \times (\mathbf{r}_O - \mathbf{r}_C)$$
$$\Rightarrow v_{xO} = -\omega_z R$$
$$\Rightarrow a_{xO} = -\alpha_z R$$

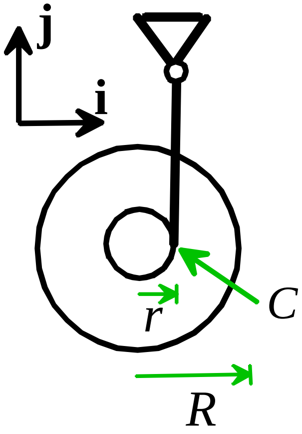


General

$$\mathbf{v}_B - \mathbf{v}_A = \omega_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A)$$
$$\mathbf{a}_B - \mathbf{a}_A = \alpha \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega^2 (\mathbf{r}_B - \mathbf{r}_A)$$



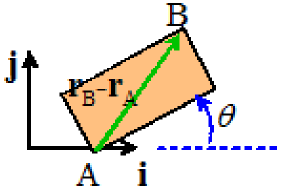
Calculating forces or accelerations



- Idealize system
- Free body diagram for each rigid body
- $\sum \mathbf{F} = M\mathbf{a}_G$ for each rigid body.
- $\sum \mathbf{r} \times \mathbf{F} + \sum Q_z \mathbf{k} = \mathbf{r}_G \times M\mathbf{a}_G + I_{Gzz}\alpha_z \mathbf{k}$ for each rigid body
- Use kinematics equations to relate $\mathbf{a}_G, \alpha_z$ for each rigid body

$$\mathbf{v}_B - \mathbf{v}_A = \omega_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A)$$

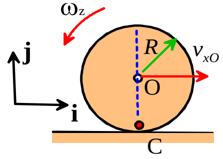
$$\mathbf{a}_B - \mathbf{a}_A = \alpha \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega^2 (\mathbf{r}_B - \mathbf{r}_A)$$



$$\mathbf{v}_O - \mathbf{v}_C = \omega_z \mathbf{k} \times (\mathbf{r}_O - \mathbf{r}_C)$$

$$\Rightarrow v_{xO} = -\omega_z R$$

$$\Rightarrow a_{xO} = -\alpha_z R$$

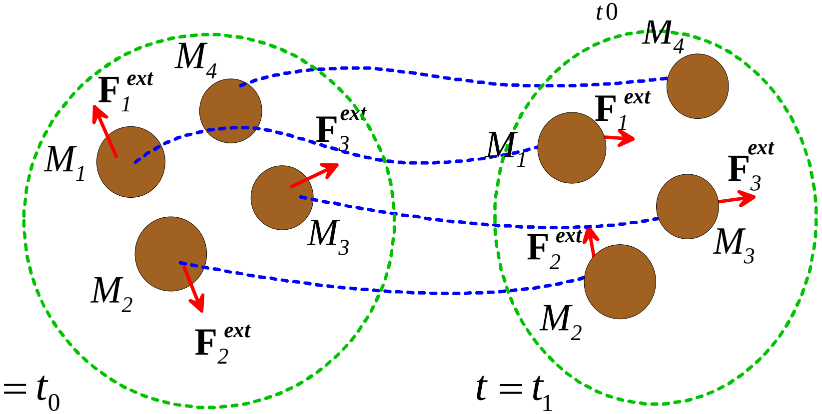


- Solve for unknown forces or accelerations

Energy equation for systems of rigid bodies

$$External\ \ Power\ \ P^{ext}(t)$$

$$External\ \ work\ \ \Delta W^{ext} = \int_{t0}^{t1} P(t)dt$$



$t = t_0$	$t = t_1$
$Total\ \ KE\ T_0^{TOT}$	$Total\ \ KE\ T_1^{TOT}$
$Total\ \ PE\ V_0^{TOT}$	$Total\ \ PE\ V_1^{TOT}$

$$\Delta W^{ext} = (T_1^{TOT} + V_1^{TOT}) - (T_0^{TOT} + V_0^{TOT}) \qquad \Delta W^{ext} = 0 \Rightarrow (T_1^{TOT} + V_1^{TOT}) = (T_0^{TOT} + V_0^{TOT})$$

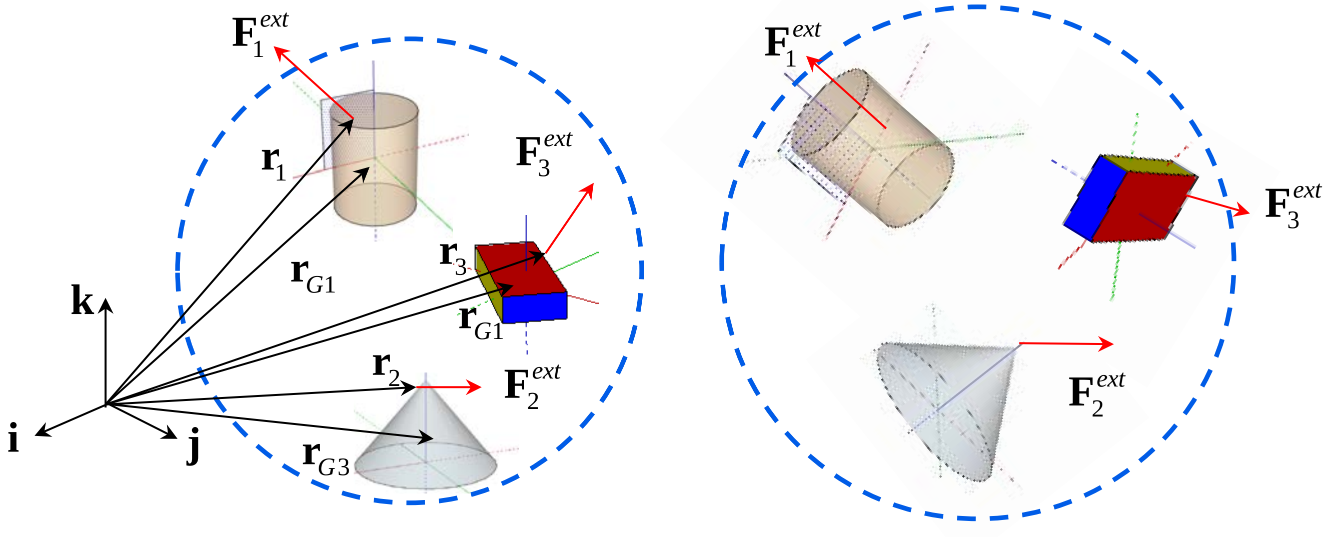
Same as systems of particles, but we now use the rigid body formula for KE

$$T = \frac{1}{2} M \left| \mathbf{v}_G \right|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega} \qquad T = \frac{1}{2} M \left| \mathbf{v}_G \right|^2 + \frac{1}{2} I_{Gzz} \omega_z^2$$

Angular Momentum equation for systems of rigid bodies

External Moment $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{Q}$

External Angular Impulse $\mathbf{A}^{ext} = \int_{t0}^{t1} \left\{ \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{Q} \right\} dt$



$t = t_0$

Total Angular Momentum $\mathbf{h}_0^{TOT}$

$t = t_1$

Total Angular Momentum $\mathbf{h}_1^{TOT}$

$$\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_i \mathbf{Q} = \frac{d\mathbf{h}^{TOT}}{dt}$$

$$\mathbf{A}^{ext} = \mathbf{h}_1^{TOT} - \mathbf{h}_0^{TOT}$$

Same as systems of particles, but we now use the rigid body formula for AM

$$\mathbf{h} = \mathbf{r} \times m\mathbf{v}_G + \mathbf{I}_G \boldsymbol{\omega} \qquad \mathbf{h} = \mathbf{r} \times m\mathbf{v}_G + I_{Gzz}\omega_z \mathbf{k}$$