Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton's laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy;
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations

5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion; Euler's laws; deriving and solving equations of motion; motion of machines

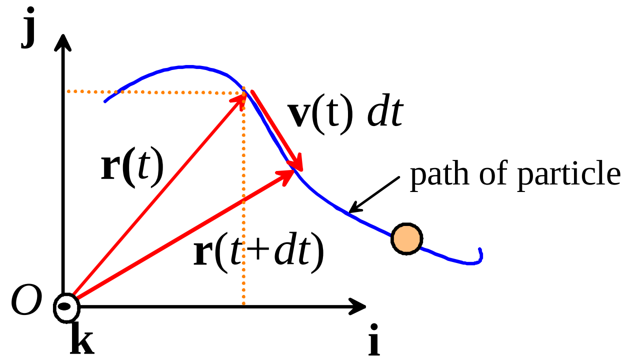
Exam topics

Particle Dynamics – concept checklist

- Understand the concept of an ‘inertial frame’
- Be able to idealize an engineering design as a set of particles, and know when this idealization will give accurate results
- Describe the motion of a system of particles (eg components in a fixed coordinate system; components in a polar coordinate system, etc)
- Be able to differentiate position vectors (with proper use of the chain rule!) to determine velocity and acceleration; and be able to integrate acceleration or velocity to determine position vector.
- Be able to describe motion in normal-tangential and polar coordinates (eg be able to write down vector components of velocity and acceleration in terms of speed, radius of curvature of path, or coordinates in the cylindrical-polar system).
- Be able to convert between Cartesian to normal-tangential or polar coordinate descriptions of motion
- Be able to draw a correct free body diagram showing forces acting on system idealized as particles
- Be able to write down Newton’s laws of motion in rectangular, normal-tangential, and polar coordinate systems
- Be able to obtain an additional moment balance equation for a rigid body moving without rotation or rotating about a fixed axis at constant rate.
- Be able to use Newton’s laws of motion to solve for unknown accelerations or forces in a system of particles
- Use Newton’s laws of motion to derive differential equations governing the motion of a system of particles
- Be able to re-write second order differential equations as a pair of first-order differential equations in a form that MATLAB can solve

Particle Kinematics

Inertial frame – non accelerating, non rotating reference frame
Particle – point mass at some position in space



Position Vector $$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Velocity Vector $$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$

$$= \frac{d}{dt}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

$$\Rightarrow v_x(t) = \frac{dx}{dt} \qquad v_y(t) = \frac{dy}{dt} \qquad v_z(t) = \frac{dz}{dt}$$

- Direction of velocity vector is parallel to path
- Magnitude of velocity vector is distance traveled / time

Acceleration Vector

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} = \frac{d}{dt}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right) = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}$$

$$\Rightarrow a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

Also $$a_x(t) = \frac{dv_x}{dx}v_x \qquad a_y(t) = \frac{dv_y}{dy}v_y \qquad a_z(t) = \frac{dv_z}{dz}v_z$$

Particle Kinematics

- Straight line motion with constant acceleration

$$\mathbf{r} = \left[X_0 + V_0 t + \frac{1}{2} a t^2 \right] \mathbf{i} \qquad \mathbf{v} = \left(V_0 + a t \right) \mathbf{i} \qquad \mathbf{a} = a \mathbf{i}$$

- Time/velocity/position dependent acceleration – use calculus

$$\mathbf{r} = \left(X_0 + \int_0^t v(t) dt \right) \mathbf{i} \qquad \mathbf{v} = \left(V_0 + \int_0^t a(t) dt \right) \mathbf{i}$$

$$a = \frac{dv}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{V_0}^{v} f(v) dv = \int_0^t g(t) dt$$

$$v = \frac{dx}{dt} = \frac{g(t)}{f(v)} \Rightarrow \int_{X_0}^{x(t)} f(x) dv = \int_0^t v(t) dt$$

$$\frac{dv}{dt} = a(x)$$

$$\Rightarrow \frac{dv}{dx} \frac{dx}{dt} = a(x) \Rightarrow v \frac{dv}{dx} = a(x)$$

$$\int_{V_0}^{v(t)} v dv = \int_0^{x(t)} a(x) dx$$

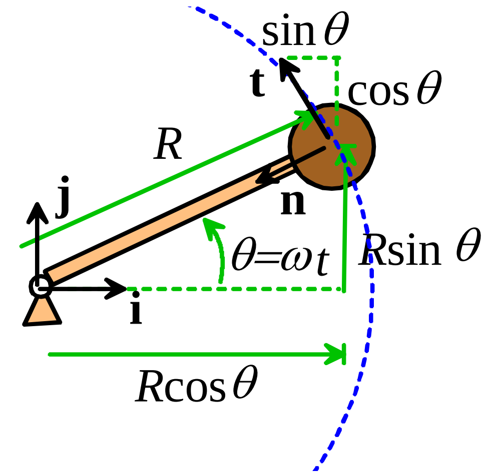
- Circular Motion at const speed

$$\theta = \omega t \quad s = R\theta \quad V = \omega R$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = -\omega^2 R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = \omega^2 R\mathbf{n} = \frac{V^2}{R}\mathbf{n}$$



- General circular motion

$$\omega = d\theta / dt \quad \alpha = d\omega / dt = d^2\theta / dt^2$$

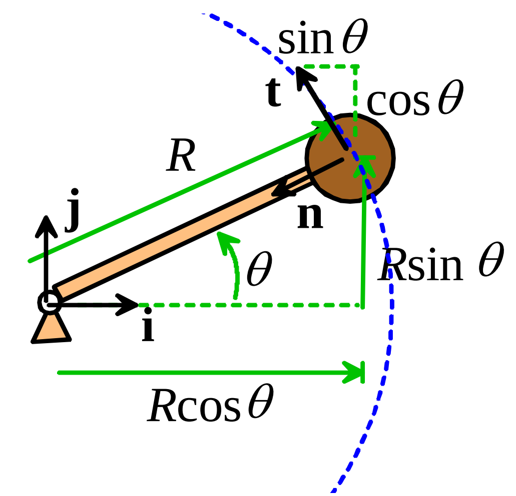
$$s = R\theta \quad V = ds / dt = R\omega$$

$$\mathbf{r} = R\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right)$$

$$\mathbf{v} = \omega R\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right) = V\mathbf{t}$$

$$\mathbf{a} = R\alpha(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \alpha R\mathbf{t} + \omega^2 R\mathbf{n} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

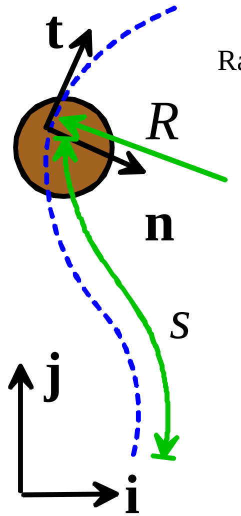


Particle Kinematics

- Motion along an arbitrary path

$$\mathbf{v} = V\mathbf{t}$$

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$



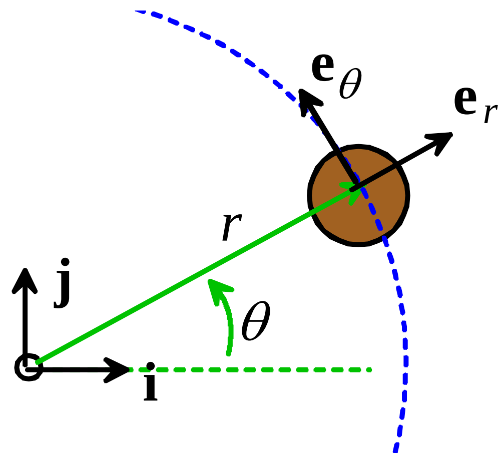
Radius of curvature $R = \dfrac{1}{\sqrt{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2}}$

$$\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$$

- Polar Coordinates

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta$$



Using Newton's laws

Calculating forces required to cause prescribed motion

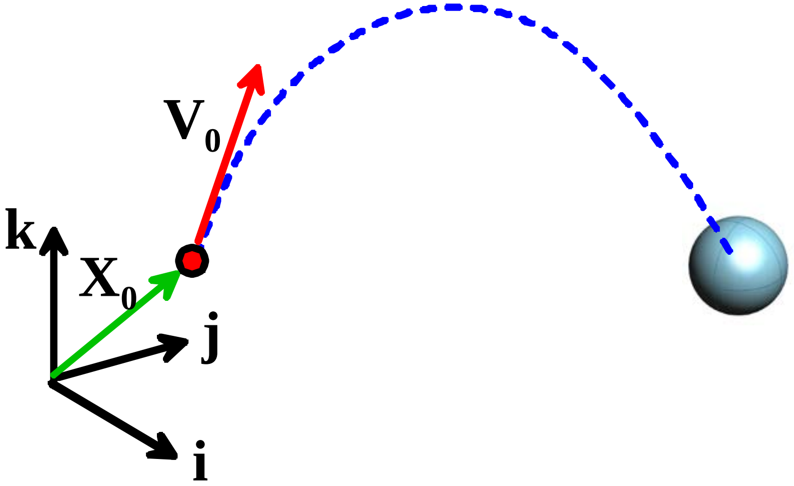
- Idealize system
- Free body diagram
- Kinematics (describe motion – usually goal is to find formula for acceleration)
- **F**=m**a** for each particle.
- $\mathbf{M}_C = \mathbf{0}$ (for steadily or non-rotating rigid bodies or frames only)
- Solve for unknown forces or accelerations (just like statics)

Using Newton's laws to derive equations of motion

1. Idealize system
2. Introduce variables to describe motion (often *x,y* coords, but we will see other examples)
3. Write down **r**, differentiate to get **a**
4. Draw FBD
5. $\mathbf{F} = m\mathbf{a}$
6. If necessary, eliminate reaction forces
7. Result will be differential equations for coords defined in (2), e.g. $m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = kY_0 \sin \omega t$
8. Identify initial conditions, and solve ODE

Motion of a projectile in earths gravity

$$\left.\begin{aligned}\mathbf{r} &= X_0\mathbf{i} + Y_0\mathbf{j} + Z_0\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= V_{x0}\mathbf{i} + V_{y0}\mathbf{j} + V_{z0}\mathbf{k}\end{aligned}\right\} t = 0$$



$$\mathbf{r} = \left(X_0 + V_{x0}t\right)\mathbf{i} + \left(Y_0 + V_{y0}t\right)\mathbf{j} + \left(Z_0 + V_{z0}t - \frac{1}{2}gt^2\right)\mathbf{k}$$

$$\mathbf{v} = \left(V_{x0}\right)\mathbf{i} + \left(V_{y0}\right)\mathbf{j} + \left(V_{z0} - gt\right)\mathbf{k}$$

$$\mathbf{a} = -g\mathbf{k}$$

Rearranging differential equations for MATLAB

- Example $$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$$
- Introduce $$v = dy / dt$$
- Then $$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} v \\ -2\zeta\omega_n v - \omega_n^2 y \end{bmatrix}$$
- This has form $$\frac{d\mathbf{w}}{dt} = f(t, \mathbf{w}) \qquad \mathbf{w} = \begin{bmatrix} y \\ v \end{bmatrix}$$

Conservation Laws – concept checklist

- Know the definitions of power (or rate of work) of a force, and work done by a force
- Know the definition of kinetic energy of a particle
- Understand power-work-kinetic energy relations for a particle
- Be able to use work/power/kinetic energy to solve problems involving particle motion
- Be able to distinguish between conservative and non-conservative forces
- Be able to calculate the potential energy of a conservative force
- Be able to calculate the force associated with a potential energy function
- Know the work-energy relation for a system of particles; (energy conservation for a closed system)
- Use energy conservation to analyze motion of conservative systems of particles

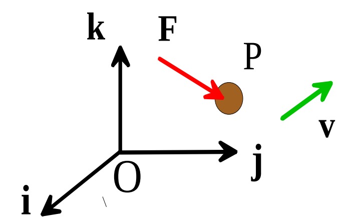
- Know the definition of the linear impulse of a force
- Know the definition of linear momentum of a particle
- Understand the impulse-momentum (and force-momentum) relations for a particle
- Understand impulse-momentum relations for a system of particles (momentum conservation for a closed system)
- Be able to use impulse-momentum to analyze motion of particles and systems of particles
- Know the definition of restitution coefficient for a collision
- Predict changes in velocity of two colliding particles in 2D and 3D using momentum and the restitution formula

- Know the definition of angular impulse of a force
- Know the definition of angular momentum of a particle
- Understand the angular impulse-momentum relation
- Be able to use angular momentum to solve central force problems/impact problems

Work-Energy relations for a single particle

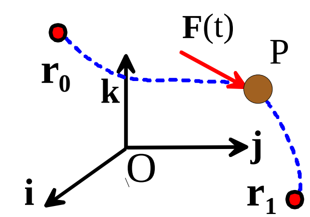
Rate of work done by a force (power developed by force)

$$P = \mathbf{F} \cdot \mathbf{v}$$



Total work done by a force

$$W = \int_0^{t_1} \mathbf{F} \cdot \mathbf{v} dt \qquad W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$



Kinetic energy

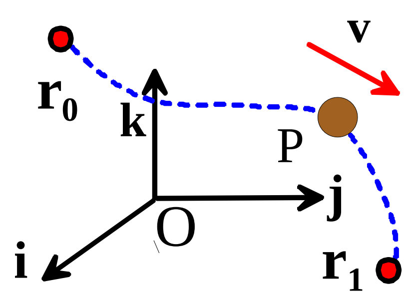
$$T = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

Power-kinetic energy relation

$$P = \frac{dT}{dt}$$

Work-kinetic energy relation

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = T - T_0$$

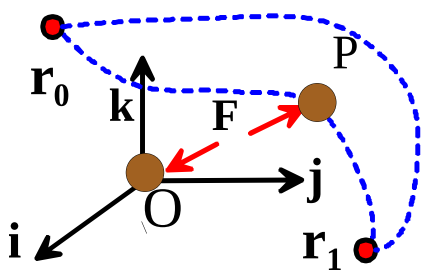


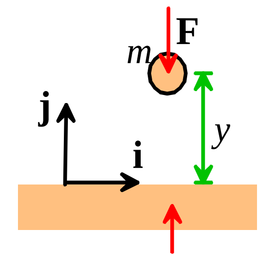
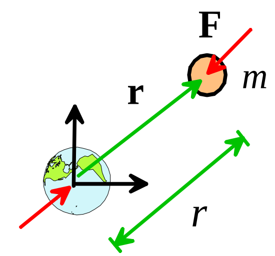
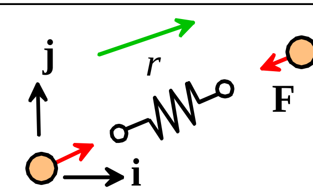
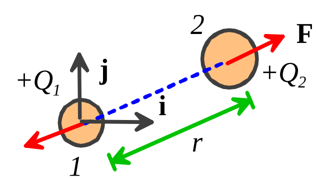
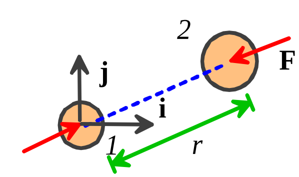
Potential Energy

Potential energy of a conservative force (pair)

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} + \text{constant}$$

$$\mathbf{F} = -\text{grad}(V)$$



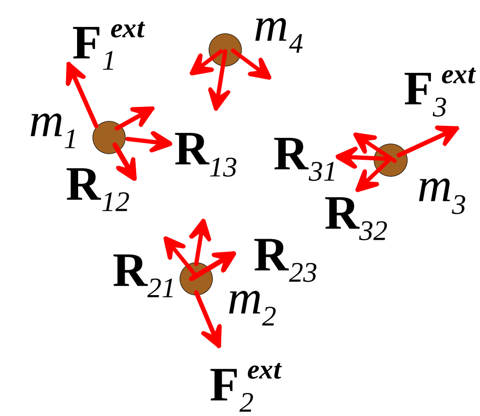
Type of force	Potential energy	
Gravity acting on a particle near earths surface	$V = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$V = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$V = \dfrac{1}{2}k(r-L_0)^2$	
Force acting between two charged particles	$V = \dfrac{Q_1Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

Energy Relation for a Conservative System

Internal Forces: (forces exerted by one part of the system on another) $\mathbf{R}_{ij}$

External Forces: (any other forces) $\mathbf{F}_i^{ext}$

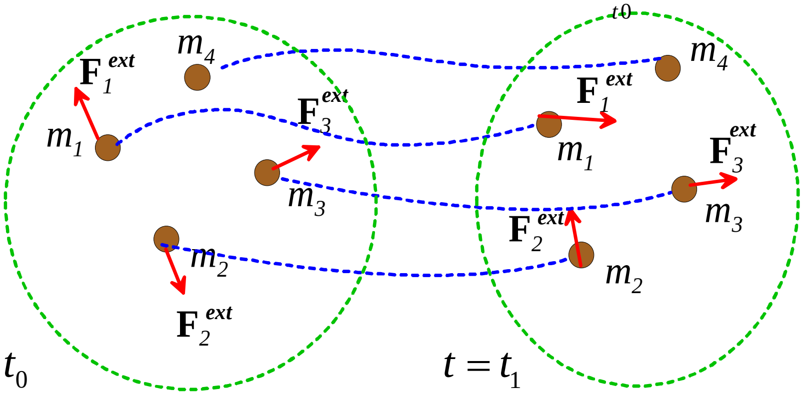
System is conservative if all internal forces are conservative forces (or constraint forces)



Energy relation for a conservative system

$$External\ \ Power\ \ P^{ext}(t)$$

$$External\ \ work\ \ \Delta W^{ext} = \int_{t0}^{t1} P(t)dt$$



$t = t_0$

$Total\ \ KE\ \ T_0^{TOT}$

$Total\ \ PE\ \ V_0^{TOT}$

$t = t_1$

$Total\ \ KE\ \ T_1^{TOT}$

$Total\ \ PE\ \ V_1^{TOT}$

$$\Delta W_{ext} = T_1^{TOT} + V_1^{TOT} - \left(T_0^{TOT} + V_0^{TOT}\right)$$

Special case – zero external work:

$$T_1^{TOT} + V_1^{TOT} = T_0^{TOT} + V_0^{TOT}$$

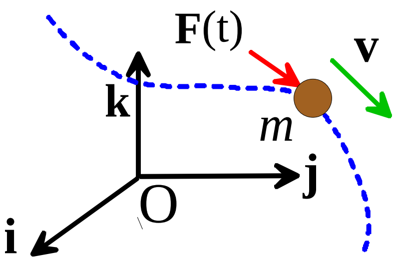
KE+PE = constant

Impulse-Momentum for a single particle

Definitions

Linear Impulse of a force $$\mathbf{I} = \int_{t_0}^{t_1} \mathbf{F}(t)\,dt$$

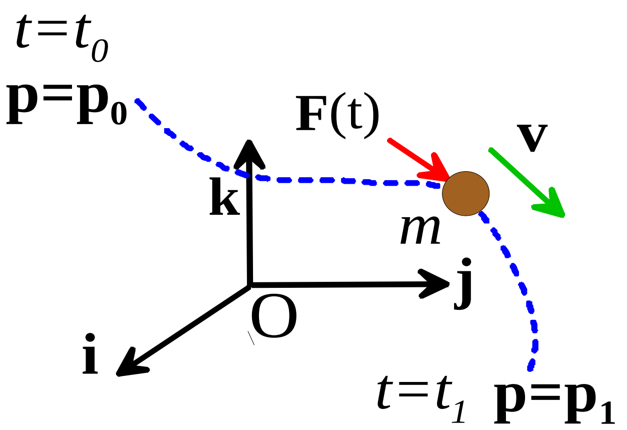
Linear momentum of a particle $$\mathbf{p} = m\mathbf{v}$$



Impulse-Momentum relations

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{I} = \mathbf{p}_1 - \mathbf{p}_0$$

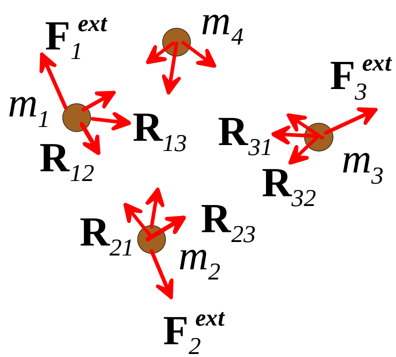


Impulse-Momentum for a system of particles

$\mathbf{R}_{ij}$ Force exerted on particle i by particle j

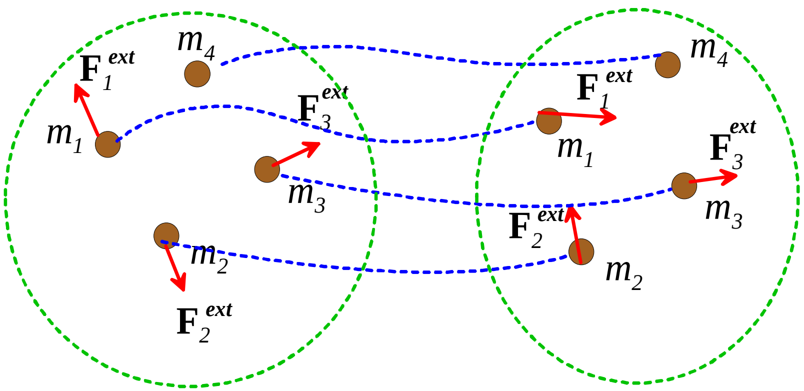
$\mathbf{F}_i^{ext}$ External force on particle i

$\mathbf{v}_i$ Velocity of particle i



$$Total\ \ External\ \ Force\ \ \mathbf{F}^{TOT}(t)$$

$$Total\ External\ Impulse\ \ \mathbf{I}^{TOT} = \int_{t0}^{t1} \mathbf{F}^{TOT}(t)dt$$



$t = t_0$

$Total\ \ momentum\ \ \mathbf{p}_0^{TOT}$

$t = t_1$

$Total\ \ momentum\ \ \mathbf{p}_1^{TOT}$

Impulse-momentum for the system:

$$\mathbf{F}^{TOT} = \frac{d\mathbf{p}^{TOT}}{dt}$$

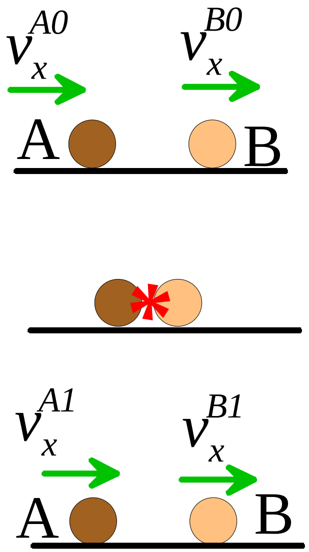
$$\mathbf{I}^{TOT} = \mathbf{p}_1^{TOT} - \mathbf{p}_0^{TOT}$$

Special case – zero external impulse:

$$\mathbf{p}_1^{TOT} = \mathbf{p}_0^{TOT}$$

(Linear momentum conserved)

Collisions

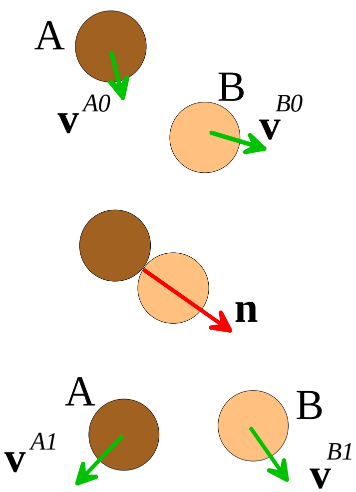


Momentum $$m_A v_x^{A1} + m_B v_x^{B1} = m_A v_x^{A0} + m_B v_x^{B0}$$

Restitution formula $$v^{B1} - v^{A1} = -e\left(v^{B0} - v^{A0}\right)$$

$$v^{B1} = v^{B0} - \frac{m_A}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$

$$v^{A1} = v^{A0} + \frac{m_B}{m_A + m_B}(1+e)\left(v^{B0} - v^{A0}\right)$$



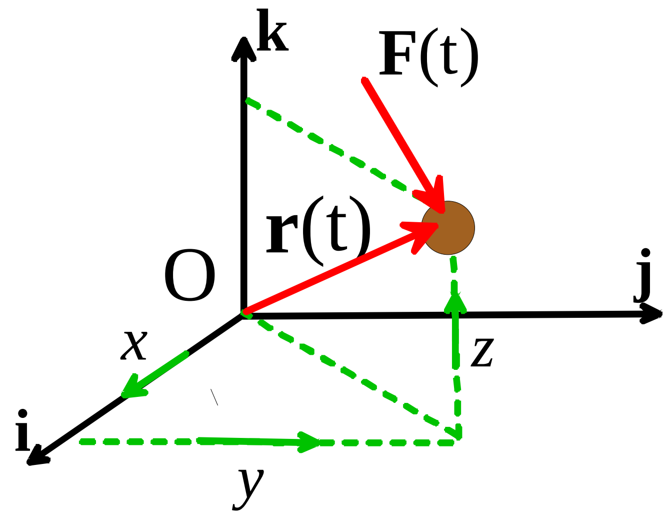
Momentum $$m_B\ \mathbf{v}^{B1} + m_A \mathbf{v}^{A1} = m_B\ \mathbf{v}^{B0} + m_A \mathbf{v}^{A0}$$

Restitution formula $$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Angular Impulse-Momentum Equations for a particle



Angular Impulse $\mathbf{A} = \int_{t_0}^{t_1} \mathbf{r}(t) \times \mathbf{F}(t) dt$

Angular Momentum $\mathbf{h} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$

Impulse-Momentum relations $\mathbf{r} \times \mathbf{F} = \frac{d\mathbf{h}}{dt}$ $\mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$

Special Case $\mathbf{A} = \mathbf{0} \Rightarrow \mathbf{h}_1 = \mathbf{h}_0$ Angular momentum conserved

Useful for central force problems (when forces on a particle always act through a single point, eg planetary gravity)