EN3: Introduction to Engineering

Teach Yourself Vectors

Division of Engineering
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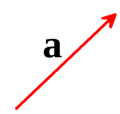
1. Definition

A vector is a mathematical object that has magnitude and direction, and satisfies the laws of vector addition. Vectors are used to represent physical quantities that have a magnitude and direction associated with them. For example,

- The velocity of an object is a vector. The direction of the vector specifies the direction of travel, and the magnitude specifies the speed.
- The force acting on an object is a vector. The direction of the vector specifies the line of action of the force, and the magnitude specifies how large the force is.

Other examples of vectors include position; acceleration; electric field; electric current flow; heat flow; the normal to a surface. Examples of quantities that are *not* vectors include mass, temperature, electric potential, volume, and energy. These can all be described by their magnitude only (they have no direction) and so are *scalars*.



A vector is often represented pictorially as an arrow (the arrow's length is its magnitude, and it points in its direction) and symbolically by an underlined letter $\underline{a}$, using bold type **a** or by an arrow symbol over a variable $\vec{a}$. The magnitude of a vector is denoted $|\underline{a}|$, $|\mathbf{a}|$ or $|\vec{a}|$. There are two special cases of vectors: the *unit vector* **n** has $|\mathbf{n}| = 1$; and the *null vector* **0** has $|\mathbf{0}| = 0$.

Problems

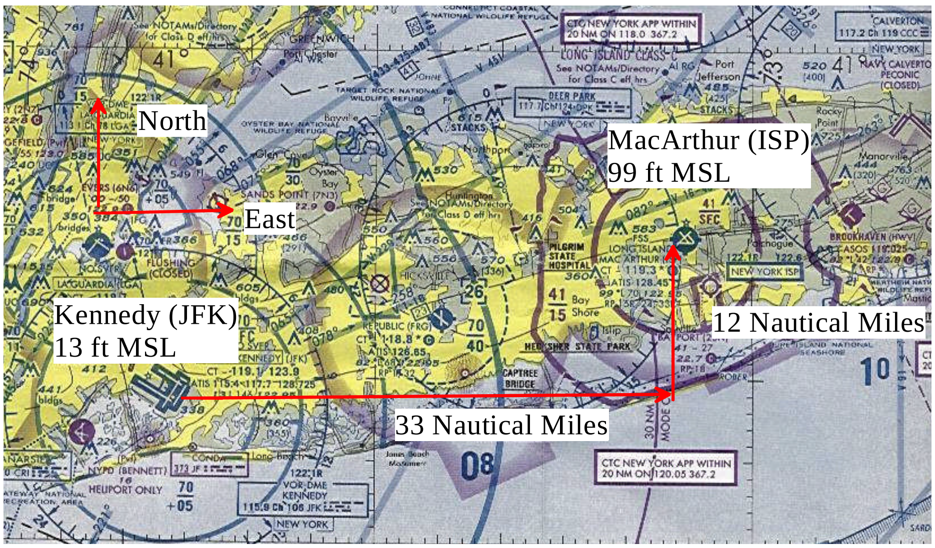
1.1 Identify whether the following physical quantities should be described as vectors or scalars

(a) Your age
(b) The distance between the Earth and the sun
(c) Forecast wind
(d) The gradient (slope) of a surface
(e) The linear momentum of an object
(f) The speed of light

2. A first look at vector components

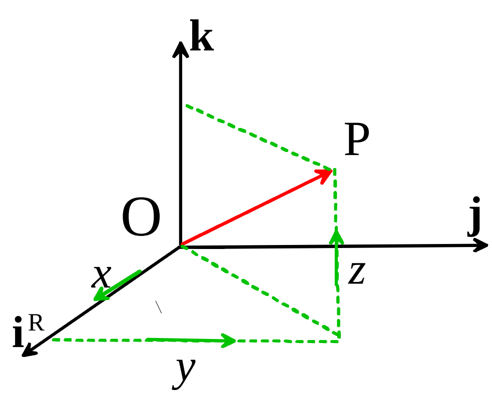
In practice, we almost always describe 3 dimensional vectors by specifying their components in a Cartesian basis.

Specifying the components of a vector is a lot like stating the position of a point on a map. For example, suppose we wish to specify the position of Long Island MacArthur airport relative to JFK airport on the map below. We might say that MacArthur is 33 Nautical miles East of JFK; 12 Nautical miles North, and and 86 feet (0.0142 Nautical Miles) above JFK.



The three distances (33 NM East, 12 NM North, 0.0142NM vertically) are the components of the position vector of ISP relative to JFK, in a Cartesian basis with its axes pointing East, North and vertically.

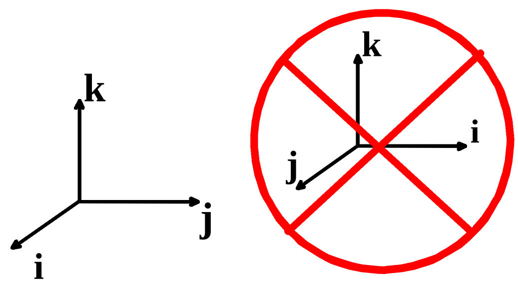
We follow the same procedure to specify the components of any vector. First, we choose three convenient, mutually perpendicular, reference directions as shown in the figure. The three reference directions are often given the symbols **i**, **j** and **k** or $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, and are denoted {**i,j,k**} or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for short. Then, we describe vectors by specifying how far you need to travel along each of the three reference directions to reach the tip of the vector from its tail, as shown in the picture. For example, to reach *P* from *O* in the figure, you need to travel a distance *x* along **i**, a distance *y* along **j** and a distance *z* along **k**. In mathematical notation this would be expressed as



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The three numbers *(x,y,z)* are called the *components* of the vector **r** in the basis {**i,j,k**}

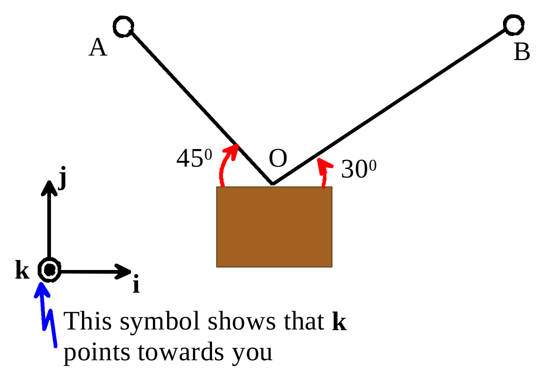
The reference directions {**i,j,k**} are arbitrary, except for two important restrictions. First, as we have already stated, the directions must be mutually perpendicular. Secondly, the directions must form a *right handed triad*, which means the arrows must be chosen so that it is possible to orient your right hand so that your thumb is parallel to **i**, your index finger is parallel to **j** and your middle finger is parallel to **k**. The figure below shows two bases: the one on the left is correct (it is a right handed triad) but the one on the right is not (it is a left handed triad)



Note that the three directions **i, j**, **k** can (and will) be regarded as vectors. Since they have direction but no well defined magnitude, we will choose them to be **unit vectors.**

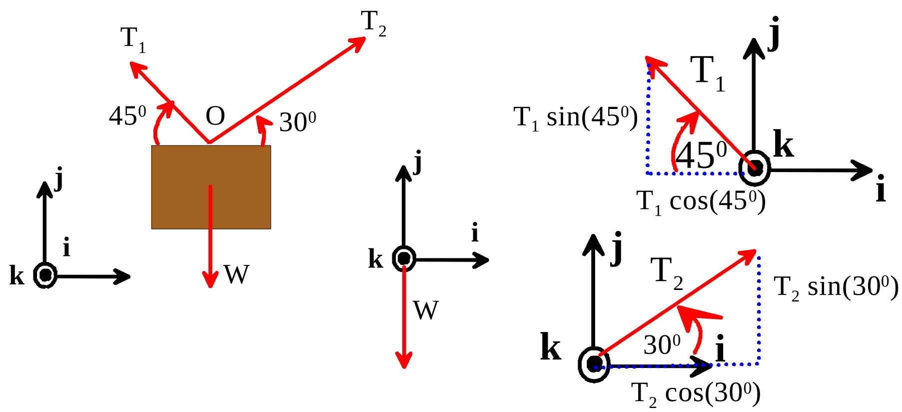
Example

The figure below shows a heavy box suspended from two cables. The box is subjected to a vertical gravitational force, and two forces of magnitude T_1, T_2 acting parallel to cables OA, and OB, respectively. Express each force as vector components in the basis shown.



Solution

A picture is always helpful



Now, remember that to write down the components of a vector, you need to specify the distance you travel in each of the **i,j,k** directions to reach the tip of the vector from its tail.

For the gravity force, we travel a distance $-W$ in the **j** direction. Therefore

$$\mathbf{F}_{gravity} = -W\mathbf{j}$$

For the tension in OA, we travel a distance $-T_1\cos(45) = -T_1/\sqrt{2}$ in the **i** direction, and $T_1\sin(45) = T_1/\sqrt{2}$ in the **j** direction. Therefore

$$\mathbf{F}_{OA} = -T_1/\sqrt{2}\ \mathbf{i} + T_1/\sqrt{2}\ \mathbf{j}$$

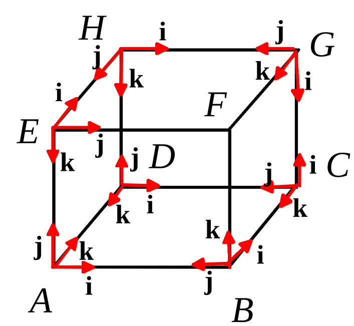
Finally, for the tension in OB, we travel a distance $T_2\cos(30) = T_2/2$ in the **i** direction, and $T_2\sin(30) = T_2\sqrt{3}/2$ in the **j** direction. Therefore

$$\mathbf{F}_{OB} = T_2/2\ \mathbf{i} + T_2\sqrt{3}/2\ \mathbf{j}$$

Writing down vector components always follows this general procedure.

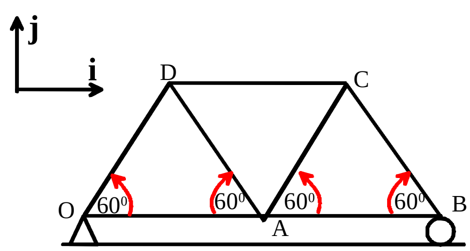
Problems

2.1 Consider the cube shown in the figure. Identify which of the {**i,j,k**} bases shown in the figure are right handed triads.

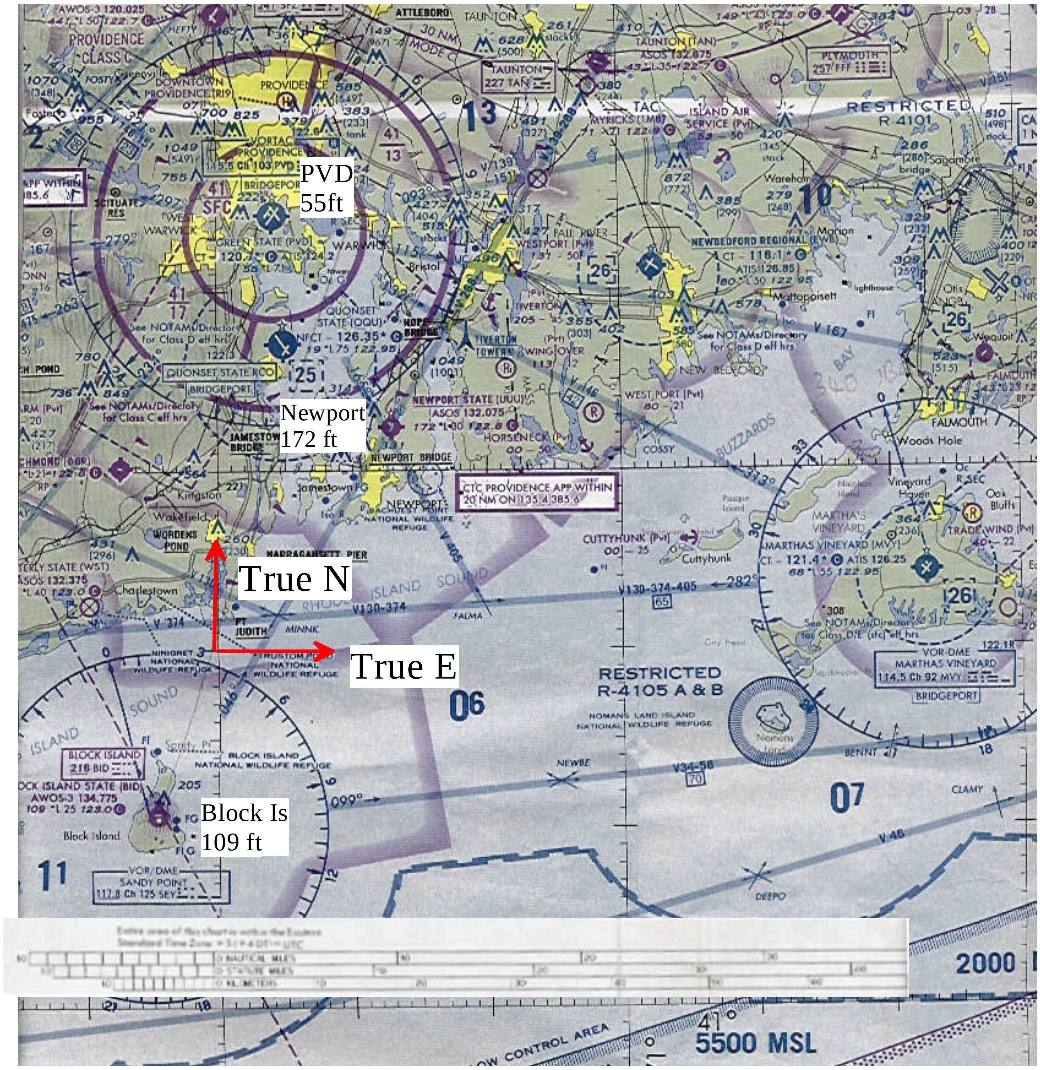


2.2 Consider the simple two-dimensional truss structure shown below. Each member has length 2m. Write down the following position vectors, expressing your answer as components in the basis shown, with physical dimensions of meters:

(a) The position vector of A relative to O (i.e. the vector pointing from O to A)
(b) The position of B relative to O
(c) The position of D relative to O
(d) The position of C relative to O
(e) The position of C relative to B
(f) The position of B relative to C



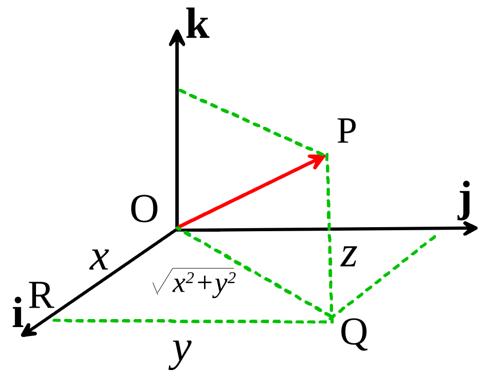
2.3 Consider the VFR Aeronautical Sectional Chart shown below. Establish a Cartesian basis with **i** pointing true East, **j** pointing true North and **k** perpendicular to the plane of the picture.



Write down the components of the following vectors in this basis, expressing your answer in Nautical Miles

(a) The position vector of Newport State Airport relative to Providence T.F. Green Airport. (The height of each airport in feet above mean sea level is shown near each airport – see figure)
(b) The position vector of Block Island State Airport relative to Newport State Airport
(c) The position vector of Block Island State Airport relative to Providence airport
(d) An aircraft at 2000 feet on a 10 mile final approach to Providence runway 23L. (An aircraft on final is aligned with the runway, and the number of the runway (23) indicates that the runway heading is 230 degrees magnetic. Magnetic variation at PVD is 15 degrees W, so 230 magnetic is 245 degrees true.)

3. How to calculate the magnitude of a vector in terms of its components



Let **r** be a vector and let

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where (x,y,z) are three numbers specifying the Cartesian components of the vector **r**. Find a formula for the length (or magnitude) of **r** in terms of (x,y,z).

Elementary geometry, my dear Watson. Consider the figure shown above. Observe that the magnitude of **r** is equal to the distance from O to P. Begin by calculating the distance from O to Q. Observe that OQR is a right angled triangle, so Pythagoras' theorem gives

$|OQ| = \sqrt{x^2 + y^2}$. Now observe further that OQP is a right angled triangle, so apply Pythagoras' theorem again to see that

$$|\mathbf{r}| = |OP| = \sqrt{|OQ|^2 + z^2} = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2}$$

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

Problems

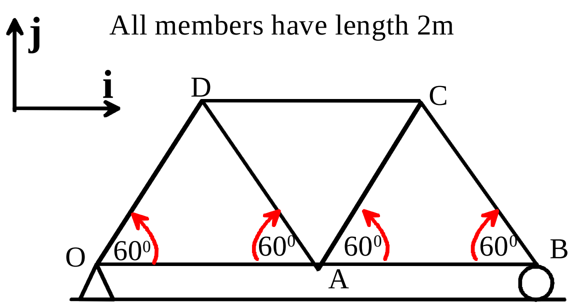
3.1 Calculate the magnitudes of each of the vectors shown below

(a) $\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

(b) $\mathbf{r} = 16\mathbf{i} + 6\mathbf{j}$

(c) $\mathbf{r} = -9.6\mathbf{j} + 2.4\mathbf{i} - 4.6\mathbf{k}$

3.2 For the truss shown below, find the magnitude of the position vector of C with respect to O.

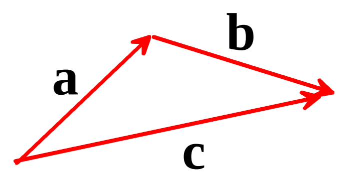


3.3 A vector has magnitude 3, and **i** and **j** components of 1 and 2, respectively. Calculate its **k** component.

3.4 Let {**i,j,k**} be a Cartesian basis. A vector **a** has magnitude 4 and subtends angles of 30 degrees and 100 degrees to the **i** and **k** directions, respectively. Calculate the components of **a** in the basis **{i,j,k}**

4. Addition of vectors

Let **a** and **b** be vectors. Then (by definition) $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is also a vector. Vector addition satisfies $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (again, by definition). The vector **c** may be shown diagramatically by placing arrows representing **a** and **b** head to tail, as shown.



4.1 Formula for the sum of two vectors in Cartesian components

Let

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

where (a_x, a_y, a_z), (b_x, b_y, b_z) are the Cartesian components of vectors **a**, **b** in a basis $\{\mathbf{i,j,k}\}$.

Let

$$\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$$

If **c=a+b** calculate (c_x, c_y, c_z) in terms of (a_x, a_y, a_z), (b_x, b_y, b_z)

Just Do It!

$$\begin{aligned} \mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k} = \ \mathbf{a} + \mathbf{b} &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \ + \ b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \\ &= (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k} \end{aligned}$$

and so comparing coefficients of **i,j** and **k**

$$c_x = a_x + b_x \qquad c_y = a_y + b_y, \qquad c_z = a_z + b_z$$

Problems

4.1 Find the sum of the vectors listed below, expressing your answer as components in the **{i,j,k}** basis. Also compute the magnitude of each vector and the magnitude of their sum.

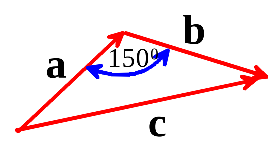
(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 6\mathbf{k}$

(c) $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{b} = \xi\mathbf{i} - \eta\mathbf{j} + \varsigma\mathbf{k}$

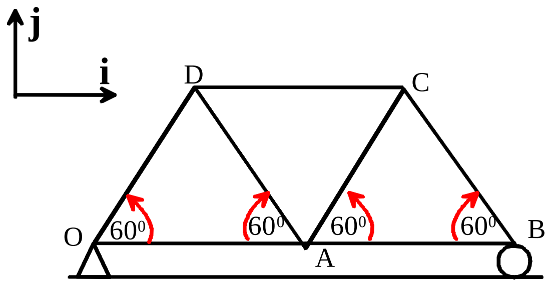
4.2 For each of the vectors listed in the preceding section, calculate **a-b**.

4.3 The vectors **a** and **b** shown in the figure have magnitudes $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$. Calculate the magnitude of the vector **c**.

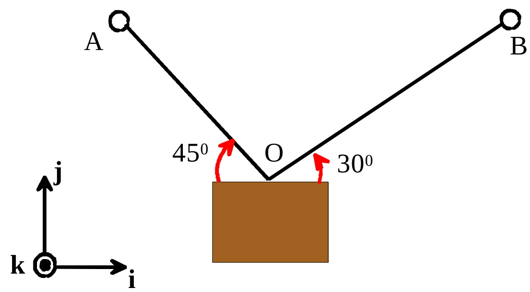


4.4 For the structure shown, write down the position vectors of B relative to O, C relative to B and C relative to O. Verify your answer by checking that

$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$



4.5 The figure shows a heavy box suspended from two cables. The box is subjected to a vertical gravitational force, and two forces of magnitude T_1, T_2 acting parallel to cables OA, and OB, respectively. Express each force as vector components in the basis shown. Write down the vector sum of the forces.



5. Multiplication of vectors

5.1 Multiplication by a scalar.

Let $\mathbf{a}$ be a vector, and α a scalar. Then $\mathbf{b} = \alpha\,\mathbf{a}$ is a vector. The direction of $\mathbf{b}$ is parallel to $\mathbf{a}$ and its magnitude is given by $|\mathbf{b}| = \alpha|\mathbf{a}|$.

Note that you can form a unit vector $\mathbf{n}$ which is parallel to $\mathbf{a}$ by setting

$$\mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

5.2 Formula for the product of a scalar and a vector in Cartesian Components

Let $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ be a vector and α a scalar. Find an expression for the components of the vector $\mathbf{b} = \alpha\mathbf{a}$ Then

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} = \alpha\mathbf{a} = \alpha\left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right) = \alpha a_x\mathbf{i} + \alpha a_y\mathbf{j} + \alpha a_z\mathbf{k}$$

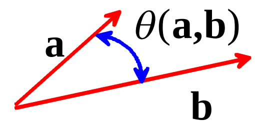
and hence

$$b_x = \alpha a_x, \quad b_y = \alpha a_y, \quad b_z = \alpha a_z$$

Problems

5.1 Find the components of a unit vector parallel to the vector $\mathbf{a} = 5\mathbf{i} + 6\mathbf{k}$
5.2 Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 6\mathbf{k}$. Find 6**a**+2**b**, and 6**a**-2**b**.

5.3 Dot Product



(also called the scalar product). Let **a** and **b** be two vectors. The dot product of **a** and **b** is a scalar denoted by $\alpha = \mathbf{a}\cdot\mathbf{b}$, and is defined by

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta(\mathbf{a},\mathbf{b}),$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ denote the magnitudes of **a** and **b**, respectively, and $\theta(\mathbf{a},\mathbf{b})$ is the angle subtended by **a** and **b**, as shown in the figure.

Note that $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$, and $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$. If $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$ then $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if $\cos\theta(\mathbf{a},\mathbf{b}) = 0$; i.e. **a** and **b** are perpendicular.

5.4 Formula for the dot product of two vectors in Cartesian Components

Let

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$$

where (a_x, a_y, a_z), (b_x, b_y, b_z) are the Cartesian components of vectors **a**, **b** in a basis{**i,j,k**}.

Calculate $\mathbf{a}\cdot\mathbf{b}$ in terms of (a_x, a_y, a_z), (b_x, b_y, b_z).

This time we have to do some real work. Substitute for **a** and **b** and see what happens

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_xb_x\mathbf{i}\cdot\mathbf{i} + a_xb_y\mathbf{i}\cdot\mathbf{j} + a_xb_z\mathbf{i}\cdot\mathbf{k} \\ &\quad + a_yb_x\mathbf{j}\cdot\mathbf{i} + a_yb_y\mathbf{j}\cdot\mathbf{j} + a_yb_z\mathbf{j}\cdot\mathbf{k} \\ &\quad + a_zb_x\mathbf{k}\cdot\mathbf{i} + a_zb_y\mathbf{k}\cdot\mathbf{j} + a_zb_z\mathbf{k}\cdot\mathbf{k}\end{aligned}$$

This is a mess. But recall that **I**, **j** and **k** are mutually perpendicular, so the angle between them is 90 degrees. Recall also that cos(90) = 0. Finally, recall the definition of the dot product. Therefore $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$.

This leaves

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x\mathbf{i}\cdot\mathbf{i} + a_yb_y\mathbf{j}\cdot\mathbf{j} + a_zb_z\mathbf{k}\cdot\mathbf{k}$$

Finally, note that the a vector is always parallel to itself, so the angle between a vector and itself is zero. Recall also that **I**, **j** and **k** are all **unit vectors**. Therefore $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}||\mathbf{i}|\cos(0) = 1$, and so on for all three remaining dot products. So, finally

$$\mathbf{a}\cdot\mathbf{b} = a_xb_x + a_yb_y + a_zb_z$$

Problems

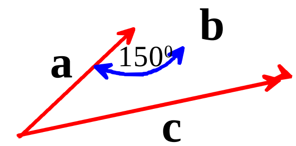
5.3 Find the dot products of the vectors listed below

(a) $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$

(b) $\mathbf{a}=3\mathbf{i}+2\mathbf{j}$, $\mathbf{b}=4\mathbf{i}+6\mathbf{k}$

(c) $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\mathbf{b}=\xi\mathbf{i}-\eta\mathbf{j}+\varsigma\mathbf{k}$

5.4 The vectors **a** and **b** shown in the figure below have magnitudes $|\mathbf{a}|=3$, $|\mathbf{b}|=5$. Calculate $\mathbf{a}\cdot\mathbf{b}$.

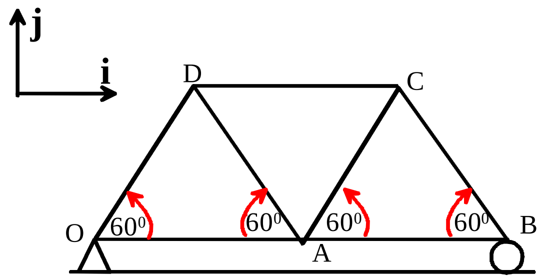


5.5 Two vectors **a** and **b** are mutually perpendicular. What is their dot product?

5.6 Calculate $\mathbf{j}\cdot(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

5.7 Calculate the angle between each pair of vectors listed in Problem 5.3 – i.e. find the angle $\theta(\mathbf{a},\mathbf{b})$ between **a** and **b** in each case

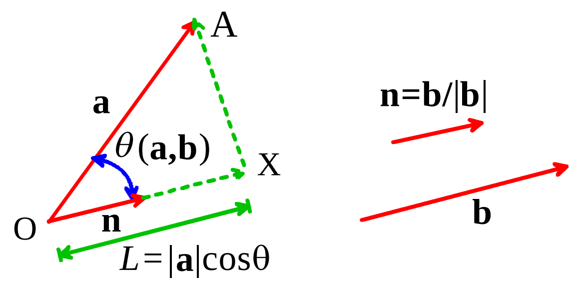
5.8 For the structure shown, calculate the angle between the vectors $\mathbf{a}=\overrightarrow{OB}$ (i.e. the vector pointing from O to B) and $\mathbf{b}=\overrightarrow{OC}$. (Use vectors – it's possible to do this by long-winded trigonometry and Pythagoras theorem but that's not the point)



5.5 Dot Product as a Projection

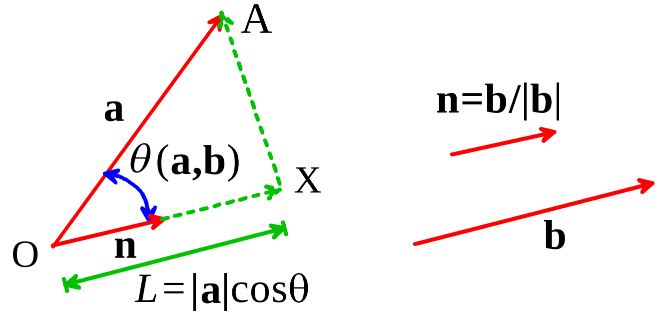
The quantity $\mathbf{a}\cdot\mathbf{b}/|\mathbf{b}|$ is sometimes referred to as *the component of* ***a*** *in a direction parallel to* ***b***. The figure shows why.

The vector **a** can be thought of as the sum of two vectors: one (OX) parallel to **b** and another (XA) perpendicular to **b**. This process of dividing **a** into two parts is known as *projecting* **a** *onto components parallel and perpendicular to* **b**.



Recall that |**a**| is the length of OA. The length of OX is therefore $L=|\mathbf{a}|\cos\theta(\mathbf{a},\mathbf{b})$. But recall that $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta(\mathbf{a},\mathbf{b})$, so that $L=\mathbf{a}\cdot\mathbf{b}/|\mathbf{b}|$, as stated.

Problems



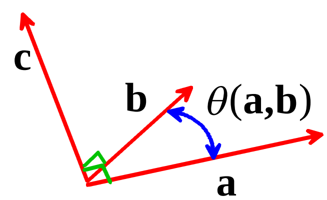
5.9 Let **a** and **b** be two vectors. Project **a** onto components parallel and perpendicular to **b** as shown in the picture.

(i) Show that the vector $\overrightarrow{XA} = \mathbf{a} - \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2} \right)\mathbf{b}$

(ii) Verify that the preceding result satisfies $\overrightarrow{XA}\cdot\mathbf{b} = 0$, as it should (why?)

(iii) Show that the component of **a** in a direction perpendicular to **b** is $|XA| = \sqrt{|\mathbf{a}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 / |\mathbf{b}|^2}$

5.5 Cross Product (also called the vector product).



Let **a** and **b** be two vectors. By definition, the cross product of **a** and **b** is a vector, denoted by $\mathbf{c} = \mathbf{a}\times\mathbf{b}$. The direction of **c** is perpendicular to both **a** and **b,** and is chosen so that (**a,b,c**) form a right handed triad, as shown. The magnitude of **c** is given by

$$|\mathbf{c}| = |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta(\mathbf{a},\mathbf{b})$$

Note that $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ and $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = 0$.

Calculating the magnitude of the cross product of two vectors is no sweat, but figuring out the direction is a pain. There are various aide-memoirs to help you do this- choose the one you find least confusing, or make up your own.

Right hand rule

To find the direction of $\mathbf{a}\times\mathbf{b}$, arrange your right hand so that your thumb is parallel to **a**, your index finger is parallel to **b**, and the angle between your thumb and index finger is θ. Now set your middle finger is perpendicular to both **a** and **b.** The direction of $\mathbf{a}\times\mathbf{b}$ is parallel to your middle finger. (This rule only really works if $\theta < 90^0$, otherwise you permanently damage your hand. Please don't do this.)

Right hand screw rule

To find the direction of $\mathbf{a}\times\mathbf{b}$, arrange your right hand so that your thumb is perpendicular to both **a** and **b,** and your fingers curl in the direction of the line joining the tip of vector **a** to the tip of vector **b**. The direction of $\mathbf{a}\times\mathbf{b}$ is parallel to your thumb.

Bottle-cap rule.

Obtain a twist-top bottle of your favorite beverage. Draw an arrow on the cap. Arrange the bottle so that, by twisting the cap through the angle θ, you can rotate the arrow from parallel to **a** to parallel to **b**. The direction of $\mathbf{a}\times\mathbf{b}$ is parallel to the direction of motion of the bottle-cap as it is turned. (Full beverage containers are not be permitted in EN3 examinations)

If none of these tricks help you

Extend your middle finger into the air. Shout your favorite expletive. This will not help, but it may make you feel better.

Problem

5.10 Let {**i,j,k**} be a Cartesian basis. Use the definition of the cross product given above to calculate all possible cross products of the basis vectors –i.e., calculate

$$\begin{array}{ccc} \mathbf{i}\times\mathbf{i}, & \mathbf{i}\times\mathbf{j} & \mathbf{i}\times\mathbf{k} \\ \mathbf{j}\times\mathbf{i} & \mathbf{j}\times\mathbf{j} & \mathbf{j}\times\mathbf{k} \\ \mathbf{k}\times\mathbf{i} & \mathbf{k}\times\mathbf{j} & \mathbf{k}\times\mathbf{k} \end{array}$$

You will find that the results are all very simple. For example, $|\mathbf{i}\times\mathbf{i}|=|\mathbf{i}||\mathbf{i}|\sin\theta(\mathbf{i},\mathbf{i})=0$, since **I** is parallel to itself. Hence $\mathbf{i}\times\mathbf{i}=\mathbf{0}$. Similarly $|\mathbf{i}\times\mathbf{j}|=|\mathbf{i}||\mathbf{j}|\sin\theta(\mathbf{i},\mathbf{j})=1$, since **I** and **j** are both unit vectors and the angle between them is 90 degrees. The rules governing the direction of a cross product also show that $\mathbf{i}\times\mathbf{j}$ is parallel to **k**. Therefore
$\mathbf{i}\times\mathbf{j}=\mathbf{k}$. See if you can work out the rest on your own.

5.6 Formula for the cross product of two vectors in Cartesian Components.

Let

$$\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$$
$$\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}$$

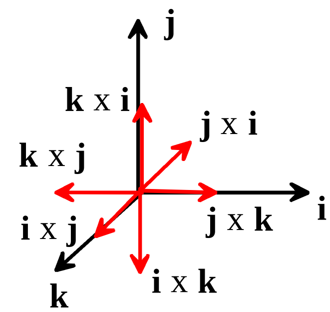
where $\left(a_x,a_y,a_z\right)$, $\left(b_x,b_y,b_z\right)$ are the Cartesian components of vectors **a**, **b** in a basis{**i,j,k**}.

Calculate $\mathbf{a}\times\mathbf{b}$ in terms of $\left(a_x,a_y,a_z\right)$, $\left(b_x,b_y,b_z\right)$.

More work for the wicked. Substitute for **a** and **b**

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \left(a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}\right)\times\left(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}\right) \\ &= a_xb_x\mathbf{i}\times\mathbf{i} + a_xb_y\mathbf{i}\times\mathbf{j} + a_xb_z\mathbf{i}\times\mathbf{k} \\ &\quad + a_yb_x\mathbf{j}\times\mathbf{i} + a_yb_y\mathbf{j}\times\mathbf{j} + a_yb_z\mathbf{j}\times\mathbf{k} \\ &\quad + a_zb_x\mathbf{k}\times\mathbf{i} + a_zb_y\mathbf{k}\times\mathbf{j} + a_zb_z\mathbf{k}\times\mathbf{k}\end{aligned}$$

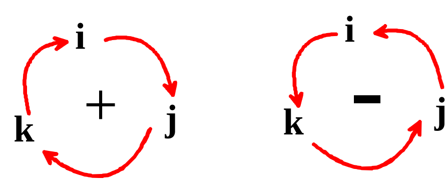
This is another mess. This time, note that $|\mathbf{i}\times\mathbf{i}| = |\mathbf{i}||\mathbf{i}|\sin\theta(\mathbf{i},\mathbf{i})$ (and similarly for $\mathbf{j}\times\mathbf{j}$ and $\mathbf{k}\times\mathbf{k}$), note that the angle $\theta(\mathbf{i},\mathbf{i})$ between **I** and itself is zero, and recall that sin(0)=0. Therefore $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$.



The remaining cross products between **i,j,k** have to be calculated laboriously one at a time, using the definition given in the preceding section. The figure shows the direction of all six possible cross products between the basis vectors (magnitudes are not shown to scale, for clarity). Thus, we conclude that

$$
\begin{array}{lll}
\mathbf{i}\times\mathbf{i} = \mathbf{0} & \mathbf{i}\times\mathbf{j} = \mathbf{k} & \mathbf{i}\times\mathbf{k} = -\mathbf{j} \\
\mathbf{j}\times\mathbf{i} = -\mathbf{k} & \mathbf{j}\times\mathbf{j} = \mathbf{0} & \mathbf{j}\times\mathbf{k} = \mathbf{i} \\
\mathbf{k}\times\mathbf{i} = \mathbf{j} & \mathbf{k}\times\mathbf{j} = -\mathbf{i} & \mathbf{k}\times\mathbf{k} = \mathbf{0}
\end{array}
$$

THESE FORMULAS ARE IMPORTANT!



You need to remember them. There is a nice little trick to help you. Write down the 3 vectors **i**, **j**, **k** in a circle, going clockwise, as shown below.

Now, to find the cross product of any pair of basis vectors, you travel around the circle. Thus, to get $\mathbf{i}\times\mathbf{j}$, you start at **i**, move to **j** and then on to **k**. If you go around the circle clockwise, the answer is positive, if you go counter-clockwise, it is negative. Thus, $\mathbf{j}\times\mathbf{k} = +\mathbf{i}$, and so on, while $\mathbf{k}\times\mathbf{k} = -\mathbf{i}$, etc.

If we substitute these results into our expression for $\mathbf{a}\times\mathbf{b}$ we determine that

$$
\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right)\times\left(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}\right) \\
&= a_x b_y\mathbf{k} - a_x b_z\mathbf{j} \\
&\quad - a_y b_x\mathbf{k} + a_y b_z\mathbf{i} \\
&\quad + a_z b_x\mathbf{j} - a_z b_y\mathbf{i} \\
&= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}
\end{aligned}
$$

Hence

$$
c_x = (a_y b_z - a_z b_y), \quad c_y = (a_z b_x - a_x b_z), \quad c_z = (a_x b_y - a_y b_x)
$$

This is not an easy formula to remember, but it is so important that you must memorize it. The following trick is sometimes used to help remember the formula – if you know how to calculate the determinant of a matrix, then you will note that

$$\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_y b_z - a_z b_y)\mathbf{i} \quad + \quad (a_z b_x - a_x b_z)\mathbf{j} \quad + \quad (a_x b_y - a_y b_x)\mathbf{k}$$

It is fairly easy to remember the cover-up rule for computing the determinant of a matrix, so this is a popular trick.

Another way to remember the formula is to notice the pattern in the indices. The indices are written out below to show the pattern more clearly

$$\begin{aligned} c_x &= (a_y b_z - a_z b_y), &\qquad &\text{indices } x,y,z - x,z,y \\ c_y &= (a_z b_x - a_x b_z), &\qquad &\text{indices } y,z,x - y,x,z \\ c_z &= (a_x b_y - a_y b_x), &\qquad &\text{indices } z,x,y - z,y,x \end{aligned}$$

There are two things to notice about this pattern. First, note that the expression for c_x involves only a_y, a_z and b_y, b_z, similarly, the expression for c_y involves only a_x, a_z and b_x, b_z, and the third expression has the same feature. Secondly, notice that the indices always appear both forwards (*x,y,z* or *y,z,x* or *z,x,y*) and backwards (*x,z,y* or *y,x,z* or *z,y,x*) in each expression. The forward terms (*x,y,z* or *y,z,x* or *z,x,y*) are all positive, while the backward terms (*x,z,y* or *y,x,z* or *z,y,x*) are all negative.

Problems

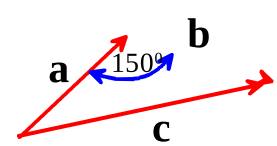
5.11 Find the cross products of the vectors listed below

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 6\mathbf{k}$

(c) $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{b} = \xi\mathbf{i} - \eta\mathbf{j} + \varsigma\mathbf{k}$

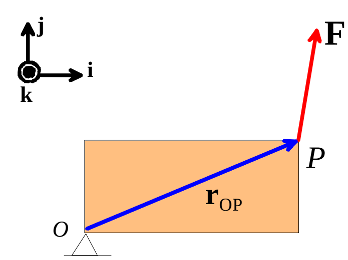
5.12 The vectors **a** and **b** shown in the figure have magnitudes $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$. Calculate $|\mathbf{a}\times\mathbf{b}|$. What is the direction of $\mathbf{a}\times\mathbf{b}$?



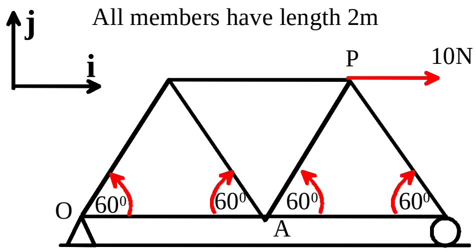
5.13 A force **F** acts at some point *P* on a solid object, as shown in the figure. By definition, the moment of the force about an arbitrary point *O* is a vector **M**, defined as

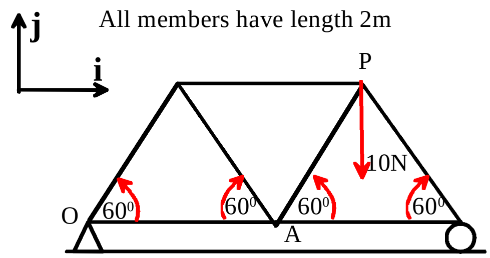
$$\mathbf{M} = \mathbf{r}_{OP} \times \mathbf{F}$$

where $\mathbf{r}_{OP}$ is the position vector of point *P* relative to *O*.



For each figure shown below, write down the force **F** and the position vector $\mathbf{r}_{OP}$ as components in the basis shown, and hence calculate the vector moment **M** of the force about the point O.

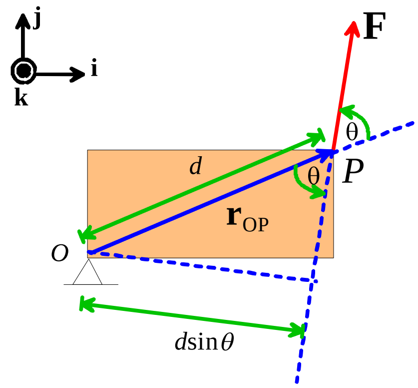




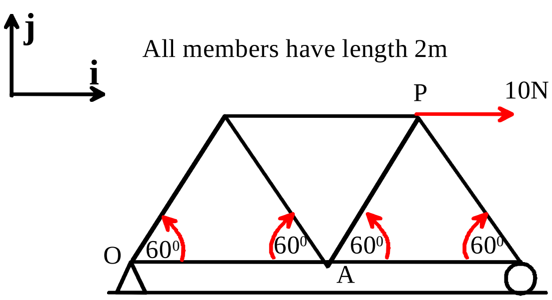
5.14 There is a more direct way to calculate the moment of a force, which avoids having to write out components of the position vector and force and then taking the cross product. Instead, recalling the definition of a cross product, we note that the magnitude of the moment is

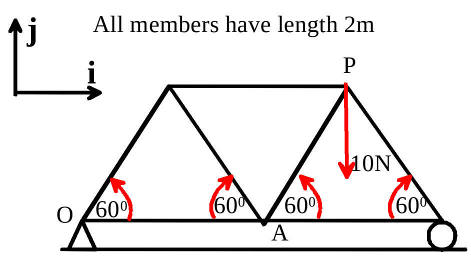
$$|\mathbf{M}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{F}||\mathbf{r}|\sin\theta(\mathbf{r},\mathbf{F})$$

Observe that $|\mathbf{r}|\sin\theta(\mathbf{r},\mathbf{F})$ is the perpendicular distance from O to a line drawn parallel to **F** through its point of action, as shown in the picture. Thus, to calculate the magnitude of a moment, you only need to find this perpendicular distance, and multiply it by the magnitude of the force. The direction of **M** can be deduced using the usual rules.



Calculate the required perpendicular distance in each problem below, and hence deduce **M** for each figure.



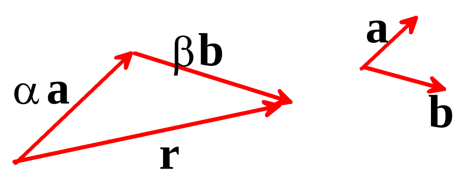


6. A New Look at Vector Components

Finally, we take a new look at what we are doing when we express vectors as components in a basis. First, two theorems.

6.1 THEOREM 1.

Let **a** and **b** be two non-collinear vectors. Then any vector **r** which is coplanar with **a** and **b** can be expressed as a linear combination of **a** and **b**, that is to say, there exist two scalar numbers α and β such that $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$.



It is easiest to see this graphically. Recall that a vector **r** can be regarded as connecting two points in a plane. If **a** and **b** lie in the same plane, it is always possible to get from one end of the vector to the other by traveling along a path parallel to **a** and **b**.

In fact, we can even find a formula for the two numbers α and β. Recall that $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$. We can turn this into two scalar equations by taking dot products of both sides with **a** and **b** in turn

$$\mathbf{r}\cdot\mathbf{a} = \alpha\mathbf{a}\cdot\mathbf{a} + \beta\mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{r}\cdot\mathbf{b} = \alpha\mathbf{a}\cdot\mathbf{b} + \beta\mathbf{b}\cdot\mathbf{b}$$

Solve for α and β.

$$\alpha = \frac{(\mathbf{r}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{r}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2}$$

$$\beta = \frac{(\mathbf{r}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{r}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{b})}{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2}$$

This is messy, but it looks a bit better if we choose **a** and **b** to be *unit vectors*, in which case $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = 1$

$$\alpha = \frac{(\mathbf{r}\cdot\hat{\mathbf{a}}) - (\mathbf{r}\cdot\hat{\mathbf{b}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})}{1 - (\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^2}$$

$$\beta = \frac{(\mathbf{r}\cdot\hat{\mathbf{b}}) - (\mathbf{r}\cdot\hat{\mathbf{a}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})}{1 - (\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})^2}$$

(We put little hats on the vectors to show that you can only use the formula for unit vectors) We really blow our minds if we also choose **a** and **b** to be mutually perpendicular so $\mathbf{a}\cdot\mathbf{b} = 0$

$$\alpha = (\mathbf{r}\cdot\hat{\mathbf{a}}) \qquad \beta = (\mathbf{r}\cdot\hat{\mathbf{b}}) \qquad \hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = 0$$

6.2 THEOREM II

The same sort of thing works in three dimensions. In this case, let **a**, **b** and **c** be three non-coplanar, non-collinear vectors. Then any vector **r** can be represented as a linear combination of **a**, **b**, and **c**, i.e. there exist three scalars α , β and γ such that $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

We could, if we really wanted to, find a general expression for α , β and γ , but the results are so complicated it's not really worth the effort. However, if we choose **a**, **b**, and **c** to be mutually perpendicular, unit vectors, we find that

$$\alpha = \mathbf{r}\cdot\hat{\mathbf{a}} = |\mathbf{r}|\cos\theta(\mathbf{r},\hat{\mathbf{a}})$$
$$\beta = \mathbf{r}\cdot\hat{\mathbf{b}} = |\mathbf{r}|\cos\theta(\mathbf{r},\hat{\mathbf{b}}) \qquad \hat{\mathbf{a}}\cdot\hat{\mathbf{b}} = \hat{\mathbf{a}}\cdot\hat{\mathbf{c}} = \hat{\mathbf{b}}\cdot\hat{\mathbf{c}} = 0$$
$$\gamma = \mathbf{r}\cdot\hat{\mathbf{c}} = |\mathbf{r}|\cos\theta(\mathbf{r},\hat{\mathbf{c}})$$

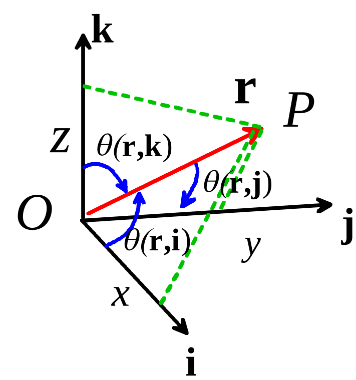
These results give us a new insight into what it means to express a vector as components in a basis. Here's the scoop.

Truth be told, we don't actually like vectors very much. (This may be the first statement in this tutorial you really appreciate). Calculating sums and products of arbitrary vectors is a pain. So, in any problem we solve, we use as few vectors as possible. In two dimensions, we pick two convenient vectors $\{\mathbf{i},\mathbf{j}\}$and then express all vectors as a sum of these two, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. In three dimensions, we need to pick three reference vectors, $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, then we can make all other vectors a sum of these $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

We know from our Theorems that it is very helpful if we pick our reference vectors to be mutually perpendicular unit vectors. In this case, we get a very convenient formula for the three numbers *x*, *y* and *z*.

$$x = \mathbf{r}\cdot\mathbf{i} = |\mathbf{r}|\cos\theta(\mathbf{r},\mathbf{i})$$
$$y = \mathbf{r}\cdot\mathbf{j} = |\mathbf{r}|\cos\theta(\mathbf{r},\mathbf{j})$$
$$z = \mathbf{r}\cdot\mathbf{k} = |\mathbf{r}|\cos\theta(\mathbf{r},\mathbf{k})$$

The figure shows the three angles $\theta(\mathbf{r},\mathbf{i}),\ \theta(\mathbf{r},\mathbf{j}),\ \theta(\mathbf{r},\mathbf{k})$.



We see that *x*, *y* and *z* correspond to the projections of **r** on the three basis vectors, precisely as we assumed when first writing down vectors as components in a basis.

This discussion has given us new insight into what we are doing in expressing vectors as components in a basis. It shows why our three reference directions can be regarded as vectors; it shows why they should be unit vectors, and why the vectors should be mutually perpendicular. It does not explain why the three basis vectors must form a right handed triad -- This is done so that we get the correct expression for the component form for a cross product (Sect 5.6)

Note, however, that mathematically speaking we did everything backwards in this tutorial. Strictly speaking, we should have started with the definition of a vector sum (if **a** and **b** are vectors then **c**=**a**+**b** is a vector), define the dot product ($\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta(\mathbf{a},\mathbf{b})$), and then deduce the existence of a basis as we did in this section. Then, finally, we can deduce expressions for vector operations in component form.

Problems

6.1 Let **a**=5**i**-6**k**; **b**=4**i**+2**k** Let **r**=5**k**. Express **r** as components parallel to **a** and **b**, i.e. find two scalars α and β such that $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$

6.3 Direction Cosines of a vector

We see from the preceding section that all vectors can be represented in a Cartesian basis as

$$\mathbf{r} = |\mathbf{r}|(\cos\theta(\mathbf{r},\mathbf{i})\ \mathbf{i}\ +\ \cos\theta(\mathbf{r},\mathbf{j})\ \mathbf{j}\ +\ \cos\theta(\mathbf{r},\mathbf{k})\ \mathbf{k})$$

The three numbers $\cos\theta(\mathbf{r},\mathbf{i})$, $\cos\theta(\mathbf{r},\mathbf{j})$, $\cos\theta(\mathbf{r},\mathbf{k})$ are known as the *direction cosines* of a vector. This is because they are cosines, and specify the direction of the vector. Duh.

It is straightforward to calculate the direction cosines of a vector if you know its components. For example, if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the three direction cosines are

$$\frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}}$$

See if you can show this for yourself.

Problems

6.2 Find the direction cosines of the following vectors

(a) $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$

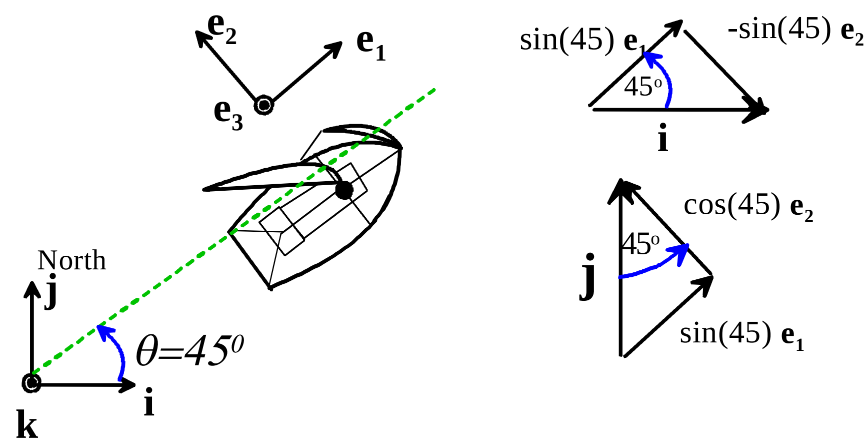
6.4 Change of basis

Next, we ask an obvious question. Supposing we are given all our vectors as components in some basis {**i**,**j**,**k**}, but for some reason we don't like this basis, and would prefer to know our vector as components in another basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (Here, the three **e**s represent mutually perpendicular unit vectors, just like **i**,**j**,**k**). How do we convert from one to the other?

For two dimensional problems, the easiest procedure is to find how to construct each of the **i**,**j**,**k** vectors by adding up $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and then substitute. We will illustrate this using an example.

In the figure shown below, the sailboat travels in a Northeasterly direction. The wind is $-5\mathbf{i}+10\mathbf{j}$ knots. Find the components of the wind vector in a basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ aligned with the sailboat.

.



To proceed, we will express **i** and **j** in terms of $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Recall that **i** and **j** have unit length, therefore we can combine the **e** basis vectors as shown in the figure to make up the **i** and **j** vectors. From the figure

$$\mathbf{i}=\cos(45)\mathbf{e}_1-\sin(45)\mathbf{e}_2=\frac{1}{\sqrt{2}}\mathbf{e}_1-\frac{1}{\sqrt{2}}\mathbf{e}_2$$

$$\mathbf{j}=\sin(45)\mathbf{e}_1+\cos(45)\mathbf{e}_2=\frac{1}{\sqrt{2}}\mathbf{e}_1+\frac{1}{\sqrt{2}}\mathbf{e}_2$$

so plug into the expression for the wind

$$\mathbf{w}=-5\mathbf{i}+10\mathbf{j}=-5\left[\frac{1}{\sqrt{2}}\mathbf{e}_1-\frac{1}{\sqrt{2}}\mathbf{e}_2\right]+10\left[\frac{1}{\sqrt{2}}\mathbf{e}_1+\frac{1}{\sqrt{2}}\mathbf{e}_2\right]$$

$$=\frac{5}{\sqrt{2}}\mathbf{e}_1+\frac{15}{\sqrt{2}}\mathbf{e}_2 \text{ knots}$$

For three dimensional problems, it pays to be more systematic. Suppose we know the components of a vector **r** in {**i**,**j**,**k**} are (*x*,*y*,*z*), and we wish to calculate the components (ξ,η,ς) in $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$.

To proceed, we go back to the fundamental definition of the basis vectors, and note that **r** can be written as

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\xi\mathbf{e}_1+\eta\mathbf{e}_2+\varsigma\mathbf{e}_3$$

A clever trick allows us to solve for (ξ,η,ς). If we take the dot product of both sides with $\mathbf{e}_1$, then

$$x\mathbf{i}\cdot\mathbf{e}_1+y\mathbf{j}\cdot\mathbf{e}_1+z\mathbf{k}\cdot\mathbf{e}_1=\xi\mathbf{e}_1\cdot\mathbf{e}_1+\eta\mathbf{e}_2\cdot\mathbf{e}_1+\varsigma\mathbf{e}_3\cdot\mathbf{e}_1$$

But recall that $\mathbf{e}_1\cdot\mathbf{e}_1=1$, $\mathbf{e}_1\cdot\mathbf{e}_2=\mathbf{e}_1\cdot\mathbf{e}_3=0$, so that

$$x\mathbf{i}\cdot\mathbf{e}_1+y\mathbf{j}\cdot\mathbf{e}_1+z\mathbf{k}\cdot\mathbf{e}_1=\xi$$

Similarly, taking dot products with the other two **e** vectors gives

$$x\mathbf{i}\cdot\mathbf{e}_1 + y\mathbf{j}\cdot\mathbf{e}_1 + z\mathbf{k}\cdot\mathbf{e}_1 = \xi$$
$$x\mathbf{i}\cdot\mathbf{e}_2 + y\mathbf{j}\cdot\mathbf{e}_2 + z\mathbf{k}\cdot\mathbf{e}_2 = \eta$$
$$x\mathbf{i}\cdot\mathbf{e}_3 + y\mathbf{j}\cdot\mathbf{e}_3 + z\mathbf{k}\cdot\mathbf{e}_3 = \varsigma$$

Finally, we have to calculate $\mathbf{i}\cdot\mathbf{e}_1$ and all the rest. This is done either by finding the angles between the appropriate vectors and using the definition of a dot product, or, if we are lucky, we know the components of each **e** vector in {**i**,**j**,**k**} in which case we can evaluate the dot product directly.

Let's try this out on our sailboat example. In this case *x=-5*, *y=10*, *z=0*, and

$$\mathbf{i}\cdot\mathbf{e}_1 = \cos(45) = \frac{1}{\sqrt{2}} \qquad \mathbf{i}\cdot\mathbf{e}_2 = -\sin(45) = -\frac{1}{\sqrt{2}} \qquad \mathbf{i}\cdot\mathbf{e}_3 = 0$$
$$\mathbf{j}\cdot\mathbf{e}_1 = \sin(45) = \frac{1}{\sqrt{2}} \qquad \mathbf{j}\cdot\mathbf{e}_2 = \cos(45) = \frac{1}{\sqrt{2}} \qquad \mathbf{j}\cdot\mathbf{e}_3 = 0$$

So, plug everything into the magic formula to see that

$\xi = \frac{5}{\sqrt{2}} \qquad \eta = \frac{15}{\sqrt{2}} \qquad \varsigma = 0$ knots, giving the same answer as before (Phew!)

Problems

6.3 Repeat the sailboat problem again, but with the boat traveling at 30 degrees to the **i** direction.

6.4 Let $\mathbf{a} = 5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{c} = 45\mathbf{i} - 30\mathbf{j} + 15\mathbf{k}$.

(a) Verify that **a**, **b** and **c** are mutually perpendicular and that $\mathbf{c} = \mathbf{a}\times\mathbf{b}$

(b) In view of (a), three unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ parallel to **a b** and **c** can form a basis. Calculate the components of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the {**i**,**j**,**k**} basis.

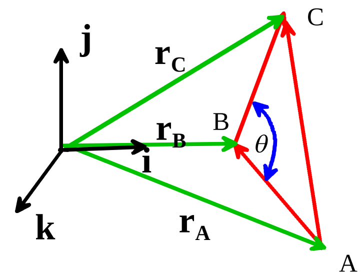
(c) Let **r**=4**i**+6**k**. Calculate the components of **r** in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. (Use your answer to (b) to calculate the required dot products in the formula)

7. Fun Activities with Vectors

You will use vectors in a majority of your engineering courses (if you plan to leave engineering after EN3, you will use vectors in *all* your engineering courses!). They are used in dynamics, fluid flow problems, mechanics of deformable solids, electric and magnetic fields, heat flow problems, among others. Here, we just illustrate a few applications.

7.1 Calculating areas.

Many geometry problems are nicely solved using vectors. We'll show one example. A triangle has corners at points $\mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_C$ as shown in the picture. Calculate the area of the triangle.



Recall that the area of the triangle is $\frac{1}{2}|AB||AC|\sin\theta$. This quantity can be found quickly using a cross product

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left|(\mathbf{r}_B - \mathbf{r}_A) \times (\mathbf{r}_C - \mathbf{r}_B)\right|$$

If you really want to look cool, you can simplify this formula to

$$A = \frac{1}{2}\left|\mathbf{r}_A \times \mathbf{r}_B + \mathbf{r}_B \times \mathbf{r}_C + \mathbf{r}_C \times \mathbf{r}_A\right|$$

7.2 Solving Vector Equations

In EN3, by far the most important application of vectors will be in solving the equations of equilibrium for structures and machines. Newton's laws (with an extension by Euler), says that if a body is subjected to forces $\mathbf{F}_i$ ($i=1\ldots N$) acting at positions $\mathbf{r}_i$, together with moments $\mathbf{M}_j$ ($j=1\ldots.M$), then the resultant force and moment on the solid must vanish if the body is at equilibrium. The equilibrium equations are

$$\sum_{i=1}^{N} \mathbf{F}_i = 0$$

$$\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i + \sum_{j=1}^{M} \mathbf{M}_j = \mathbf{0}$$

In a typical statics problem, our task will be to identify all the forces acting on our system of interest, express them all as vectors, and then use the equations of equilibrium to solve for any unknown forces.

Example 1

A typical force equilibrium equation looks like this

$$T\mathbf{i} + \frac{R}{5}(4\mathbf{i} - 3\mathbf{j}) + W\mathbf{k} - 10(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{0}$$

Here, T R and W are three forces whose direction is known, but whose magnitude is not known. We need to solve the equation for T, R and W.

To do so, we collect together all the **i**, **j** and **k** components

$$\left(T + \frac{4R}{5} - 10\right)\mathbf{i} + \left(\frac{-3R}{5} - 10\right)\mathbf{j} + (W - 10)\mathbf{k} = \mathbf{0}$$

then, we note that if the vector is zero, all its components must be zero, so that

$$\left(T+\frac{4R}{5}-10\right)=0 \qquad \left(\frac{-3R}{5}-10\right)=0 \qquad (W-10)=0$$

giving us three equations to solve (the anwer is $R=\frac{-50}{3} \quad T=\frac{40}{3} \quad W=10$ Newtons)

Example 2
Here is a typical force-and-moment equilibrium problem, in two dimensions

$$R\mathbf{j}+(T_x\mathbf{i}+T_y\mathbf{j})-10\mathbf{j}=\mathbf{0}$$

$$(4\mathbf{i}+6\mathbf{j})\times R\mathbf{j}+(-\mathbf{i}-\mathbf{j})\times(T_x\mathbf{i}+T_y\mathbf{j})=\mathbf{0}$$

The first equation is force equilibrium, the second is moment equilibrium. R**j** is a force with known direction but unknown magnitude, $T_x\mathbf{i}+T_y\mathbf{j}$ is a force with unknown magnitude *and* direction.

To proceed, we have to multiply out the cross product in the second equation (remember the rules for cross products of the basis vectors?)

$$4R\mathbf{k}+T_x\mathbf{k}-T_y\mathbf{k}=\mathbf{0}$$

Then, collect all the vector components from both equations to see that

$$T_x=0 \qquad R+T_y-10=0 \qquad 4R+T_x-T_y=0$$

which are easily solved to get $T_x=0, \quad R=2 \quad T_y=8$ Newtons.

The same procedure works for three dimensional problems. Usually, in a 3D problem we end up with 6 equations for 6 unknowns instead of just 3, so things can get really messy. But that's what computers (or entry level engineers in your firm) are for.

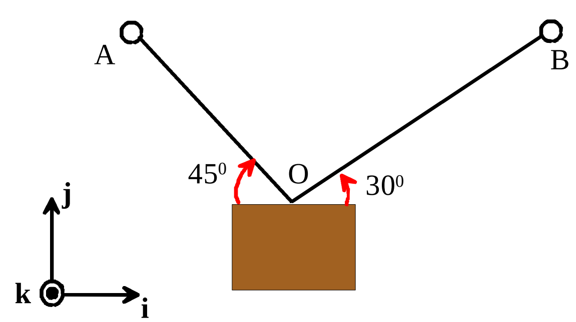
Problems

7.1 Solve the following vector equations

(a) $\frac{1}{2}x\mathbf{i}+7\mathbf{j}+\sqrt{2}\mathbf{i}-y\mathbf{j}=0$

(b) $\mathbf{k}\times\left(\frac{1}{2}x\mathbf{i}+7\mathbf{j}\right)+y\mathbf{i}-\sqrt{2}\mathbf{j}=0$

7.2 The figure shows a heavy box suspended from two cables. The box is subjected to a vertical gravitational force W, and two forces of magnitude T_1, T_2 acting parallel to cables OA, and OB, respectively. Express each force as vector components in the basis shown. Write down the vector sum of the forces. Use the fact that the vector sum of the forces is zero to calculate T_1, and T_2 in terms of W



Summary Checklist

Before taking the EN3 Vector Proficiency Exam, you should make sure you can accomplish the following tasks:

(i) Identify physical quantities as scalars or vectors
(ii) Set up a Cartesian basis
(iii) Identify whether a triad of vectors is right handed
(iv) Using geometry, write down the components of vectors such as position, force, etc as components in a Cartesian basis
(v) Calculate the magnitude of a vector whose Cartesian components are given
(vi) Add and subtract vectors, both graphically and using components
(vii) Multiply a vector by a scalar
(viii) Calculate the dot product of two vectors;
(ix) Use dot products to calculate angles between vectors
(x) Calculate the component of a vector in a direction parallel to another vector
(xi) Calculate the cross product of two vectors
(xii) Calculate moments of forces about a given point, both using the cross product method and using the perpendicular distance method.
(xiii) Calculate the direction cosines of a vector
(xiv) Given vector components in one basis, compute new components in a second basis
(xv) Calculate the area of a triangle given vector expressions for the position of its corners
(xvi) Solve vector equations for unknown components or magnitudes of vectors
(xvii) Derive any formula or expression which is derived for you in this tutorial
(xviii) Compose an epic poem extolling the joys of vectors and recite it while walking on water.