ENGN2530 Matlab Project 1
Sampling and Reconstruction

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Sept. 23, 2009

Due Oct. 7, 2009

1 Sampling Theorem

We define a periodic function $duf(x)$ that has a period,

$$duf(x) = \begin{cases} 1 & 0 \le x < \frac{\pi}{2} \\ \cos(x - \frac{\pi}{2}) & \frac{\pi}{2} \le x < \frac{3\pi}{2} \\ -1 & \frac{3\pi}{2} \le x < 2\pi \end{cases} . \quad (1)$$

One period of $duf(x)$ is shown in the figure 1 below.

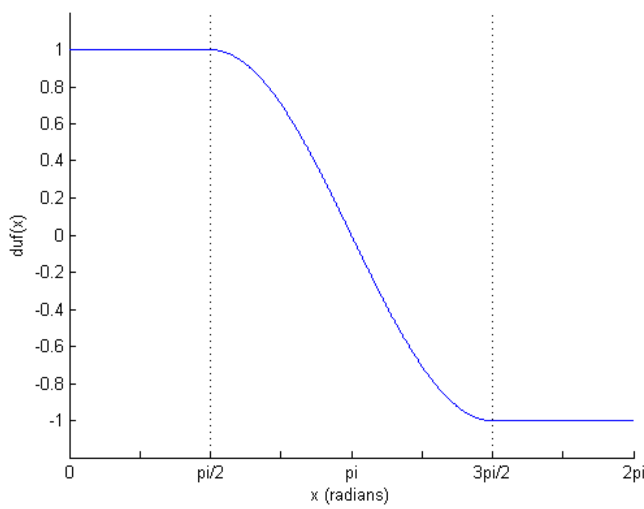

Figure 1: one period of $duf(x)$

Let $x(t)$, $y(t)$, and $z(t)$ be the continuous time functions described below:

$$x(t) = \begin{cases} 4duf(2300\pi t) & t \le 1 \\ 0 & t > 1 \end{cases} \quad (2)$$

$$y(t) = \begin{cases} 2duf(100\pi t) & t \le 1 \\ 0 & t > 1 \end{cases} \quad (3)$$

$$z(t) = \begin{cases} x(t) - y(t) & t \leq 1 \\ 0 & t > 1 \end{cases} \tag{4}$$

You are to sample $x(t)$, $y(t)$, and $z(t)$ at three different rates: $f_{s_1} = 200$, $f_{s_2} = 2000$, and $f_{s_3} = 20000$, obtaining $x_1[n]$, $x_2[n]$, $x_3[n]$, $y_1[n]$, $y_2[n]$, $y_3[n]$, $z_1[n]$, $z_2[n]$, and $z_3[n]$, respectively.

1.1 Directions

Assume $d(t) = duf(\omega_0 t)$ is a periodic function, repeating the above $duf(x)$ period every time the argument $\omega_0 t$ reaches a multiple of 2π.

1. It will help for the rest of this exercise to know the Fourier Series for $d(t)$, so derive this mathematically and include the derivation.
2. Use MATLAB to plot all the time functions in the range $-0.5 \leq t \leq 1.5$. Use the figures to comment on the sampling effects. Make sure the figures are correctly labeled with time as the abscissa.
3. Program your own DTFT MATLAB function (DO NOT use any "canned" functions for this). Include a copy of the code in your writeup. Hmmmm! How can you do this when the DTFT produces a function of a continuous frequency variable?
4. Use your own DTFT program to draw the frequency domain of each time function. You will have 9 plots for $X_k(e^{j\omega T_k})$, $Y_k(e^{j\omega T_k})$, and $Z_k(e^{j\omega T_k})$ for $k = 1, 2, 3$. Show both magnitude and principal phase plots. Comment on the three corresponding frequency domain signals. (Make sure you use the appropriate frequency resolution and draw up to at least twice the sampling frequency.)

1.2 Hand in

1. Your derivation of the Fourier series for the $duf(x)$ function.
2. Nine separate plots of the time functions.
3. Nine separate plots of the frequency-domain functions, as a function of frequency in Hertz.
4. Your DTFT Matlab program.
5. Your comments on all plots, explaining what you see, especially with respect to the sampling theorem.

2 Reconstruction Theorem

Consider a continuous-time signal $x_c(t)$. This signal is sampled with sampling interval T to form the discrete-time signal $x[n] = x_c(nT)$. The reconstruction theorem states that, as long as $x_c(t)$ was appropriately sampled (faster than the Nyquist rate), $x_c(t)$ can be exactly reconstructed from the samples $x[n]$.

We will simulate this reconstruction process in the following exercise. It is important to note that we will not actually be following the steps detailed in the reconstruction theorem. The reconstruction theorem deals with discrete-to-continuous (D/C) conversion, however, we can only work with discrete signals on a computer. We will model the continuous signal $x_c(t)$ with a discrete signal sampled at a very high rate and "pretend" that this is a continuous signal.

You have been given the periodic signal $x_c(t)$ in a file called ***xc.mat***. Although we will proceed as if this is our continuous-time signal, it is actually a discrete signal sampled at 64 kHz.

2.1 Theory

Reconstruction is a two-step processes. The first step is to form a continuous-time representation of the sampled signal $x[n]$. We convert the sequence $x[n]$ to a continuous-time impulse train $x_s(t)$.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \tag{5}$$

In the equation above, $\delta(t)$ is the continuous-time unit impulse.

Sampling a continuous signal creates, in the frequency domain, periodic repetitions of the frequency response of the original signal. The periodic repetitions occur at multiples of the sampling frequency as shown in figure 2, where f_s is the sampling frequency and f_x is the bandwidth of $x_c(t)$.

In the second step of reconstruction, we apply a low-pass filter $h_r(t)$ to remove the unwanted frequencies created by the sampling process. According to the reconstruction theorem, if $h_r(t)$ is designed appropriately, the output of this filter is exactly equal to $x_c(t)$. Your objective is to design this filter and apply it to the sampled signal $x[n]$. This process should give you some insight into the design of interpolation filters, which will come up later in the course.

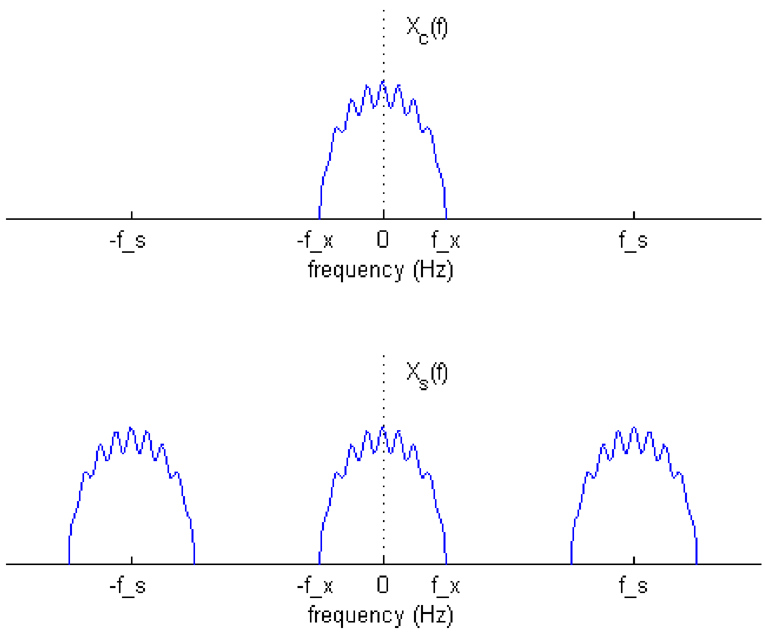

Figure 2: Effect of sampling on frequency spectrum

2.2 Directions

1. Plot and comment on the frequency spectrum of $x_c(t)$. Can you make a guess as to the functional form of $x_c(t)$?

2. Obtain the discrete-time signal $x[n]$ by downsampling to 8 kHz. Are we losing any information by doing this?

3. Plot $x[n]$ and comment on any differences you see between it and $x_c(t)$. Look closely at the high-frequency content of both $x[n]$ and $x_c(t)$.

4. Design a reconstruction filter. Refer to [Oppenheim, Schafer and Buck, Section 4.3] for more information on the ideal reconstruction filter. There are two main differences between the true ideal reconstruction filter and the one you will be designing. First, you will construct a discrete-time filter. This filter will be a sequence of numbers, sampled at 64 kHz, but you will treat it as if it were a continuous-time filter. Second, the ideal filter response extends from $-\infty$ to ∞, and therefore cannot be represented exactly in a computer. Your filter will be a finite approximation of the ideal filter. Choose a number for the length of your filter (how many 64 kHz samples it will contain). You should choose an odd filter length, and your filter will be symmetric around $t = 0$. Why did you choose that filter length? What tradeoffs are involved in selecting a filter length?

5. Plot the frequency response of the reconstruction filter. How does the frequency response compare to that of the ideal filter? As an experiment, design a filter with a different length and plot its frequency response. How does the filter length affect the frequency response of the filter?

6. Construct the signal $x_s(t)$, an impulse train representing the signal $x[n]$ in a "continuous" time domain (remember, $x_s(t)$ will actually be discrete, sampled at 64 kHz).

7. Apply your filter to $x_s(t)$ to get a new, continuous-time signal $\hat{x}_c(t)$. How does this signal compare to the original $x_c(t)$. Plot the difference $x_c(t) - \hat{x}_c(t)$ to give you a sense of the error induced by the reconstruction process.

8. Compute the mean-square-error (MSE) between $\hat{x}_c(t)$ and $x_c(t)$. Try applying filters of different lengths, and comment on the relationship between filter length and MSE.