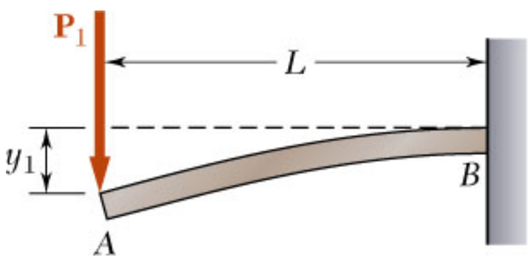
Work and Energy Under a Single Load

Strain energy may be found from the work of other types of single concentrated loads.

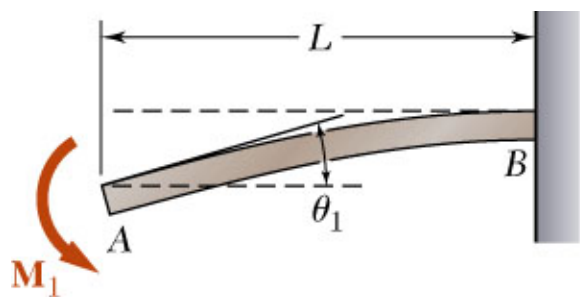
Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$= \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI}\right) = \frac{P_1^2 L^3}{6EI}$$

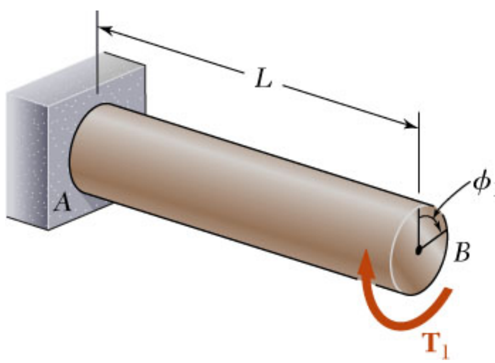
Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$= \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI}\right) = \frac{M_1^2 L}{2EI}$$

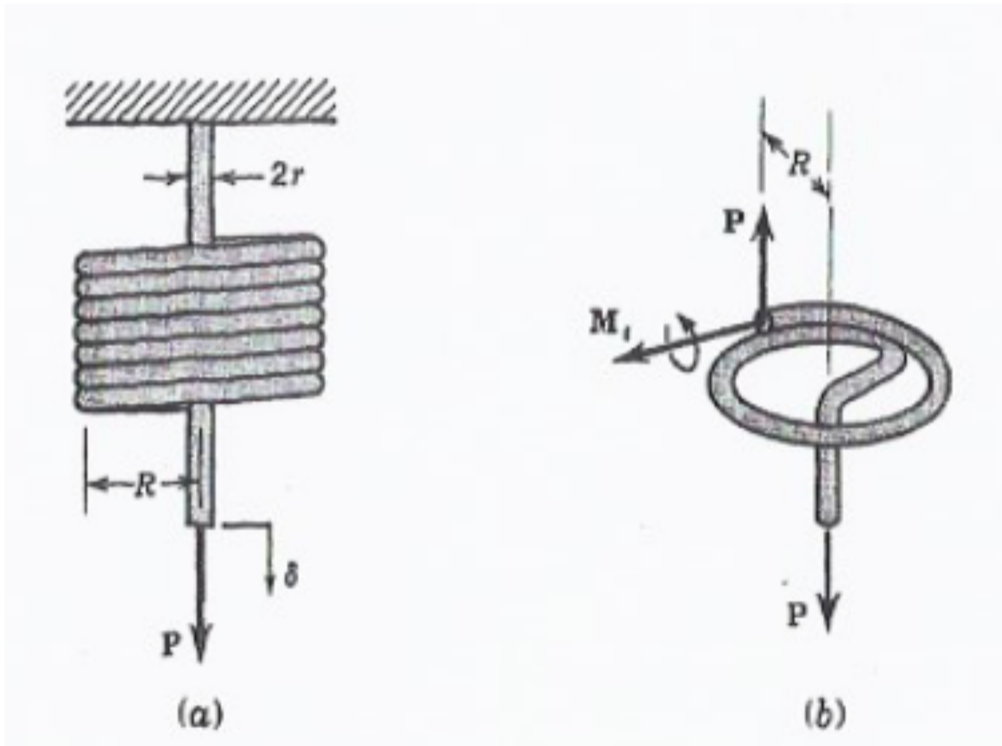
Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

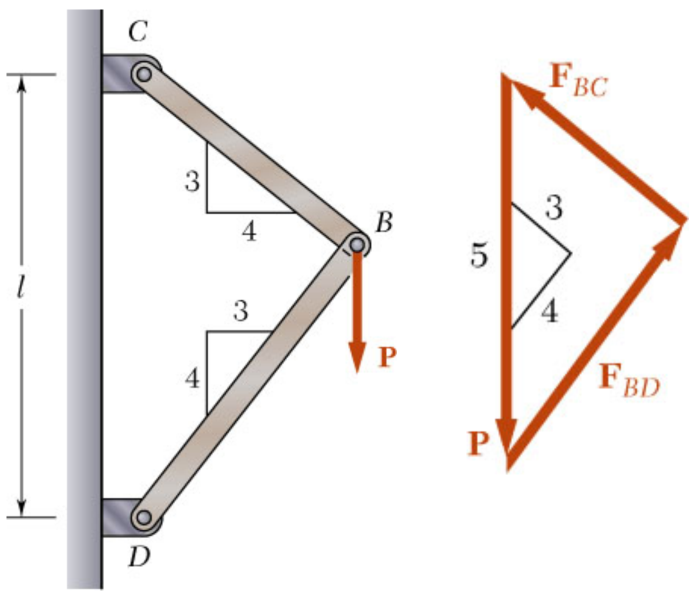
$$= \tfrac{1}{2} T_1 \left(\frac{T_1 L}{JG}\right) = \frac{T_1^2 L}{2JG}$$

Deflection of a spring



Compute the deflection (which gives the spring constant) in terms of r, R and n, the number of turns.

Deflection Under a Single Load



From the given geometry,

$$L_{BC} = 0.6l \quad L_{BD} = 0.8l$$

From statics,

$$F_{BC} = +0.6P \quad F_{BD} = -0.8P$$

If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.

Strain energy of the structure,

$$U = \frac{F_{BC}^2 L_{BC}}{2AE} + \frac{F_{BD}^2 L_{BD}}{2AE}$$

$$= \frac{P^2 l\left[(0.6)^3 + (0.8)^3\right]}{2AE} = 0.364\frac{P^2 l}{AE}$$

Equating work and strain energy,

$$U = 0.364\frac{P^2 L}{AE} = \tfrac{1}{2}P y_B$$

$$y_B = 0.728\frac{Pl}{AE}$$

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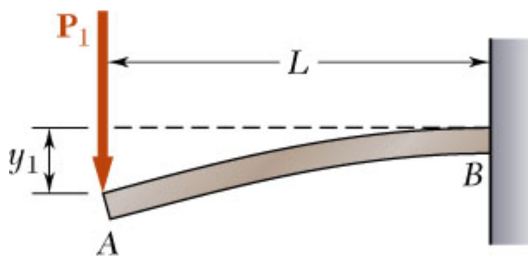
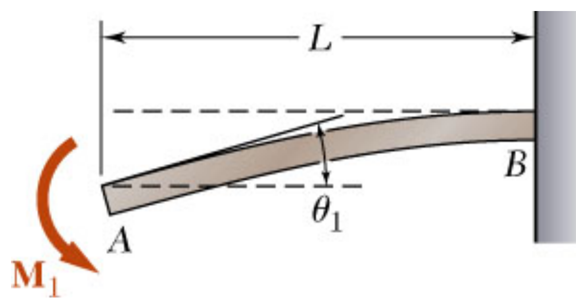


$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$= \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI} \right) = \frac{P_1^2 L^3}{6EI}$$

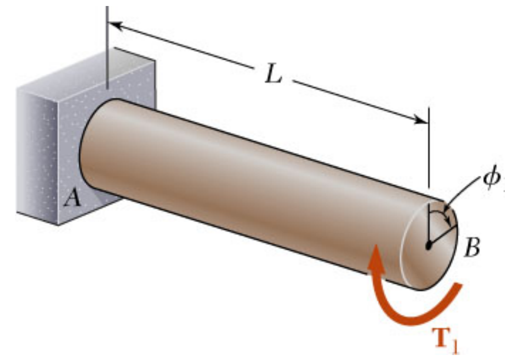
Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$= \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI} \right) = \frac{M_1^2 L}{2EI}$$

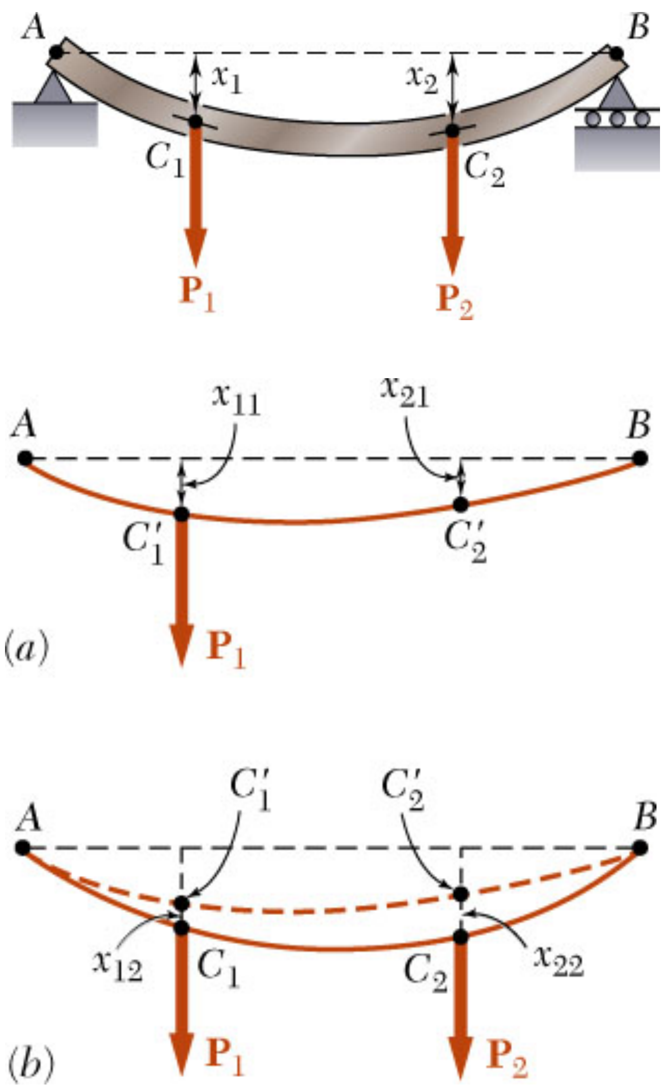
Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

$$= \tfrac{1}{2} T_1 \left(\frac{T_1 L}{JG} \right) = \frac{T_1^2 L}{2JG}$$

Work and Energy Under Several Loads



Deflections of an elastic beam subjected to two concentrated loads,

$$x_1 = x_{11} + x_{12} = \alpha_{11} P_1 + \alpha_{12} P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21} P_1 + \alpha_{22} P_2$$

Compute the strain energy in the beam by evaluating the work done by slowly applying P_1 followed by P_2,

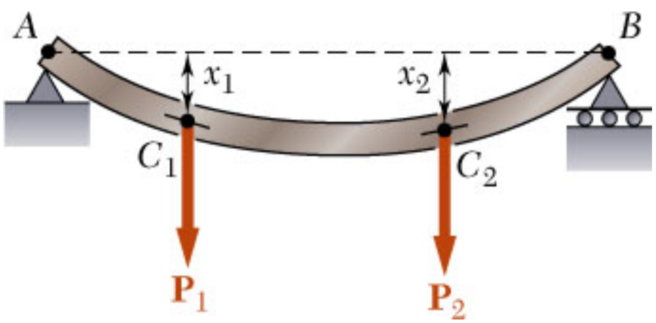
$$U = \tfrac{1}{2}\left(\alpha_{11} P_1^2 + 2\alpha_{12} P_1 P_2 + \alpha_{22} P_2^2\right)$$

Reversing the application sequence yields

$$U = \tfrac{1}{2}\left(\alpha_{22} P_2^2 + 2\alpha_{21} P_2 P_1 + \alpha_{11} P_1^2\right)$$

Strain energy expressions must be equivalent. It follows that $\alpha_{12}=\alpha_{21}$ (*Maxwell's reciprocal theorem*).

Castigliano's Theorem



Strain energy for any elastic structure subjected to two concentrated loads,

$$U = \frac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

Differentiating with respect to the loads,

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1$$

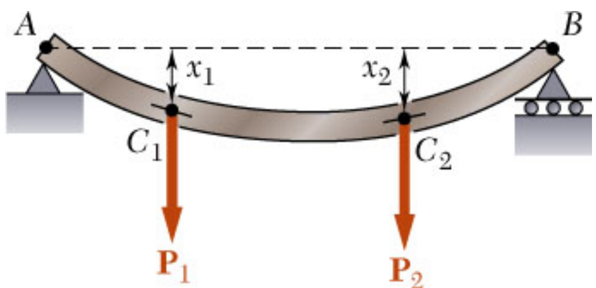
$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2$$

Castigliano's theorem: For an elastic structure subjected to *n* loads, the deflection x_j of the point of application of P_j can be expressed as

$$x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \qquad \phi_j = \frac{\partial U}{\partial T_j}$$

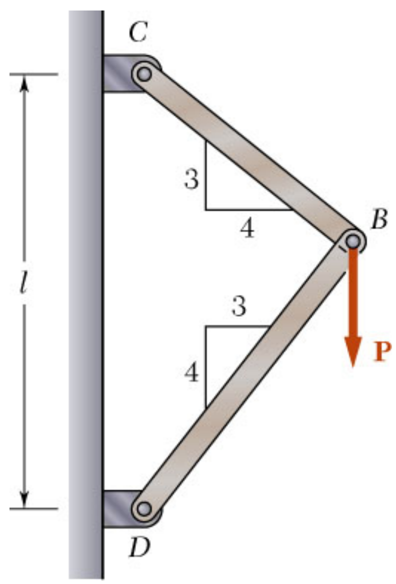
Deflections by Castigliano's Theorem

Application of Castigliano's theorem is simplified if the differentiation with respect to the load P_j is performed before the integration or summation to obtain the strain energy U.



In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_j} dx$$



For a truss,

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2 A_i E} \qquad x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j}$$