The purpose of this homework is to gain facility with indicial notation in vector analysis. Three concepts are especially crucial:

- The *Kronecker delta* \( \delta_{ij} \), defined by
  \[
  \delta_{ij} = \begin{cases} 
  1 & \text{if } i = j, \\
  0 & \text{if } i \neq j. 
  \end{cases}
  \]

- The *alternating symbol* \( \epsilon_{ijk} \), defined by
  \[
  \epsilon_{ijk} = \begin{cases} 
  1 & \text{if } \{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \text{ or } \{3, 1, 2\}, \\
  -1 & \text{if } \{i, j, k\} = \{2, 1, 3\}, \{1, 3, 2\}, \text{ or } \{3, 2, 1\}, \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- The *Einstein summation convention*, which implies summation over the range 1, 2, 3 for any repeated index (the appearance of an index more than twice in any term is not allowed), i.e.
  \[
  u_i v_j = u_1 v_1 + u_2 v_2 + u_3 v_3.
  \]

Throughout, \( \varphi \) and \( \psi \) refer to scalar fields and \( u, v, \) and \( w \) refer to vector fields. (Boldface notation is equivalent to the under-squiggle on the blackboard.)

1. By employing the summation convention, simplify the following expressions involving the Kronecker delta: (a) \( \delta_{ii} \), (b) \( \delta_{ij}\delta_{ij} \), (c) \( \delta_{ik}\delta_{kj} \), (d) \( \delta_{ij}\delta_{ik}\delta_{jk} \), and (e) \( \delta_{ij}u_j \).

2. Simplify the following expressions involving the alternating symbol: (a) \( \epsilon_{ijk}\delta_{jk} \), (b) \( \epsilon_{ijk}u_ju_k \), and (c) \( \epsilon_{ij3}u_i\delta_{2j} \).

3. Consider the identity
  \[
  \epsilon_{ijk}\epsilon_{pqrs} = \det \begin{bmatrix} 
  \delta_{ip} & \delta_{iq} & \delta_{ir} \\
  \delta_{jp} & \delta_{jq} & \delta_{jr} \\
  \delta_{kp} & \delta_{kq} & \delta_{kr} 
  \end{bmatrix}.
  \]
  (a) Verify the identity for
  \begin{align*}
  (i) & \ i = 1, j = 2, k = 3, p = 3, q = 1, r = 2, \\
  (ii) & \ i = 1, j = 2, k = 3, p = 3, q = 2, r = 1, \\
  (iii) & \ i = 1, j = 2, k = 3, p = 1, q = 2, r = 1.
  \end{align*}
  (b) Show that
  \[
  \epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}.
  \]
  This is referred to as the epsilon-delta identity and is especially useful when working with cross products.
(c) Using the epsilon-delta identity, evaluate $\epsilon_{ijp}\epsilon_{ijq}$ and $\epsilon_{ijk}\epsilon_{ijk}$.

4. Using indicial notation, show that

(a) $u \cdot (v \times w) = v \cdot (w \times u) = w \cdot (u \times v)$,
(b) $u \times v = -v \times u$,
(c) $u \times u = 0$,
(d) $u \cdot (u \times v) = 0$,
(e) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$,
(f) $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0$,
(g) $(u \times v) \cdot (u \times v) = (u \cdot u)(v \cdot v) - (u \cdot v)^2$.

5. (from Fung) Write out the following equation in unabridged form:

$$\mu u_{i,jj} + (\mu + \lambda)u_{j,ij} + F_i = 0,$$

where $\mu$ and $\lambda$ are scalar constants. As we shall see later in the course, these are the Navier-Cauchy equations of elasto-statics, where $u_i$ is the displacement and $F_i$ is the body force.

6. Using indicial notation, establish the following identities:

(a) $\nabla \cdot (\varphi u) = \varphi \nabla \cdot u + u \cdot \nabla \varphi$,
(b) $\nabla \cdot (u \times v) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v)$,
(c) $\nabla \times (\varphi u) = \varphi (\nabla \times u) + \nabla \varphi \times u$,
(d) $\nabla \cdot (\nabla \varphi \times \nabla \psi) = 0$,
(e) $\nabla \times (\nabla \varphi) = 0$,
(f) $\nabla \times (\nabla \times u) = \nabla (\nabla \cdot u) - \nabla^2 u$, where $\nabla^2 () = \nabla \cdot (\nabla ())$.

7. (adapted from Fung) Let $\mathbf{r}$ be the position vector and $r$ be its magnitude $\sqrt{\mathbf{r} \cdot \mathbf{r}}$. Using indicial notation, show that

(a) $\nabla \cdot \mathbf{r} = 3$,
(b) $\nabla r = \frac{\mathbf{r}}{r}$,
(c) $\nabla \mathbf{r} = \mathbf{1}$,
(d) $\nabla \cdot (r^n \mathbf{r}) = (n + 3)r^n$,
(e) $\nabla \times (r^n \mathbf{r}) = 0$,
(f) $\nabla^2 (r^n \mathbf{r}) = n(n + 3)r^{n-2} \mathbf{r}$.