ENGN 1750: Advanced Mechanics of Solids
Homework 1

Due: Friday, September 13, 2013

The purpose of this homework is to gain facility with indicial notation in vector analysis. Three concepts are especially crucial:

- The *Kronecker delta* δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- The *alternating symbol* ϵ_{ijk}, defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \text{ or } \{3,1,2\}, \\ \text{-1} & \text{if } \{i,j,k\} = \{2,1,3\}, \{1,3,2\}, \text{ or } \{3,2,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

- The *Einstein summation convention*, which implies summation over the range 1, 2, 3 for any repeated index (the appearance of an index more than twice in any term is not allowed), i.e.

$$u_i v_j = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Throughout, φ and ψ refer to scalar fields and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ refer to vector fields. (Boldface notation is equivalent to the under-squiggle on the blackboard.)

1. By employing the summation convention, simplify the following expressions involving the Kronecker delta: (a) δ_{ii}, (b) $\delta_{ij}\delta_{ij}$, (c) $\delta_{ik}\delta_{kj}$, (d) $\delta_{ij}\delta_{ik}\delta_{jk}$, and (e) $\delta_{ij}u_j$.

2. Simplify the following expressions involving the alternating symbol: (a) $\epsilon_{ijk}\delta_{jk}$, (b) $\epsilon_{ijk}u_j u_k$, and (c) $\epsilon_{ij3}u_i\delta_{2j}$.

3. Consider the identity

$$\epsilon_{ijk}\epsilon_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}.$$

   (a) Verify the identity for

   (i) $i = 1, j = 2, k = 3, p = 3, q = 1, r = 2$,
   (ii) $i = 1, j = 2, k = 3, p = 3, q = 2, r = 1$,
   (iii) $i = 1, j = 2, k = 3, p = 1, q = 2, r = 1$.

   (b) Show that

$$\epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}.$$

   This is referred to as the epsilon-delta identity and is especially useful when working with cross products.

(c) Using the epsilon-delta identity, evaluate $\epsilon_{ijp}\epsilon_{ijq}$ and $\epsilon_{ijk}\epsilon_{ijk}$.

4. Using indicial notation, show that

   (a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$,

   (b) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,

   (c) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,

   (d) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$,

   (e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$,

   (f) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$,

   (g) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2$.

5. (from Fung) Write out the following equation in unabridged form:

$$\mu u_{i,jj} + (\mu + \lambda) u_{j,ij} + F_i = 0,$$

   where μ and λ are scalar constants. As we shall see later in the course, these are the Navier-Cauchy equations of elasto-statics, where u_i is the displacement and F_i is the body force.

6. Using indicial notation, establish the following identities:

   (a) $\nabla \cdot (\varphi \mathbf{u}) = \varphi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \varphi$,

   (b) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$,

   (c) $\nabla \times (\varphi \mathbf{u}) = \varphi(\nabla \times \mathbf{u}) + \nabla \varphi \times \mathbf{u}$,

   (d) $\nabla \cdot (\nabla \varphi \times \nabla \psi) = 0$,

   (e) $\nabla \times (\nabla \varphi) = \mathbf{0}$,

   (f) $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$, where $\nabla^2(\,) = \nabla \cdot \nabla(\,)$.

7. (adapted from Fung) Let $\mathbf{r}$ be the position vector and r be its magnitude $\sqrt{\mathbf{r} \cdot \mathbf{r}}$. Using indicial notation, show that

   (a) $\nabla \cdot \mathbf{r} = 3$,

   (b) $\nabla r = \dfrac{\mathbf{r}}{r}$,

   (c) $\nabla \mathbf{r} = \mathbf{1}$,

   (d) $\nabla \cdot (r^n \mathbf{r}) = (n+3) r^n$,

   (e) $\nabla \times (r^n \mathbf{r}) = \mathbf{0}$,

   (f) $\nabla^2 (r^n \mathbf{r}) = n(n+3) r^{n-2} \mathbf{r}$.