A Game Theory Analysis of Social Cost

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Abstract: Environmental regulators desire truthful reporting, but it is not necessarily the interest of industry to report truthfully. A solution to this problem of incentive compatibility is the Groves regulatory mechanism. Under the Groves mechanism, the regulated party maximizes his or her payoff when he or she reports truthfully; however, the mechanism also has its shortcomings.

This paper examines an alternative to the Groves mechanism. I derived the mechanism specifically for managing common property resources, such as fisheries and the atmosphere. I dub the alternative, the "mean reporter" mechanism, because it bases the tax or transfer utility rule on the report of the player with the mean amount of action, henceforth known as the mean player. This mechanism has a Nash equilibrium where the report of the mean player is the net social benefit maximizing report; however, this Nash equilibrium requires specific initial conditions. In this Nash equilibrium, the mechanism is budget-balanced but not efficient*.

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1. Introduction: The regulation of common property resources is an information intensive procedure. Some types of information can be directly measured, such as smoke or number of grazing cattle, but other types, such as health costs, cannot. Without this so-called "private information," the regulator has the problem of incomplete information. If the regulator tries to elicit or infer this information, he may instead have the problem of imperfect information. How does the regulator ensure correct information?

One cause of imperfect information is forward-looking behavior. People who are about to be regulated, the players, are expected to act in a manner that maximize their future payoff. Consequently, if a player is asked to report his private information, he will think of the future consequences of his report before doing so. If lying or its euphemistic cousin, misreporting, maximizes a player's future payoff, then he may misreport and give the regulator imperfect information. Likewise, if the regulator tries to infer this information through a proxy, e.g., health costs through medical bills, a forward looking player can be expected to manipulate the proxy to his advantage. There are no simple solutions to this problem. Forward-looking behavior must be accounted for before this problem can be satisfactorily resolved.

In the real world, players do not necessarily have enough information to be able to maximize their future payoff. Acting furtively, the regulator may very well obtain perfect information and "shock" the players with regulation in the future. However, each time the regulator wishes to update the regulation, he must devise another surreptitious method to try and obtain perfect information. A similar problem arises when new players enter the regulatory game. The regulator can no longer expect to trick the new entrants into truthful reporting. Forward-looking behavior is an accurate assumption, even when players are initially naïve.

This paper describes one solution to the problem of regulation and information. The regulator announces a regulatory mechanism with the knowledge that it will be updated periodically based on the players' actions. After some specified time interval, the regulator tallies everyone's actions and selects a player to determine the regulation for the next time period. The players know exactly the process that determines selection, but they also know that there is no way to manipulate the process to their benefit.[1] If the mechanism "converges", i.e., becomes more accurate with each period, then the regulator can be confident that the regulations will improve as time passes. Convergence has yet to be determined for this model.

The rest of the paper is a formalization of the problem: Section 2 contains a formal introduction to mechanism design. Section 3 introduces notation, definitions, and a simple game theory model. Section 4 provides a Nash equilibrium solution to the two round game that requires a specific initial condition. Section 5 is the balanced Groves mechanism for the same framework. Section 6 shows the budget balance and approximates efficiency. Section 7 gives numerical examples. Concluding remarks form Section 8 and works cited are listed in Section 9.

2. Background: Generally, the class of mechanisms that are both efficient and truth revealing coincides with the class of Groves mechanisms (Groves 1973, Green and Laffont 1979). The transfer utility rule of the Groves mechanism for player *j* is based on the report of every player but *j*. Consequently, player *j* has no incentive to misreport based on the transfer utility rule. The net social benefit maximization rule of the Groves mechanism, usually in terms of provision of a public good, for player *j* coincides with player *j*'s own benefit maximization rule [see Section 4]. Thus, player *j* has an incentive to report truthfully based on the maximization rule.

A drawback of the Groves mechanism is that the transfer utility goes to an outside player, commonly government. The government accrues a budget surplus, which can be returned to the players only under specific conditions. There has been a great deal of research on the class of benefit functions that result in balanced Groves mechanisms (Groves and Loeb 1975, Tian 1996, Liu and Tian 1999). Another approach has been to replace the dominant strategies characteristic of the Groves mechanism with the weaker Nash equilibrium (Groves and Ledyard 1977). A

[1] The selection process is proxy-based. While there is no cost associated with misreporting, players that wish to misreport must first be selected, which is costly due to the proxy. The Groves mechanism is not

balanced dominant strategy Groves mechanism can be easily constructed for the class of benefit functions used in this paper [see Section 4].

One form of the Groves mechanism, the Clarke or pivotal mechanism (Clarke 1970), is especially important. The transfer utility rule of the Clarke mechanism for player *j* is a tax equal to the marginal cost imposed by player *j*'s participation on the rest of society. In other words, player *j* pays the externality associated with his action.[2] There is no similar refund rule for player *j* that says his compensation is equal to the marginal cost imposed on him by the rest of society, the marginal compensation principle. This rule depends on player *j*'s report and thus violates the definition of a Groves mechanism. Under the "mean reporter" mechanism, the refund rule for player *j* does depend on player *j*'s "report", although only partially. (I place report in quotation marks because it is implicit under the "mean reporter" mechanism, where only one person reports, instead of explicit under the Groves mechanism, where everyone reports.) The tax rule for the "mean reporter" mechanism is the same as that of the Clarke mechanism.

If, hypothetically, there was a mechanism that compensates the externality imposed by the rest of society, that mechanism could not be a Groves mechanism, because player *j*'s refund depends on player *j's* report. In this case, the incentives are directly opposite the incentives of a Groves mechanism. This hypothetical mechanism would be, if not impossible, very difficult to design for a significant class of benefit functions. In the "mean reporter" mechanism, the refund for each player is equal; thus player *j*'s refund is only partially dependent on player *j*'s "report". Although "misreporting" is only possible by changing one's action, which can be injurious, the "mean reporter" mechanism still has significant incentive compatibility issues that are illustrative of non-Groves mechanism design in general.

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proxy-based.

[2] The efficient tax on an externality is also known as a Pigovian tax, for the economist, A.C. Pigou, who pioneered the study of externalities. Mathematically, the externality tax is set according to the social marginal damage or marginal cost schedule.

In addition to the compensation improvement, the "mean reporter" mechanism has the advantage of only requiring one person to report instead of requiring everyone to report, as is the case under the Groves mechanism. Experimentally, there can be a number of human errors associated with reporting, such as inconsistency, irrationality, forgetfulness, etc. This is not so much a problem when the reporters are professionally trained to correctly quantify their private information. The disadvantage of relying on a "mean reporter" is existence, because the mean reporter exists only when there is a continuum of players and actions.

3. Framework: I follow a game theory framework with quadratic benefits and linear costs. The benefit function is the same across all players, where "all players" excludes the government. The cost multiplier or c_i is privately held information. The actions can be monitored and are publicly known. The only difference between this framework and a utility function for private goods is that the cost term depends on the total action (or production) instead of the individual's own action (or production). This difference is the crux of the social cost problem in allocation as opposed to its private cost counterpart. The government's role is to maximize the net social benefit (NSB).

$i = 1 \ldots N$	The players.
$i = N + 1$	The government.
$x_i : x_i \geq 0$	The *i*th player's action.
$c_i : c_i \geq 0$	The *i*th player's private cost.[3]
$100x_i - \frac{x_i^2}{2}$	The benefit function for the *i*th player.[4]
$-c_i \sum_{j=1}^{N} x_j$	The social cost function for the *i*th player.

[3] Here, private cost is analogous to a player's private type, as defined in a Bayesian game.

[4] Any expression of the form, $Ax_i - Bx_i^2$, is acceptable, where *A* and *B* are unknown to the regulator but identical and known to the players. Generally, any benefit function is valid for this model as long as it is identical for all the players.

$$U_i = 100x_i - \frac{x_i^2}{2} - c_i \sum_{j=1}^{N} x_j$$ The payoff function for the *i*th player.

$$U_{N+1} = \sum_{j=1}^{N} U_j$$ The payoff function for the government.

The identity $N\overline{x} = \sum_{j=1}^{N} x_j$ will be used.

Assume a standard noncooperative simultaneous one round game. Each player, excluding the government, chooses an action [2] to maximize his utility or payoff function [1].

In a slightly modified game, the payoff functions have now been altered by the Clarke mechanism [3]. Each player again chooses an action to maximize his new utility function [4].

The Clarke mechanism results in efficient actions. Because the Clarke action [4] and the unregulated action [2] are unequal, the unregulated game must be inefficient.

$$U_i = 100x_i - \frac{x_i^2}{2} - c_i \sum_{j=1}^{N} x_j \qquad [1]$$

$$x_i = 100 - c_i \qquad [2]$$

$$U_i = 100x_i - \frac{x_i^2}{2} - c_i \sum_{j=1}^{N} x_j - x_i \sum_{j \neq i}^{N} c_j \qquad [3]$$

$$x_i = 100 - N\overline{c} \qquad [4]$$

To improve upon the inefficient allocation [2], I pose a mechanism [5], where *r* equals the "mean report", or the report from the mean player.

$$U_i = 100x_i - \frac{x_i^2}{2} - c_i \sum_{j=1}^{N} x_j + Nr(\overline{x} - x_i) \qquad [5]$$

4. Nash Equilibria: The "mean reporter" mechanism can be analyzed as a two round game. The Nash equilibria can be revealed using backwards induction. In the second round, the value *r* has already been chosen and each player chooses an x_i. Since there are no more rounds, the choice of x_i only affects the second round utility. In this case, the following is a Nash equilibrium (NE) for the players in the second round:

$$x_i^* = 100 - c_i + r - Nr$$
$$\bar{x}^* = 100 - \bar{c} + r - Nr$$

[6a]

Differentiating the utility function [5] with respect to *x* and setting the first order condition to zero, one can find x^*. Summing the actions over all the players and dividing by *N* results in the average action [See Appendix 1]. In this NE, the mean player happens to have the mean cost.

Suppose that in the first round NE, the mean player also happens to have the mean cost. I select this player to report *r* to initialize the second round. Before reporting, the mean player looks ahead to the players' actions in the second round [6a]. Then, he chooses the *r* that maximizes his second round utility. To derive this, substitute his action (in terms of his report) back into his payoff function [5]. Then, differentiate with respect to the report and calculate the value that maximizes his payoff [See Appendix 2a] in the second round. This report *r* is also the *r* that maximizes the government's second round utility [See Appendix 3]. Consequently, the following is a NE for all parties in the second round if the first round mean player also has the mean cost:

$$r^* = \bar{c}$$
$$x_i^* = 100 - c_i + r - Nr$$
$$\bar{x}^* = 100 - \bar{c} + r - Nr$$

[6b]

At first, it may appear possible to calculate r^* directly from the $\bar{x}$ equation [6a]. The only unknown in the $\bar{x}$ equation is $\bar{c}$, which is equal to r^* [6b]. However, there are at least two obstacles to this calculation. First, the constants may be unknown in the more general model [See Footnote 4], which increase the number of unknowns from one to three. Second, the players may choose different NE strategies if they know the regulator's intent to calculate r^* from $\bar{x}$. It will only be possible to calculate r^* directly when the constants are known and the players act myopically, e.g., the shock effect. Generally, the game must be analyzed for at least two rounds.

The first round of this two round game is difficult to analyze. The NE actions, $x_1^*, \ldots, x_N^*$, in the first round affect the first round mean player's report or r^* [6b], which in turn affects the second round actions. By changing my first round action, I affect the second round initialization, r^* [6b], and thus may improve my payoff in the second round. It can be shown, however, that there is a two round NE where the first round mean player has the mean cost [6c], thus satisfying the condition on r^* [6b].

The complete Nash equilibrium for the two round game was found with the aid of Maple [See Appendix 2b]. The NE has an initial condition requirement. I could not determine the exact relationship between the initial condition and the equilibrium, because the calculations became too cumbersome [See Appendix 4]. In the most optimistic scenario, the value for r would converge, i.e., the r reported in this round would be more accurate then the r reported in the previous round.

Initial Condition:

$$r^* = \bar{c}$$

Round 1:

$$x_i^* = \frac{200N^2 - 200N + Nc_i - N\bar{c} + 3N^2\bar{c} - N^2c_i - 2N^3\bar{c} + (\bar{c} - c_i)\sqrt{A}}{2N(N-1)}$$

$$A = 4N^5 - 7N^4 + 10N^3 - 7N^2 + 4N$$

$$\bar{x}^* = 100 - N\bar{c}$$

Round 2:

$r_j^* = \bar{c}$ such that $x_j = \bar{x}$ from Round 1.

$x_i^* = 100 - c_i + r_j^* - Nr_j^*$

$\bar{x}^* = 100 - \bar{c} + r_j^* - Nr_j^*$

[6c]

5. Balanced Groves Mechanism: Using the same framework, a budget balanced Groves mechanism in dominant strategies can also be derived. Take the Clarke mechanism from before [3], and assume that truthful reporting has not yet been determined by placing hats on the *c* values [7]. Rewrite the government's utility function [8] and expand it [9]. The government's utility function and the player's utility function differ only by a constant [10]. Hence, the value that maximizes the player's utility function [7] also maximizes the government's utility function [10] and truthful reporting is the dominant strategy.

$$U_i = 100x_i - \frac{x_i^2}{2} - c_i \sum_{j=1}^{N} x_j - x_i \sum_{j \neq i}^{N} \hat{c}_j \quad [7]$$

$$U_{N+1} = \sum_{i=1}^{N} \left[100x_i - \frac{x_i^2}{2} \right] - \sum_{i=1}^{N} c_i \sum_{i=1}^{N} x_i \quad [8]$$

$$U_{N+1} = 100x_i - \frac{x_i^2}{2} - \hat{c}_i \sum_{j}^{N} x_j - x_i \sum_{j \neq i}^{N} \hat{c}_j + \sum_{j \neq i}^{N} \left[100x_j - \frac{x_j^2}{2} \right] - \sum_{j \neq i}^{N} \hat{c}_j \sum_{j \neq i}^{N} x_j$$

[9]

$$U_{N+1} = 100x_i - \frac{x_i^2}{2} - \hat{c}_i \sum_{j}^{N} x_j - x_i \sum_{j \neq i}^{N} \hat{c}_j + K$$

[10]

The move from a Clarke to a balanced Groves mechanism requires partitioning the budget surplus [11] and returning it in an allowable form [See Appendix 5]. I suggest one compensation rule [12] and one form of a balanced Groves mechanism for the framework in this paper [13].

$$BS = \sum_{i=1}^{N}\left[x_i \sum_{j\neq i}^{N} c_j \right] \qquad [11]$$

$$TU_i = \frac{1}{N-1}\sum_{j\neq i}^{N} c_j \sum_{j\neq i}^{N} x_j \qquad [12]$$

$$- x_i \sum_{j\neq i}^{N} \hat{c}_j + \frac{1}{N-1}\sum_{j\neq i}^{N} c_j \sum_{j\neq i}^{N} x_j \qquad [13]$$

6. Budget Balance and Efficiency: The budget balance is trivial to show [14]. The efficiency loss is easiest to calculate using substitutions in Maple [See Appendix 6]. I give the absolute loss for Round 1 [15] and Round 2 [16]. On orders of magnitude approximations, the Round 1 loss is *N* times as great as the Round 2 loss. Percentage based estimates are not useful when the distribution of *c* is unknown, but are also in Appendix 6.

$$\sum_{i=1}^{N} N\bar{c}\left(\bar{x} - x_i\right) = N\bar{c}\left(N\bar{x} - \sum_{i=1}^{N} x_i \right) \qquad [14]$$

$$N\bar{c}\left(N\bar{x} - \sum_{i=1}^{N} x_i \right) = N\bar{c}\left(N\bar{x} - N\bar{x}\right) = 0$$

$$R_1 = \frac{1}{4}\sum_{i=1}^{N}\left(c_i - \bar{c}\right)^2 * \frac{2N^4 - 3N^3 + 4N^2 - 3N + 2 + N\sqrt{A} - \sqrt{A}}{N\left(N-1\right)^2}$$

$$A = 4N^5 - 7N^4 + 10N^3 - 7N^2 + 4N \qquad [15]$$

$$R_2 = \frac{1}{2}\sum_{i=1}^{N}\left(c_i - \bar{c}\right)^2 \qquad [16]$$

7. Numeric Examples: Due to the complexity of the equations, it was not always feasible to verify them. I tested the equations numerically to see if the predicted and actual values matched. Table 1 is the two round Nash equilibrium derived in Section 3. Tables 2 and 3 illustrate Player 1's attempt to deviate in Round 1 to influence the Round 2 report. Note that his utility is lower in both cases, showing that the candidate Nash equilibrium is indeed a Nash equilibrium. In Table 4, the new mean action in Round 1 matches the action level of Player 2. The predicted cost of the reporting player should then equal the private cost of Player 2, the reporter. They match!

	Player	Cost	Action 2	Action 1	(NEW) Action 1	Utility
	1	5	44.15	45.93	**45.93**	**2454.59**
	2	5.5	43.65	44.06	44.06	2022.58
	3	4	45.15	49.66	49.66	3307.39
	4	5	44.15	45.93	45.93	2454.59
	5	6	43.15	42.19	42.19	1586.83
	6	7	42.15	38.46	38.46	704.12
	7	8	41.15	34.72	34.72	-193.56
	8	5	44.15	45.93	45.93	2454.59
	9	4	45.15	49.66	49.66	3307.39
	10	7	42.15	38.46	38.46	704.12
Average	10	5.65	43.5	43.5	43.5	
	Actual	5.65				
	Report	5.65				

Table 1: The candidate Nash equilibrium.

	Player	Cost	Action 2	Action 1	(NEW) Action 1	Utility
	1	5	44.1233231	45.93	**45.83**	**2454.58**
	2	5.5	43.6233231	44.06	44.06	2022.53
	3	4	45.1233231	49.66	49.66	3306.78
	4	5	44.1233231	45.93	45.93	2454.35
	5	6	43.1233231	42.19	42.19	1586.96
	6	7	42.1233231	38.46	38.46	704.61
	7	8	41.1233231	34.72	34.72	-192.69
	8	5	44.1233231	45.93	45.93	2454.35
	9	4	45.1233231	49.66	49.66	3306.78
	10	7	42.1233231	38.46	38.46	704.61
Average	10	5.65	43.4733231	43.5	43.49014232	
	Actual	5.65263837				
	Report	5.6529641				

Table 2: Player 1 attempts a negative deviation.

	Player	Cost	Action 2	Action 1	(NEW) Action 1	Utility
	1	5	44.1774472	45.93	**46.03**	**2454.58**
	2	5.5	43.6774472	44.06	44.06	2022.64
	3	4	45.1774472	49.66	49.66	3308.01
	4	5	44.1774472	45.93	45.93	2454.84
	5	6	43.1774472	42.19	42.19	1586.70
	6	7	42.1774472	38.46	38.46	703.61
	7	8	41.1774472	34.72	34.72	-194.45
	8	5	44.1774472	45.93	45.93	2454.84
	9	4	45.1774472	49.66	49.66	3308.01
	10	7	42.1774472	38.46	38.46	703.61
Average	10	5.65	43.5274472	43.5	43.51014232	

	Actual	5.64728544
	Report	5.64695031

Table 3: Player 1 attempts a positive deviation.

	Player	Cost	Action 2	Action 1	(NEW) Action 1	Utility
	1	5	45.6658589	45.93	51.53	2437.75
	2	**5.5**	45.1658589	44.06	**44.06**	2024.57
	3	4	46.6658589	49.66	49.66	3340.77
	4	5	45.6658589	45.93	45.93	2467.04
	5	6	44.6658589	42.19	42.19	1578.36
	6	7	43.6658589	38.46	38.46	674.72
	7	8	42.6658589	34.72	34.72	-243.89
	8	5	45.6658589	45.93	45.93	2467.04
	9	4	46.6658589	49.66	49.66	3340.77
	10	7	43.6658589	38.46	38.46	674.72
Average	10	5.65	45.0158589	43.5	**44.06014232**	

	Actual	**5.50007989**
	Report	5.48157123

Table 4: Theoretical private cost and actual private cost.

8. Discussion: The purpose of the "mean reporter" mechanism was to improve upon the budget balance property of the Groves mechanism. This goal, however, lead to the creation of a non-Groves mechanism and all the difficulties associated with non-Groves mechanisms. By using the "mean reporter" mechanism, which has only one reporting player, I avoided the problem of truthful revelation for all the players, a major obstacle, but resigned the mechanism to continuity, constant marginal benefits, and inefficiency. In practice, a working mechanism that requires constant marginal benefits and has an efficiency loss is still better than no working mechanism at all, but the continuity requirement keeps the mechanism from practical applications. Without placing restrictions on the action levels a player can choose, the only way I can think of around the continuity issue is to give up the budget balance. However, with this capitulation, it is not known whether the "approximately mean reporter" has the same incentives as the "mean reporter." Additionally, the Nash equilibrium characterization in this scenario is another matter altogether.

The "mean reporter" mechanism does not have a solution in dominant strategy, like the Groves mechanism does. In Groves, the government has the power to ultimately determine the public good provision, but in "mean reporter", the government's only real power is in initializing the mechanism. The "mean player" has the power to ultimately determine the public good provision analogue in this framework. While the second round Nash equilibrium [6] is almost as good as a dominant strategies equilibrium, the first round NE [see Section 3] in the two round game appears very weak due to its intricate calculations. Experimentally, Round 1 may not consistently resolve to the same Nash equilibrium, weakening the predictive powers of this thesis.

The bright spot was, save for the unknown convergence properties, the successful two round Nash equilibrium implementation. Although inefficient, the equilibrium demonstrates that there exists at least one set of symmetric strategies that is non-manipulable. The development of non-Groves mechanisms hinges on whether it's possible to have incentive compatibility when utility transfer and reporting go hand in hand. Generally (Gibbard 1973, Sattherwaite 1975), this is not the case, but for restricted preference profiles, it may be possible. I intuit that symmetric strategies in a pure Nash equilibrium exist for incorrect initialization values, i.e., convergence exists, although I was unable to find it. Numeric examples could reinforce or disprove this conjecture. Finally, this mechanism could be generalized for a much wider class of functions than the ones I used, especially in the cost term. Generalization, along with discreteness and convergence, could help this mechanism, which has a unique lax reporting requirement, to have niche uses.

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