PL162 QUANTUM MECHANICS ASSIGNMENT 3 ANSWERS

Suppose we are describing quantities with an inertial rectilinear coordinate system using the axes $\{t, x, y, z\}$. Specifically you have vectors that are tangent to the world-lines of ordinary subluminal particles—time-like vectors v_1 with coordinates $(5, 1, -2, 2)$ and v_2 with coordinates $(5, -4, 0, 0)$. To calculate their inner product, you treat one of the vectors (it doesn't matter which) as a row matrix and the other as a column matrix, and you stick a matrix representing the metric in between them.

$$\begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

And to find their relative speed, v, one calculates by way of a function of the spacetime angle $cosh(\theta)$:

$$\frac{1}{\sqrt{1-v^2}} = cosh(\theta) = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$

where $|v_i|$ is the length of v_i, i.e., the square root of the number formed by multiplying v_i as a row vector times the metric matrix times v_i as a column matrix.

1. (10 points) What is the speed of v_1 and v_2 relative to something at rest in the coordinate system?

Short answer:

$$\text{speed of } v_1 = \frac{\sqrt{1^2 + (-2)^2 + 2^2}}{5} = \frac{3}{5} \text{ of the speed of light.}$$

and

$$\text{speed of } v_2 = \frac{\sqrt{(-4)^2}}{5} = \frac{4}{5} \text{ of the speed of light.}$$

Long answer:

$$|v_1| = \sqrt{5^2 - 1^2 - (-2)^2 - 2^2} = 4$$

$$|v_2| = \sqrt{5^2 - (-4)^2} = 3$$

A sample vector at rest is v_3 is $(1, 0, 0, 0)$.

$$|v_3| = \sqrt{1^2} = 1$$

$$\begin{aligned}
\frac{1}{\sqrt{1-v^2}} &= \frac{v_1 \cdot v_3}{|v_1||v_3|} = \frac{5}{4 \cdot 1} \\
1 - v^2 &= \frac{4^2}{5^2} \\
v^2 &= 1 - \frac{16}{25} = \frac{9}{25} \\
v &= \frac{3}{5}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{1-v^2}} &= \frac{v_2 \cdot v_3}{|v_2||v_3|} = \frac{5}{3 \cdot 1} \\
1 - v^2 &= \frac{3^2}{5^2} \\
v^2 &= 1 - \frac{9}{25} = \frac{16}{25} \\
v &= \frac{4}{5}
\end{aligned}$$

2. (10 points) Calculate the relative speed between v_1 and v_2.

$$\begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix} = 5^2 - (1)(-4) - (-2)(0) - (2)(0) = 29$$

$$\begin{aligned}
\frac{1}{\sqrt{1-v^2}} &= \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{29}{4 \cdot 3} \\
1 - v^2 &= \frac{144}{29^2} \\
v^2 &= 1 - \frac{144}{29^2} = \frac{29^2 - 144}{29^2} = \frac{697}{29^2} \\
v &= \frac{\sqrt{697}}{29} \text{ of the speed of light.}
\end{aligned}$$

Now, consider the same system described from the point of view of another inertial rectilinear coordinate system that is rotated by an angle θ with respect to the x-y plane of the first. To get the coordinates for a rotated row vector, we multiply it on the right by a special matrix that represents a rotation. For example,

$$\begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Or, for column vectors, we multiply it on the left by a slightly different matrix that represents the same rotation. For example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

3. (10 points) Calculate the coordinates for v_1 and v_2 in the coordinate system rotated by $\pi/4$ (counterclockwise).

$$\begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\pi/4) & sin(\pi/4) & 0 \\ 0 & -sin(\pi/4) & cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & cos(\pi/4) + 2sin(\pi/4) & sin(\pi/4) - 2cos(\pi/4) & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & \frac{3\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\pi/4) & -sin(\pi/4) & 0 \\ 0 & sin(\pi/4) & cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4cos(\pi/4) \\ -4sin(\pi/4) \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{bmatrix}$$

4. (15 points) Prove that *any* inner products in a rotated coordinate system are the same as in the old coordinate system. (This verifies that relative velocities are invariant under rotations.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -cos\theta & -sin\theta & 0 \\ 0 & sin\theta & -cos\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -cos^2\theta - sin^2\theta & cos\theta sin\theta - sin\theta cos\theta & 0 \\ 0 & sin\theta cos\theta - cos\theta sin\theta & -sin^2\theta - cos^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now, consider an inertial rectilinear coordinate system that is boosted by a velocity v in the x direction with respect to the first coordinate system. To get the coordinates for a boosted row vector, we multiply it on the right by a special matrix,

$$\begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \quad \begin{bmatrix} cosh\theta & sinh\theta & 0 & 0 \\ sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and for column vectors, we multiply it on the left by the same matrix.

$$\begin{bmatrix} cosh\theta & sinh\theta & 0 & 0 \\ sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

Some useful relationships:
$cosh\theta = 1/\sqrt{1-v^2}$, $sinh\theta = v/\sqrt{1-v^2}$, $cosh^2\theta - sinh^2\theta = 1$.

5. (10 points) Calculate the components for v_1 in a coordinate system that has been boosted by $v = 3/5$ towards the negative end of the x axis. (Hint, this is equivalent to boosting v_1 in the positive x direction, so the new x-component should be more positive than the old x-component.)

$$cosh\theta = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \sqrt{\frac{25}{25-9}} = \frac{5}{4}$$

$$sinh\theta = \frac{v}{\sqrt{1-v^2}} = \frac{\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{3}{5}\sqrt{\frac{25}{25-9}} = \frac{3}{4}$$

$$\begin{aligned} \begin{bmatrix} 5 & 1 & -2 & 2 \end{bmatrix} \quad & \begin{bmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 5(\frac{5}{4}) + \frac{3}{4} & 5(\frac{3}{4}) + \frac{5}{4} & -2 & 2 \end{bmatrix} \\ = \begin{bmatrix} 7 & 5 & -2 & 2 \end{bmatrix} \end{aligned}$$

6. (15 points) Prove that *any* inner products in a boosted coordinate system are the same as in the old coordinate system. (This verifies that relative velocities are invariant under boosts.)

$$\begin{bmatrix} cosh\theta & sinh\theta & 0 & 0 \\ sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cosh\theta & sinh\theta & 0 & 0 \\ sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cosh\theta & -sinh\theta & 0 & 0 \\ sinh\theta & -cosh\theta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cosh\theta & sinh\theta & 0 & 0 \\ sinh\theta & cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cosh^2\theta - sinh^2\theta & cosh\theta sinh\theta - sinh\theta cosh\theta & 0 & 0 \\ sinh\theta cosh\theta - cosh\theta sinh\theta & sinh^2\theta - cosh^2\theta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We can think of the tangent vectors as representing mass, energy, and momentum. Suppose that the length of the vector v_1 is the first particle's mass and v_2 is the second particle's mass. Then, the t-component of each is its energy, and the x, y, z-components of each are its momentum. The properties of the combined system composed of the two particles is given by $v_1 + v_2$.

7. (8 points) What is the total energy of the two particles in the initial coordinate system? Momentum? Mass?

$$v_1 + v_2 = \begin{bmatrix} 5+5 & 1-4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -3 & -2 & 2 \end{bmatrix}$$

Energy is 10.
Momentum is $-3\hat{x} - 2\hat{y} + 2\hat{z}$.
Mass is $\sqrt{100 - (-3)^2 - (-2)^2 - 2^2} = \sqrt{83}$.

8. (12 points) What is the total energy, momentum, and mass relative to the boosted coordinate system in problem 5?

$$\begin{bmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25/4 - 3 \\ 15/4 - 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 13/4 \\ -5/4 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = \begin{bmatrix} 7 + 13/4 & 5 - 5/4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 41/4 & 15/4 & -2 & 2 \end{bmatrix}$$

Energy is 41/4.
Momentum is $15/4\hat{x} - 2\hat{y} + 2\hat{z}$.
Mass is $\sqrt{41^2/16 - 15^2/16 - (-2)^2 - 2^2} = \sqrt{83}$. (This verifies the invariance of mass under boosts. It has to be the same the same mass as in problem 7.)

A note here on the mass calculation. I defined the mass of the total system to be the length of $v_1 + v_2$. Why is this the mass instead of just adding the masses of the individual components? It's really a matter of making decisions about how best to subdivide the conceptual landscape. Let $m_1 = |v_1 + v_2|$ and $m_2 = |v_1| + |v_2|$. The question is now, "Which quantity, m_1 or m_2, best deserves the label 'mass of the system'?" In favor of m_1: If the two particles fuse into a single particle, its mass is m_1, and doing so lets us say that mass is conserved (as well as energy and momentum). Concerning the role of mass as inertia, if the pair is subject to a constant force, the center of mass of the system will bend as if it had inertia m_1. Putting the pair in a perfectly elastic box and then putting them on a scale, one would measure the mass as m_1. In favor of m_2: The role of mass in generating gravitational attraction depends on where the individual masses are located: m_1 plays no role in generating the spacetime curvature so is unneeded; on the other hand it is really the stress-energy tensor that generates spacetime curvature and that includes more than just mass. Adding the masses is simpler. Although m_1 is the mass of the fusion of the pair, we can always say that there is no conservation of mass and have as an extra rule that in particle fusion and decay, the total energy-momentum vector is conserved, and the total mass is just the sum of the lengths of the individual particles' energy-momentum vectors. It doesn't matter fundamentally which definition of mass is used.