Electromagnetic scattering of vector mesons in the Sakai-Sugimoto model

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Summary

- Introduction: photon-hadron scattering
- The Sakai-Sugimoto model
- The $\rho$ meson form factors in the Sakai-Sugimoto model
- The $\rho$ meson structure functions
- Conclusions and perspectives
1. Introduction: photon-hadron scattering

Consider the electromagnetic scattering

\[ e^-(k) + H(p) \rightarrow e^-(k') + X \]  \hspace{1cm} (1)

The electron and hadron exchange a \textit{virtual photon} with momentum \( q^\mu = k^\mu - k'^\mu \).
Kinematical variables

- Mass of the initial hadron and lepton: $M$, $m$
- **Virtuality**: $q^2 = -q_0^2 + \vec{q}^2 > 0$
- Energy transfer: $\nu = (p \cdot q)/M$
- **Bjorken variable**: $x = -q^2/(2p \cdot q) = -q^2/(2M\nu)$
- Photon-hadron center of mass energy squared:
  \[ W^2 = -(p + q)^2 = M^2 + q^2 \left( \frac{1}{x} - 1 \right) \]  

The scattering **physical region** is given by $0 < x \leq 1$.

An **elastic** (inelastic) scattering corresponds to $x = 1$ ($x < 1$).
Unpolarized cross section and the hadronic tensor

\[ d\sigma = \left( \frac{1}{4ME} \right) \frac{d^3k}{(2\pi)^32E'} \left( \frac{1}{2} \sum_{\sigma,\sigma'} \right) \left( \frac{1}{2s_H + 1} \sum_{\sigma_H} \sum_X \right) \]
\[ \times (2\pi)^4\delta^4(p + q - p_X)|\mathcal{M}|^2, \]

where \( \mathcal{M} \) is the scattering amplitude:

\[ \mathcal{M} = e^2\bar{u}(k', \sigma')\gamma^\mu u(k, \sigma) \frac{\eta_{\mu\nu}}{q^2} \langle X|J_H'^\mu(0)|p, \sigma_H \rangle. \]

\( \sum_X \): sum over final states \( X \) (inclusive cross section),

\( p_X \): final momentum for each state \( X \).

\( \sigma, \sigma' \): lepton polarizations,

\( s_H, \sigma_H \): spin and polarization of the initial hadron.
It is convenient to write the cross section as

\[ d\sigma = \left( \frac{1}{4ME} \right) \frac{d^3 k}{(2\pi)^3 2E'} \frac{e^4}{q^4} (4\pi) L_{\mu\nu} \, W^{\mu\nu}, \]  

(4)

where

\[ L_{\mu\nu} = \frac{1}{2} \sum_{\sigma, \sigma'} \bar{u}(k, \sigma) \gamma_{\mu} u(k', \sigma') \bar{u}(k', \sigma') \gamma_{\nu} u(k, \sigma) \]
\[ = 2 \left[ k_{\mu} k'_{\nu} + k_{\nu} k'_{\mu} + (m^2 - k \cdot k') \eta_{\mu\nu} \right], \]  

(5)

is the **leptonic tensor** and

\[ W^{\mu\nu} = \frac{1}{4\pi} \left( \frac{1}{2s_H + 1} \right) \sum_{\sigma_H} \sum_{X} (2\pi)^4 \delta^4(p + q - px) \]
\[ \times \langle p, \sigma_H | J^\mu_H(0) | X \rangle \langle X | J^\nu_H(0) | p, \sigma_H \rangle, \]  

(6)

is the **hadronic tensor**.
Structure functions

We can use current conservation, $P$, $T$ and Lorentz invariance to decompose the hadronic tensor.

In the unpolarized case it takes the form

$$W^{\mu\nu} = F_1(x, q^2) \left( \eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{2x}{q^2} F_2(x, q^2) \left( p^\mu + \frac{q^\mu}{2x} \right) \left( p^\nu + \frac{q^\nu}{2x} \right),$$

where $F_1(x, q^2)$ and $F_2(x, q^2)$ are known as the structure functions of the hadron.

To calculate the structure functions we need to sum over all possible final states.
Form factors

In the hadronic tensor, each final state $X$ contributes through the current element $\langle p, \sigma_H | J_H^{\mu}(0) | X \rangle$. The final state $X$ can be one particle or many particles.

Here we are interested in the case $X = H$, where the final state is one hadron with the same mass and spin as the initial hadron. In that case the current matrix element can be decomposed as

$$\langle p, n, \sigma_H | J_H^{\mu}(0) | p + q, n_X, \sigma_X \rangle = \sum_i \Gamma_i^{\mu}(p, q) F_{n, n_X}^i(q^2),$$

where $n$ ($n_X$) is an index associated to the mass of the initial (final) hadron. The functions $F_{n, n_X}^i(q^2)$ are the hadronic form factors.
Example: The elastic $\rho$ meson form factor

In this case we have that $n_X = n = 1$ and the current can be decomposed as

$$\langle p, \epsilon | J^\mu(0) | p + q, \epsilon' \rangle = \epsilon \cdot \epsilon' (2p + q)^\mu F_1(q^2)$$

$$+ [\epsilon^\mu \epsilon' \cdot q - \epsilon'^\mu \epsilon \cdot q] [F_1(q^2) + F_2(q^2)] - \frac{q \cdot \epsilon' q \cdot \epsilon}{M^2} (2p + q)^\mu F_3(q^2).$$

(9)

It is also convenient to define the electric, magnetic, and quadrupole form factors

$$G_E = F_1 - \frac{q^2}{6M^2} [F_2 - (1 + \frac{q^2}{4M^2}) F_3] \quad , \quad G_M = F_1 + F_2,$$

$$G_Q = -F_2 + \left(1 + \frac{q^2}{4M^2}\right) F_3.$$  

(10)
2. The Sakai-Sugimoto model

Motivation

- In the regime of **low momentum transfer** ($\sqrt{q^2}$ lower than some $\text{GeV}s$) **soft processes** become dominant and **perturbative QCD** fails.
- In this region we need to construct **non-perturbative models** since the precision of lattice QCD is still very far from the experiments.
- The **AdS/QCD models** provide new insights into this old problem. These models are motivated by the **AdS/CFT** correspondence.
The D4-brane background (Witten 1998)

$N_c$ coincident **D4-branes in type IIA Supergravity** generate the metric

$$ds^2 = \frac{u^{3/2}}{R^{3/2}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{u^{3/2}}{R^{3/2}} f(u) d\tau^2 + \frac{R^{3/2}}{u^{3/2}} \frac{du^2}{f(u)} + R^{3/2} u^{1/2} d\Omega_4^2,$$

$$f(u) = 1 - \frac{u^3}{u^3},$$

(11)

where $R = (\pi g_s N_c)^{1/3} \sqrt{\alpha'}$.

In addition, we have a dilaton and a four-form

$$e^\phi = g_s \frac{u^{3/4}}{R^{3/4}}, \quad F_4 = \frac{(2\pi l_s)^3 N_c}{V_{s^4}} \epsilon_4.$$

(12)
The $\tau$ coordinate is compactified in a circle with period

$$\delta \tau = \frac{4\pi}{3} \frac{R^{3/2}}{U_{\Lambda}^{1/2}} \equiv \frac{2\pi}{M_{\Lambda}},$$

(13)

where $M_{\Lambda}$ is a 4-d mass scale.

Imposing anti-periodic conditions for the fermionic states we arrive to a 4-d non-supersymmetric strongly coupled $U(N_c)$ theory at large $N_c$.

The 4-d 't Hooft constant is given by

$$\lambda = g_{YM}^2 N_c = (2\pi M_{\Lambda}) g_s N_c l_s.$$

(14)
The D4-D8 brane intersection (Sakai & Sugimoto 2004)

Consider $N_f$ coincident D8-$\overline{\text{D}8}$ probe branes living in the background generated by the $N_c$ D4-branes. The probe approximation is guaranteed by the condition $N_f \ll N_c$.

The D8 and $\overline{\text{D}}8$ branes separated in the UV region merge in the infrared. This is a geometrical realization of chiral symmetry breaking.

$$U(N_f) \times U(N_f) \rightarrow U(N_f)$$
The **DBI action** describing 9-d gauge field fluctuations in the D8-\overline{D8} branes can be integrated in $S^4$ leading to

$$S_{YM} = -\kappa \int d^4x dz \text{Tr} \left[ \frac{1}{2} K_z^{-1/3} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + M_{\Lambda}^2 K_z \eta^{\mu\nu} F_{\mu z} F_{\nu z} \right]$$

(15)

where $K_z = 1 + z^2$ and $\kappa = \lambda N_c / (216\pi^3)$.

The 5-d gauge field can be expanded, in the $A_z = 0$ gauge, as

$$A_\mu(x, z) = \hat{V}_\mu(x) + \hat{A}_\mu(x) \psi_0(z) + \sum_{n=1}^{\infty} v_\mu^n(x) \psi_{2n-1}(z) + \sum_{n=1}^{\infty} a_\mu^n(x) \psi_{2n}(z),$$

where the $\psi_n(z)$ modes satisfy

$$\kappa \int dz K_z^{-1/3} \psi_n(z) \psi_m(z) = \delta_{nm}, \quad -K_z^{1/3} \partial_z [K_z \partial_z \psi_n(z)] = \lambda_n \psi_n(z).$$

(16)
Integrating the $z$ coordinate we get a 4-d effective lagrangian. The vector (axial) fields $v_\mu^n(x)$ $(a_\mu^n(x))$ correspond to the modes $\psi_{2n-1}(z)$ $(\psi_{2n}(z))$ and the pion $\pi(x)$ is related to $\psi_0(z)$.

The 4-d fields $\hat{V}_\mu(x)$ and $\hat{A}_\mu(x)$ can be expressed in terms of the

$$\hat{V}_\mu(x) = \frac{1}{2} U^{-1} \left[ A_L^\mu + \partial_\mu \right] U + \frac{1}{2} U \left[ A_R^\mu + \partial_\mu \right] U^{-1}$$

$$\hat{A}_\mu(x) = \frac{1}{2} U^{-1} \left[ A_L^\mu + \partial_\mu \right] U - \frac{1}{2} U \left[ A_R^\mu + \partial_\mu \right] U^{-1}$$

where

$$U(x) = e^{i \frac{\pi(x)}{f_\pi}}, \quad A_L^\mu(R) = V_\mu \pm A_\mu$$

(17)

with $V$ interpreted as the photon.
In order to have a diagonal kinetic term, the vector mesons are redefined as \( \tilde{v}_\mu^n = v_\mu^n + \left( g_{v^n}/M_{v^n}^2 \right) \gamma_\mu \) (similarly for the axial mesons).

The quadratic terms in the vector sector take the form

\[
\mathcal{L}_2 = \frac{1}{2} \sum_n \left[ \text{Tr} \left( \partial_\mu \tilde{v}_\nu^n - \partial_\nu \tilde{v}_\mu^n \right)^2 + 2M_{v^n}^2 \text{Tr} \left( \tilde{v}_\mu^n - \frac{g_{v^n}}{M_{v^n}^2} \gamma_\mu \right)^2 \right],
\]

where

\[
M_{v^n}^2 = \lambda_{2n-1} M_A^2, \quad g_{v^n} = \kappa M_{v^n}^2 \int dz K_z^{-1/3} \psi_{2n-1}(z).
\]

The term \( g_{v^n} \tilde{v}_\mu^n \gamma_\mu \) represents the decay of the photon into vector mesons realizing vector meson dominance.
From the cubic terms in the 4-d effective meson theory discussed above we can extract the vector meson interaction term

\[ \mathcal{L}_{\text{vvv}} = \sum_{n,\ell,m} g_{v^n v^\ell v^m} \text{Tr}\left\{ (\partial^\mu \tilde{v}^\nu n - \partial^\nu \tilde{v}^\mu n) [\tilde{v}_\mu^\ell, \tilde{v}_\nu^m] \right\} \]  

(18)

where \( g_{v^n v^\ell v^m} \) are 4-d effective couplings given by the integral

\[ g_{v^n v^\ell v^m} = \kappa \int dz \, K_z^{-1/3} \psi_{2n-1}(z) \psi_{2\ell-1}(z) \psi_{2m-1}(z). \]  

(19)
Using the Feynman rules associated to the interaction term we find the current matrix element for vector mesons

\[
\langle v^m(p), \epsilon | J^{\mu}(0) | v^\ell(p+q), \epsilon' \rangle = \sum_{n=1}^{\infty} g_{vn} g_{vm} v^n v^\ell \Delta^{\mu\sigma}(q, m^2_n) \epsilon^\nu \epsilon'^\rho \left[ \eta_{\sigma\nu}(q - p)_\rho + \eta_{\nu\rho}(2p + q)_\sigma - \eta_{\rho\sigma}(p + 2q)_\nu \right]
\]

where \( \Delta^{\mu\sigma}(q, m^2_n) \) is the propagator of a massive vector particle with momentum \( q \) and mass \( m^2_n \).

\( v^m \): initial vector meson

\( v^\ell \): final vector meson

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Using the holographic sum rule

\[
\sum_{n=1}^{\infty} \frac{g_{vn} g_{vn} m_{\nu} \ell}{M_{vn}^2} = \delta_{m\ell}
\]  

(21)

and the transversality of the initial and final polarizations we get the simple expression

\[
\langle v^m(p), \epsilon| J^\mu(0)| v^\ell(p + q), \epsilon' \rangle = \epsilon^\nu \epsilon'^\rho \left[ \eta_{\nu\rho}(2p + q)_{\sigma} + 2(\eta_{\sigma\nu} q_{\rho} - \eta_{\rho\sigma} q_{\nu}) \right] \left( \eta^{\mu\sigma} - \frac{q_{\mu} q_{\sigma}}{q^2} \right) F_{v^m v^\ell}(q^2).
\]  

(22)

where

\[
F_{v^m v^\ell}(q^2) = \sum_{n=1}^{\infty} \frac{g_{vn} g_{vn} m_{\nu} \ell}{q^2 + M_{vn}^2}.
\]  

(23)

is a generalized vector meson form factor.
The elastic $\rho$ meson form factors can be extracted from the case $m = \ell = 1$:

\[
\langle \nu^1(p), \epsilon | J^\mu(0) | \nu^1(p + q), \epsilon' \rangle = \left\{ (\epsilon \cdot \epsilon')(2p + q)\mu + 2 \left[ \epsilon_\mu (\epsilon' \cdot q) - \epsilon'\mu (\epsilon \cdot q) \right] \right\} \mathcal{F}_{\nu^1\nu^1}(q^2), \tag{24}
\]

Comparing this result with the expansion (9) we find the $\rho$ meson form factors:

\[
F_1(q^2) = F_2(q^2) = \mathcal{F}_{\nu^1\nu^1}(q^2), \quad F_3(q^2) = 0, \quad (25)
\]
Hence the electric, magnetic and quadrupole form factors predicted by the D4-D8 brane model for the \( \rho \) meson are

\[
G_E = (1 + \frac{q^2}{6M^2})F_1 \quad , \quad G_M = 2F_1 \quad , \quad G_Q = -F_1 . \quad (26)
\]

From these form factors, we can extract the \( \rho \) meson electric radius

\[
\langle r_\rho^2 \rangle = -6\frac{d}{dq^2}G_E(q^2)|_{q^2=0} = 0.5739 , \quad (27)
\]

and the magnetic and quadrupole moments

\[
\mu = \frac{1}{2M}G_M(q^2)|_{q^2=0} = \frac{1}{M} \quad , \quad D \equiv \frac{1}{M^2}G_Q(q^2)|_{q^2=0} = -\frac{1}{M^2} . \quad (28)
\]
4. The $\rho$ meson structure functions

(\textit{BB, Boschi-Filho, Braga, Torres 2010})

The \textbf{optical theorem} relates the hadronic tensor to the \textit{imaginary part} of the tensor associated to the \textbf{Compton forward scattering}. We consider the case where the \textbf{final state is one vector meson}. 
Using the Feynman rules corresponding to this diagram, we find

\[
\text{Im } T^{\mu\nu} = \frac{1}{3} \sum_\epsilon \epsilon^{\lambda_1} \epsilon^{\lambda_2} \sum_{n_x} \left\{ \sum_n g_{\nu} g_{\nu_1} v_{nx} v_n \frac{[\eta^{\mu} \sigma_1 + q^{\mu} q_{\sigma_1}]}{q^2 + M_{vn}^2} \right\} \\
\times \left\{ \sum_m g_{\nu} g_{\nu_1} v_{nx} v_m \frac{[\eta^{\nu} \sigma_2 + q^{\nu} q_{\sigma_2}]}{q^2 + M_{vm}^2} \right\} \\
\times f^{0ab} \left[ \eta_{\sigma_1 \rho_1} (2q) \rho_1 + \eta_{\lambda_1 \rho_1} (2p) \sigma_1 + \eta_{\sigma_1 \rho_1} (-2q) \lambda_1 \right] \\
\times f^{0ab} \left[ \eta_{\sigma_2 \rho_2} (2q) \rho_2 + \eta_{\lambda_2 \rho_2} (2p) \sigma_2 + \eta_{\sigma_2 \rho_2} (-2q) \lambda_2 \right] \\
\times \int \frac{d^4 p_x}{2\pi^4} (2\pi) \delta(p_x^2 + M_{vn_x}^2) \left[ \eta^{\rho_1 \rho_2} + \frac{p_{x}^{\rho_1} p_{x}^{\rho_2}}{M_{vn_x}^2} \right] (2\pi)^4 \delta^4(p + q - p_x) \\
\text{(29)}
\]
The polarization vectors satisfy

$$\sum_\varepsilon \epsilon^{\lambda_1} \epsilon^{\lambda_2} = \eta^{\lambda_1 \lambda_2} + \frac{p^{\lambda_1} p^{\lambda_2}}{M_{\nu_1}^2}. \quad (30)$$

After some algebra we obtain

$$\text{Im} \, T^{\mu \nu} = \frac{N_f}{3} \left[ \eta^{\mu \sigma_1} - \frac{q^{\mu} q^{\sigma_1}}{q^2} \right] \left[ \eta^{\nu \sigma_2} - \frac{q^{\nu} q^{\sigma_2}}{q^2} \right]$$

$$\times \left[ \eta^{\lambda_1 \lambda_2} + \frac{p^{\lambda_1} p^{\lambda_2}}{M_{\nu_1}^2} \right] \left[ \eta^{\rho_1 \rho_2} + \frac{(p + q)^{\rho_1} (p + q)^{\rho_2}}{W^2} \right]$$

$$\times \left[ \eta_{\sigma_1 \lambda_1} (2q)^{\rho_1} + \eta_{\lambda_1 \rho_1} (2p)_{\sigma_1} + \eta_{\sigma_1 \rho_1} (-2q)_{\lambda_1} \right]$$

$$\times \left[ \eta_{\sigma_2 \lambda_2} (2q)^{\rho_2} + \eta_{\lambda_2 \rho_2} (2p)_{\sigma_2} + \eta_{\sigma_2 \rho_2} (-2q)_{\lambda_2} \right]$$

$$\times \sum_{n_x} \left[ \mathcal{F}_{\nu_1, \nu_2} (q^2) \right]^2 (2\pi) \delta[M_{\nu_1 \nu_2}^2 - W^2]. \quad (31)$$
We can approximate the sum over the delta functions by an integral

\[
\sum_{n_x} \delta[M_{v{n_x}}^2 - W^2] \equiv \sum_{n_x} \delta[M_{v{n_x}}^2 - M_{v\bar{n}}^2] = \int dn_x \left[ \frac{\partial M_{v{n_x}}^2}{\partial n_x} \right]^{-1} \delta(n_x - \bar{n})
\]

\[
\equiv f(\bar{n}).
\]

Then, the structure functions are

\[
F_1 = \frac{4N_f}{3} f(\bar{n}) \left[ F_{v^1 v\bar{n}}(q^2) \right]^2 q^2 \left[ 2 + \frac{q^2}{4x^2 M_{v^1}^2} + \frac{q^2}{W^2 x^2} (x - \frac{1}{2})^2 \right]
\]

\[
F_2 = \frac{4N_f}{3} f(\bar{n}) \left[ F_{v^1 v\bar{n}}(q^2) \right]^2 \frac{q^2}{2x} \left[ 3 + \frac{q^2}{M_{v^1}^2} + \frac{(q^2)^2}{M_{v^1}^2 W^2 x^2} (x - \frac{1}{2})^2 \right]
\]

(33)
Numerical results for $N_f = 1$:

$x = 0.5$ (blue), $0.3$ (green), $0.1$ (red)

$q^2 = 5$ (blue), $q^2 = 10$ (green), $q^2 = 15$ (red) in GeV$^2$
Near $x = 0.5$ we find an approximate Callan-Gross relation.
Conclusions and Perspectives

- The Sakai-Sugimoto model offers a new approach for the non-perturbative regime of hadronic interactions. In particular, it realizes in a simple way the property of vector meson dominance.

- It would be interesting to investigate the regime of high energies but low momentum transfer where the soft pomeron is relevant.

- The perturbative QCD approach to hadron scattering involves factorization between hard parton scattering and soft parton distribution functions. It would be interesting to understand this factorization in the AdS/QCD approach.