READ THESE INSTRUCTIONS CAREFULLY

1. The time allowed to complete the exam is 12:00–5:00 PM.

2. All work is to be done without the use of books or papers and without help from anyone.

3. Use a separate answer book for each question, or two books if necessary.

4. **DO NOT write your name in your booklets.** Each student has been assigned a letter code which is on the outside front cover of each exam booklet. This letter code provides anonymity to the student for faculty grading. Please make sure that this letter is listed on ALL exam booklets that you use.

5. Write the problem number in the center of the outside front cover. **Write nothing else** on the inside or outside of the front and back covers. Note that there are separate graders for each question.

6. Answer one (and **only** one) problem from each of the five pairs of questions. The pairs are labeled as follows:

   - Classical Mechanics CM1, CM2
   - Electricity and Magnetism EM1, EM2
   - Statistical Mechanics SM1, SM2
   - Quantum Mechanics QM1, QM2
   - Quantum Mechanics QM3, QM4

   Note that there are two pairs of Quantum Mechanics problems. You have to do **one** problem from each pair.

7. All problems have equal weight.
1. CM–1

Consider a "simplified solar system" containing only the Sun and Jupiter. Assume that both are in circular orbits about the common center of mass, with separation $a$. In parts (c) and (d) of this problem, we will be working in the coordinate system which rotates with the two bodies (at an angular rate $\Omega$).

(a) Write an expression for the rotation rate $\Omega$ as a function of $a$, $M_{\text{Sun}}$ and $M_{\text{Jup}}$ (2 points)

(b) In the rotating frame, the Sun and Jupiter are at rest. If we define $\mu = M_{\text{Sun}}/(M_{\text{Sun}} + M_{\text{Jup}})$, give an expression for the positions of the two objects in terms of $\mu$ and $a$. For simplicity, take the x-axis to be the direction connecting the Sun and Jupiter. (2 points)

(c) Write down an expression for the Lagrangian of a small test particle of mass $m$ in this system in the frame co-rotating with the objects. Derive the equations of motion of this test particle. (5 points)

(d) What are the conditions for the test particle (let’s call it an asteroid) to be in equilibrium? Does any term in the equations of motion not contribute to the Lagrangian in equilibrium? (1 point)
A wheel with fine teeth is attached to the end of a spring with constant $k$ and unstretched length, $l$. For $x > l$, the wheel slips freely on the surface, but for $x < l$, the teeth mesh with the teeth on the ground so that it cannot slip. Assume that all the mass of the shell is in its rim.

i) The wheel is pulled to $x = l + b$ and released. How close will it come to the wall on its first trip?

ii) How far will it go out after it leaves the wall?
3. EM-1

An uncharged conducting sphere of radius $a$ is coated with a thick insulating shell (dielectric constant $\epsilon_r$) out to radius $b$. The object is now placed in an external uniform electric field $E_0$. Find the electric field in the insulator.

Useful info:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A\ell r^\ell + \frac{B\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$
a) Given a grounded conducting sphere of radius $a$ and a point charge $q$ located at a distance $b$ from the center of the sphere ($b > a$), what is the image charge distribution needed to find the field outside the sphere? (Specify the total image charge, its shape and location relative to the center of the sphere).

b) A grounded conducting plane has a hemispherical bump of radius $a$ on its surface as shown. A point charge $q$ is placed as shown. Find the electrostatic potential everywhere on the right side of the plane. Use the center of the hemispherical bump as the origin of your coordinate system.
In this problem, we will consider the stretching of a simplified polymer under the influence of an external force. The polymer is taken to be a chain of \( N \) rod-like segments, each of length \( l \), that are connected head-to-tail. The orientation of each segment is independent, and to simplify things further, we will assume that a segment can point either up or down. We stretch the polymer by first fixing one end at height \( z = 0 \), and then attaching a small weight to the other end. Gravity exerts a force \( f \) on the weight.

a) Show that the partition function for the stretched polymer is given by:

\[
Z = \left( e^{\beta fl} + e^{-\beta fl} \right)^N
\]

where \( \beta \equiv 1/k_B T \). (Hint: this model is similar to the Ising spin model) [5 points]

b) Calculate the mean extension of the polymer (the distance the weight falls below \( z = 0 \)) at force \( f \). [3 points]

c) Show that at the limit of low force, i.e. \( f \ll k_B T/l \), the polymer behaves as a Hookean (linear) spring, with a spring constant given by \( k_B T/Ll \), where \( L \) is the contour length of the polymer (the sum of all the segment lengths). [2 points]
In a ferromagnet, the total magnetization $M(T)$ depends on temperature due to the generation of spin waves, whose quantized form is called magnons. For a cubic lattice, the magnon dispersion relation is given by

$$\hbar \omega = (2JSa^2)k^2$$

where $J$ is the exchange constant, $S$ the spin, and $a$ the lattice constant. The quantization of spin waves proceeds exactly as for photons and phonons. The energy of a mode of frequency $\omega_k$ with $n_k$ magnons is given by

$$\epsilon_k = (n_k + \frac{1}{2})\hbar \omega_k$$

The excitation of a magnon corresponds to the reversal of one spin $\frac{1}{2}$. Calculate the temperature dependence of the magnetization $M(T)$ (at $T = 0K$, $M(0) = NS$).
Consider the hydrogen atom in its ground state. At $t = 0$ an electric field perturbation is turned on, determined by the potential

$$V = e \vec{E}_0 \cdot \vec{r} \sin \omega t$$

What should be the minimal frequency for ionization? Compute using first order perturbation theory the ionization probability per unit time. (Assume the final state wavefunction for the electron to be free).
Consider a two level system with a Hamiltonian given by
\[ H = \begin{pmatrix} E_a & V e^{i\omega t} \\ V^* e^{-i\omega t} & E_b \end{pmatrix} \]
in the basis set \{ |a\rangle, |b\rangle \}. The state vector at time \( t \) is expressed in the form
\[ \begin{pmatrix} c_a(t) e^{-iE_a t/\hbar} \\ c_b(t) e^{-iE_b t/\hbar} \end{pmatrix} \]

(a) Show that the coefficients \( c_a(t) \) and \( c_b(t) \) obey the equations
\[ \dot{c_a}(t) = -\frac{i}{\hbar} V e^{i\omega t} e^{-i\omega_0 t} c_b(t); \quad \dot{c_b}(t) = -\frac{i}{\hbar} V^* e^{-i\omega t} e^{i\omega_0 t} c_a(t), \quad \omega_0 \equiv \frac{E_b - E_a}{\hbar} \]

(b) Solve the simultaneous equations in part (a) and show that the solution satisfying the initial conditions \( c_a(0) = 1, c_b(0) = 0 \) has the form
\[ c_b(t) = -\frac{i}{\hbar \omega_r} V^* e^{i(\omega - \omega_0)t/2} \sin(\omega_r t), \quad \text{and} \]
\[ c_a(t) = e^{i(\omega - \omega_0)t/2} \left[ \cos(\omega_r t) + i \left( \frac{\omega_0 - \omega}{2 \omega_r} \right) \sin(\omega_r t) \right] \]

where \( \omega_r = \sqrt{\frac{\omega - \omega_0)^2}{4} + \left( \frac{|V|}{\hbar} \right)^2} \) is the Rabi flopping frequency

(c) Determine the transition probability \( P_{a\to b}(t) \), and show that it never exceeds 1. Confirm that
\[ |c_a(t)|^2 + |c_b(t)|^2 = 1 \]

(d) Expand the result for \( P_{a\to b}(t) \) to first order in \(|V|\), and determine the regime where this is a good approximation.

(e) At what time does the system first return to its initial state?
9. QM-3

(a) (2 points) Consider a spin 1/2 system. Determine the eigenvalues and eigenvectors of the matrix $S_y$ in the $|+\rangle$, $|-\rangle$ basis (1 point).

Two non-interacting spin 1/2 particles, both prepared in their $|+\rangle$ state, are now placed in a uniform magnetic field directed along the $x$-axis.

(b) (3 points) Determine $|\psi(t)\rangle$.

(c) (3 points) Determine the average value of $S_x$, $S_y$ and $S_z$ (note that $S_z = S_{1z} + S_{2z}$).

(d) (2 points) Give a physical interpretation of your answer to part (c).
Consider a particle of mass $m$ moving in one spatial dimension ($x$) and in a state described by the wavefunction:

$$\psi(x) = e^{-(x/a)4}$$

where constant $a$ has the dimension of length. The wavefunction as written is not normalized but the normalization is not needed to answer the questions below.

a) (4 points) Find the potential energy $V(x)$ for which $\psi(x)$ is an energy eigenstate. For definiteness define $V(x)$ such that $V(x = 0) = 0$. What is its (eigen) energy?

b) (2 points) Is $\psi(x)$ the ground state, or an excited state? Explain your reasoning.

c) (2 points) Sketch the potential energy $V(x)$ that you found in part (a). It should have two degenerate minima – where are these located? On the other hand, the eigenfunction $\psi(x)$ is non-degenerate and reflection symmetric about the point $x = 0$. Use your understanding of quantum mechanics to reconcile these two observations. A sentence or two will suffice.

c) (2 points) Consider the limit $|x| \gg a$. Is the functional form of $\psi(x)$ qualitatively consistent with the WKB formula as applied to the potential energy $V(x)$ that you found in part (a)?