Physics 206

Homework #1.

13.1.1.

(13.1.8); (13.1.9); (13.1.10):

$$\frac{d^2 v}{d\rho^2} - 2\frac{dv}{d\rho} + \left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right]v = 0 \qquad \left(\lambda = \sqrt{\frac{2m}{\hbar^2 W}}\right)$$

$$v = \rho^{l+1}\sum_{k=0}^{\infty} C_k \rho^k$$

$$\frac{d^2}{d\rho^2}\left(\sum_{k=0}^{\infty} C_k \rho^{k+l+1}\right) - 2\frac{d}{d\rho}\left(\sum_{k=0}^{\infty} C_k\rho^{k+l+1}\right) +$$

$$+\left[\frac{e^2\lambda}{\rho} - \frac{l(l+1)}{\rho^2}\right]\sum_{k=0}^{\infty} C_k \rho^{k+l+1} = 0;$$

$$\sum_{k=0}^{\infty} C_k \rho^{k+l-1}(k+l+1)(k+l) - 2\sum_{k=0}^{\infty} C_k (k+l+1)\rho^{k+l} +$$

$$+\sum_{k=0}^{\infty} e^2\lambda C_k \rho^{k+l} - l(l+1)\sum_{k=0}^{\infty} C_k \rho^{k+l-1} = 0;$$

$$\cancel{\sum_{k=0}^{0} C_0 \rho^{l-1}(l+1)l} + \sum_{k=1}^{\infty} C_k \rho^{(k-1)+l}(k+1+l)(k+l) -$$

$$-2\sum_{k=0}^{\infty} C_k (k+l+1)\rho^{k+l} + \sum_{k=0}^{\infty} e^2\lambda C_k \rho^{k+l} \cancel{- l(l+1)\sum_{k=0}^{0} C_0 \rho^{l-1}} -$$

$$-\sum_{k=1}^{\infty} l(l+1) C_k \rho^{k-1+l} =$$

$$= \sum_{q=k-1=0}^{\infty} C_{q+1}\rho^{q+l}\left[(q+l+2)(q+l+1) - l(l+1)\right] -$$

$$-\sum_{k=0}^{\infty} C_k \rho^{k+l}\left[2(k+l+1) - e^2\lambda\right] =$$

$$= \sum_{k=0}^{\infty} C_{k+1}\rho^{k+l}\left[(k+l+2)(k+l+1) - l(l+1)\right] -$$

$$-\sum_{k=0}^{\infty} C_k \rho^{k+l}\left[2(k+l+1) - e^2\lambda\right] = 0$$

$$C_{k+1} = C_k \frac{2(k+l+1) - e^2\lambda}{(k+l+2)(k+l+1) - l(l+1)} \qquad (13.1.11)$$

$$\frac{C_{k+1}}{C_k} \approx \frac{2}{k} \quad (k \to \infty) \Rightarrow C_{k+1} \approx \frac{2}{k}\frac{2}{k-1}\cdots\frac{2}{1} C_1 \approx \frac{2^k}{k!} C_1$$

$$\rho \sum^{l+1} C_k \rho^k \approx C_0 \rho^{l+1} + \rho^{l+1} C_1 \sum \frac{(2\rho)^k}{k!} \sim \rho^{l+1} e^{2\rho}.$$

$$U \sim v e^{-\rho} \to \infty \quad \text{for } \rho \to \infty.$$

This is not what we want.

The only way to avoid the problem:

$$2(k+l+1) - e^2 \lambda = 0 \Rightarrow \lambda = \frac{2(k+l+1)}{e^2} \Rightarrow$$

$$\sqrt{\frac{2m}{\hbar^2}W} = \frac{2(k+l+1)}{e^2} \Rightarrow E = -W = -\frac{me^4}{2\hbar^2(k+l+1)^2}$$

(13.1.14)

13.1.5. $\left\langle \frac{d}{dt}(\hat{\vec{R}} \cdot \hat{\vec{P}}) \right\rangle = 0.$

$$\frac{d}{dt}(\vec{R} \cdot \vec{P}) = \frac{d\vec{R}}{dt}\vec{P} + \vec{R}\frac{d\vec{P}}{dt} = \frac{\partial H}{\partial \vec{P}}\vec{P} - \vec{R}\frac{\partial H}{\partial \vec{R}} =$$

$$= \frac{\vec{P}}{m}\vec{P} - \vec{R}\frac{e^2}{R^2}\frac{\vec{R}}{R} = \frac{P^2}{2m} + \frac{P^2}{2m} - \frac{e^2}{R} = \hat{T} + \hat{H}$$

$$\langle \hat{T} + \hat{H} \rangle = 0 \Rightarrow \langle \hat{T} \rangle = -\langle \hat{H} \rangle \Rightarrow \langle \hat{T} \rangle = -\frac{1}{2}\langle \hat{V} \rangle$$

$$(\hat{H} = \hat{T} + \hat{V})$$

13.4.1. $E = -\frac{m(Ze^2)^2}{2\hbar^2 n^2} \sim Z^2.$

$$U(z) \sim \frac{Ze}{z}, \quad z \sim \frac{1}{Z} \Rightarrow E \sim Z^2.$$

13.4.2. Let us estimate n_{max} for Uranium. ($Z = 92$)

$2 + 2\cdot 4 + 2\cdot 9 + 2\cdot 16 = 60$

$2 + 2\cdot 4 + 2\cdot 9 + 2\cdot 16 + 2\cdot 25 = 110.$

Thus, $n_{max} \sim 5$ (strictly speaking, Uranium has the configuration $5f^3 6d^1 7s^2$ [and $n = 1, 2, 3, 4, 5s, 5p, 5d, 6s, 6p$ are filled])

For $n_{max} \sim 5$ one expects $Z_U \sim \frac{n_{max}^2 Z_H}{Z_{unscreened}}$, where $Z_{unscreened}$ is the unscreened charge seen by the ~~[illegible]~~ outermost electrons. (1-2)

Z unscreened ~ # of the outermost electrons.

There are several such electrons.

Hence, $n_{max} \sim Z$ unscreened $\Rightarrow r_u \sim 5 r_H$

(Actually, the empirical atomic radius $r_u \sim 1.75\,\text{Å} \sim 3.5 r_H$)

$$Ry = \frac{me^4}{2\hbar^2} = \frac{0.91 \cdot 10^{-27} (4.8 \cdot 10^{-10})^4}{2 \cdot (1.05 \cdot 10^{-27})^2} \text{ cgs} \approx 220 \frac{10^{-27} \cdot 10^{-40}}{10^{-54}} = \underline{\underline{2.2 \cdot 10^{-11} \text{ erg}}}$$

$$= 2.2 \cdot 10^{-18}\,\text{J} = eV, \quad V = \frac{2.2 \cdot 10^{-18}\,\text{J}}{1.6 \cdot 10^{-19}\,\text{C}} \approx \underline{\underline{13.6\,\text{eV}}}$$

$$2.2 \cdot 10^{-18}\,\text{J} = k_B T$$

$$T = \frac{2.2 \cdot 10^{-18}\,\text{J}}{k_B} = \frac{2.2 \cdot 10^{-18}\,\text{J}}{1.38 \cdot 10^{-23}\,\text{J/K}} \approx \underline{\underline{1.6 \cdot 10^{5}\,\text{K}}}.$$