Physics 206.
Homework #2

14.3.2 1). $|\vec{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix}$

$|\vec{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi/2} \\ \cos\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix}$

$$\vec{n}\hat{\vec{\sigma}} = \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix}$$

$$\vec{n}\hat{\vec{\sigma}}\,|\vec{n}+\rangle = \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \cos\frac{\theta}{2}\cos\theta\, e^{-i\varphi/2} + \sin\frac{\theta}{2}\sin\theta\, e^{-i\varphi} \\ \cos\frac{\theta}{2}\sin\theta\, e^{i\varphi/2} - \cos\theta\sin\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \cos(\theta - \frac{\theta}{2})\, e^{-i\varphi/2} \\ \sin(\theta - \frac{\theta}{2})\, e^{i\varphi/2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix} = \frac{1}{2}|\vec{n}+\rangle$$

$$\langle \vec{n}- | \vec{n}+\rangle = -\sin\frac{\theta}{2}\, e^{+i\varphi/2}\cos\frac{\theta}{2}\, e^{-i\varphi/2} + \cos\frac{\theta}{2}\, e^{-i\varphi/2}\sin\frac{\theta}{2}\, e^{i\varphi/2} =$$

$= 0.$

Since eigenvectors of $\vec{n}\vec{\sigma}$ are orthogonal, $|\vec{n}-\rangle$ is the second eigenvector $\Rightarrow$ $\vec{n}\vec{\sigma}|\vec{n}-\rangle = -\frac{1}{2}|\vec{n}-\rangle$.

2). $\langle \vec{n}\pm|\vec{S}|\vec{n}\pm\rangle = \langle \vec{n}\pm|(\vec{n}\vec{S})\vec{n} + (\vec{k}\vec{S})\vec{k} + (\vec{l}\vec{S})\vec{l}\,|\vec{n}\pm\rangle$

where $\vec{n}, \vec{k}, \vec{l}$ form a basis.

In the $|\vec{n}\pm\rangle$ basis $(\vec{n}\vec{S}) = S_z$, $(\vec{k}\vec{S}) = S_x$, $(\vec{l}\vec{S}) = S_y$.

$$\langle \vec{n}\pm|\vec{S}|\vec{n}_\pm\rangle = \langle \pm|\, S_z\vec{n} + \vec{k}S_x + \vec{l}S_y\,|\pm\rangle =$$

$$= \langle\pm|S_z|\pm\rangle\vec{n} \quad (\langle\pm|S_x|\pm\rangle = \langle\pm|S_y|\pm\rangle = 0 \leftarrow \text{diagonal elements})$$

$$\langle\pm|S_z|\pm\rangle = \pm\frac{1}{2}\vec{n}.$$

14.3.3. $\sigma_z = -i\,\sigma_x\sigma_y$ (and cyclic permutations) $\Rightarrow$

$$\mathrm{Tr}\,\sigma_z = -i\,\mathrm{Tr}(\sigma_x\sigma_y) \quad \text{(and permutations)}$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \Rightarrow \quad \mathrm{Tr}\,\sigma_z = \frac{1}{2}(-i)\,\mathrm{Tr}(\sigma_x\sigma_y + \sigma_y\sigma_x) =$$

$$= -\frac{i}{2}\,\mathrm{Tr}\,0 = 0$$

14.3.4. 1). $(\vec{A}\,\vec{\sigma})(\vec{B}\,\vec{\sigma}) = \sum A_\alpha \sigma_\alpha B_\beta \sigma_\beta = \sum A_\alpha B_\beta \sigma_\alpha \sigma_\beta =$

$$= \frac{1}{2}\sum A_\alpha B_\beta\left([\sigma_\alpha, \sigma_\beta] + [\sigma_\alpha, \sigma_\beta]_+\right) = \frac{1}{2}\sum_\alpha A_\alpha B_\alpha [\sigma_\alpha, \sigma_\beta]_+ +$$

$$+ \frac{1}{2}\sum_{\alpha\neq\beta} A_\alpha B_\beta [\sigma_\alpha, \sigma_\beta] = (\vec{A}\,\vec{B}) + \sum_{\alpha\neq\beta} A_\alpha A_\beta\, i\,\varepsilon_{\alpha\beta\gamma}\sigma_\gamma =$$

$$= (\vec{A}\,\vec{B}) + \sum i\,\varepsilon_{\alpha\beta\gamma} A_\alpha A_\beta \sigma_\gamma = (\vec{A}\,\vec{B}) + i\,\vec{\sigma}[\vec{A}\times\vec{B}]$$

2) $(\vec{A}\,\vec{\sigma})(\vec{B}\,\vec{\sigma}) = \sum m_\alpha \sigma_\alpha$

$$m_\beta = \frac{1}{2}\,\mathrm{Tr}\left[(\vec{A}\vec{\sigma})(\vec{B}\vec{\sigma})\right]\sigma_\beta .$$

$$m_0 = \frac{1}{2}\mathrm{Tr}\,(\vec{A}\vec{\sigma})(\vec{B}\vec{\sigma}) = \frac{1}{2}\,\mathrm{Tr}\sum_{\alpha\beta} A_\alpha B_\beta \sigma_\alpha\sigma_\beta =$$

$$= \frac{1}{2}\,2\sum_\alpha A_\alpha B_\alpha = \vec{A}\,\vec{B}$$

$$m_{\alpha=x,y,z} = \frac{1}{2}\mathrm{Tr}\left[(\vec{A}\vec{\sigma})(\vec{B}\vec{\sigma})\sigma_\alpha\right] = \frac{1}{2}\sum_{\beta\gamma}\mathrm{Tr}\,A_\beta B_\gamma \sigma_\beta\sigma_\gamma\sigma_\alpha =$$

$$= \frac{1}{2}\sum_{\beta\neq\gamma}\mathrm{Tr}\,A_\beta B_\gamma\,\sigma_\beta\sigma_\gamma\sigma_\alpha = \frac{1}{2}\sum_{\substack{\beta\neq\gamma\\ \delta}}\mathrm{Tr}\,A_\beta B_\gamma\, i\,\varepsilon_{\beta\gamma\delta}\sigma_\delta\sigma_\alpha =$$

$$= \sum_{\beta\gamma\delta}\delta_{\alpha\delta} A_\beta B_\gamma\, i\,\varepsilon_{\beta\gamma\delta} = \sum_{\beta\gamma} i\,\varepsilon_{\alpha\beta\gamma} A_\beta B_\gamma = i[\vec{A}\times\vec{B}]_\alpha$$

14.3.7. 1). $\sqrt{I + i\sigma_x} = \sqrt{a}\left(I\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,\vec{n}\,\vec{\sigma}\right)$

$\vec{n} = \vec{e}_x$

$$\left.\begin{array}{l} a\cos\frac{\theta}{2} = 1 \\ -a\sin\frac{\theta}{2} = 1 \end{array}\right\} \Rightarrow \quad \begin{array}{l} \theta = -\frac{\pi}{2} \Rightarrow \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}},\ \sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}, \\ a = \sqrt{2}. \end{array}$$

$$\sqrt{I + i\sigma_x} = \sqrt{\sqrt{2}\,\exp\left(-i\left(-\frac{\pi}{4}\right)\sigma_x\right)} = 2^{1/4}\exp\left(\frac{\pi i}{8}\sigma_x\right) =$$

$$= 2^{1/4}\left[\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\,\sigma_x\right]$$

2) $(2I+\sigma_x)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{bmatrix} \frac{2}{2\cdot 2-1\cdot 1} & -\frac{1}{2\cdot 2-1\cdot 1} \\ -\frac{1}{2\cdot 2-1\cdot 1} & \frac{2}{2\cdot 2-1\cdot 1} \end{bmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} =$

$= \frac{2}{3}\hat{I} - \frac{1}{3}\hat{\sigma}_x$.

3) $\sigma_x^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{0}{0-1\cdot 1} & -\frac{1}{0-1\cdot 1} \\ -\frac{1}{0-1\cdot 1} & \frac{0}{0-1\cdot 1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$ //

14.4.1. $H = -\gamma \bar{L}\cdot\bar{B}$

$\frac{d\langle \bar{L}\rangle}{dt} = \frac{d}{dt}\langle\psi(t)|\hat{\bar{L}}|\psi(t)\rangle = \frac{d}{dt}\langle\psi(0)\exp(\frac{i\hat{H}t}{\hbar})|\hat{\bar{L}}|$

$\exp[-\frac{i\hat{H}t}{\hbar})\psi(0)\rangle = \langle\psi(0)\exp(\frac{i\hat{H}t}{\hbar})\ \hat{H}\frac{i}{\hbar}|\hat{L}|\exp(-\frac{i\hat{H}t}{\hbar})\psi(0)\rangle -$

$-\langle\psi(0)\exp(\frac{i\hat{H}t}{\hbar})|\hat{L}|(-\frac{i\hat{H}}{\hbar})\exp(-\frac{i\hat{H}t}{\hbar})|\psi(0)\rangle =$

$= \frac{i}{\hbar}\langle[\hat{H},\hat{\bar{L}}]\rangle$

$[H,\hat{L}] = -\gamma[\bar{B}\bar{L},\bar{L}] = -\gamma\sum_{\alpha\beta}B_\alpha[L_\alpha,L_\beta]e_\beta =$

$= -\gamma\hbar i\sum_{\alpha\beta\delta}B_\alpha\bar{e}_\beta\varepsilon_{\alpha\beta\delta}L_\delta = -i\gamma\hbar\sum_{\beta\delta\alpha}\varepsilon_{\beta\delta\alpha}\bar{e}_\beta L_\delta B_\alpha =$

$= -i\gamma\hbar[\vec{L}\times\bar{B}]$.

$\frac{d\langle\vec{L}\rangle}{dt} = \frac{\gamma\hbar}{\hbar}\langle[\bar{L}\times\bar{B}]\rangle = \langle\gamma\bar{L}\rangle\times\bar{B} = \langle\vec{\mu}\rangle\times\vec{B}$

→ Our trick is known as the Heisenberg picture.

14.3.8. 1) $[A,\vec{\sigma}] = 0 \Rightarrow [\sum A_\alpha\sigma_\alpha,\vec{\sigma}] = 0 \Rightarrow [A_0\hat{I},\vec{\sigma}] +$

$+[\sum_{\alpha=xyz}A_\alpha\sigma_\alpha,\vec{\sigma}] = 0 \Rightarrow \sum_{\alpha=xyz}A_\alpha[\sigma_\alpha,\vec{\sigma}] = 0 \Rightarrow$ for any δ and β

$Tr\ \sigma_\delta\sum_{\alpha=x,y,z}A_\alpha[\sigma_\alpha,\sigma_\beta] = 0 \Rightarrow 2Tr\ \sigma_\delta\sum_{\alpha\gamma}A_\alpha i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma = 0 \Rightarrow$

$\Rightarrow 2\cdot 2\sum_\alpha iA_\alpha\varepsilon_{\alpha\beta\delta} = 0 \Rightarrow A_x = A_y = A_z = 0$

(e.g. $\beta = y, \delta = z \Rightarrow \sum_\alpha A_\alpha\varepsilon_{\alpha yz} = 0 \Rightarrow A_x = 0$).

2). $A\sigma_\alpha + \sigma_\alpha A = 0 \Rightarrow \sum_{\beta=0,xyz}A_\beta(\sigma_\beta\sigma_\alpha + \sigma_\alpha\sigma_\beta) = 0 \Rightarrow Tr\sum_\beta A_\beta(\sigma_\beta\sigma_\alpha + \sigma_\alpha\sigma_\beta)$

$= 0 \Rightarrow A_\alpha = 0$ for $\alpha = x,y,z$ since $Tr(\sigma_\beta\sigma_\alpha) = Tr(\sigma_\alpha\sigma_\beta) \neq 0$ only for $\alpha = \beta$. Thus, $A = A_0\hat{I}$. However, $A_0\hat{I}\hat{\sigma}_\alpha + \hat{\sigma}_\alpha A_0\hat{I} =$

$= 2A_0\hat{\sigma}_\alpha$. This is equal to zero for $A_0 = 0$ only. Hence, $A = 0$.

HW 2-3

14.4.3. $i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}|\psi(t)\rangle$; $|\psi_2(t)\rangle = e^{-i\omega S_z t}|\psi(t)\rangle$

$$i\hbar \frac{d|\psi_2(t)\rangle}{dt} = i\hbar[-i\omega\hat{S}_z e^{-i\omega\hat{S}_z t}|\psi(t)\rangle + e^{-i\omega\hat{S}_z t}|\dot{\psi}(t)\rangle];$$

$$i\hbar|\dot{\psi}_2(t)\rangle = i\hbar[-i\omega\hat{S}_z|\psi_2(t)\rangle + e^{-i\omega\hat{S}_z t}(-\gamma\hbar\vec{\hat{S}}\vec{B})\frac{1}{i\hbar}|\psi(t)\rangle]$$

$$i\hbar|\dot{\psi}_2(t)\rangle = i\hbar[-i\omega\hat{S}_z|\psi_2(t)\rangle + e^{-i\omega\hat{S}_z t}(i\gamma\vec{\hat{S}}\vec{B})e^{+i\omega\hat{S}_z t}e^{-i\omega\hat{S}_z t}|\psi(t)\rangle]$$

$$i\hbar|\dot{\psi}_2(t)\rangle = i\hbar[-i\omega\hat{S}_z|\psi_2(t)\rangle + e^{-i\omega\hat{S}_z t}(i\gamma\vec{\hat{S}}\vec{B})e^{i\omega\hat{S}_z t}|\psi_2(t)\rangle]$$

$$e^{-i\omega\hat{S}_z t}(i\gamma\vec{\hat{S}}\vec{B})e^{i\omega\hat{S}_z t} = i\gamma e^{-i\omega\hat{S}_z t}[B_0\hat{S}_z + B\hat{S}_x\cos\omega t - B\hat{S}_y\sin\omega t]e^{i\omega\hat{S}_z t}$$

$$= i\gamma[B_0\hat{S}_z] + i\gamma e^{-i\omega\hat{S}_z t}[B\hat{S}_x\cos\omega t - B\hat{S}_y\sin\omega t]e^{i\omega\hat{S}_z t} = i\gamma B_0\hat{S}_z + i\gamma B(\cos\frac{\omega t}{2}\hat{I} - i\hat{S}_z 2\sin\frac{\omega t}{2})\times$$

$$\times[\hat{S}_x\cos\omega t - S_y\sin\omega t](\cos\frac{\omega t}{2}\hat{I} + 2i\hat{S}_z\sin\frac{\omega t}{2}) =$$

$$= i\gamma B_0\hat{S}_z + i\gamma B[\hat{S}_x\cos\omega t\cos\frac{\omega t}{2} + \hat{S}_y\cos\omega t\sin\frac{\omega t}{2} - \hat{S}_y\sin\omega t\cos\frac{\omega t}{2} + \hat{S}_x\sin\omega t\sin\frac{\omega t}{2}](\cos\frac{\omega t}{2}\hat{I} + 2i\hat{S}_z\sin\frac{\omega t}{2}) =$$

$$= i\gamma B_0\hat{S}_z + i\gamma B[\hat{S}_x\cos\frac{\omega t}{2} - \hat{S}_y\sin\frac{\omega t}{2}][\cos\frac{\omega t}{2}\hat{I} + 2i\hat{S}_z\sin\frac{\omega t}{2}] =$$

$$= i\gamma B_0\hat{S}_z + i\gamma B[\hat{S}_x\cos^2\frac{\omega t}{2} - \hat{S}_y\sin\frac{\omega t}{2}\cos\frac{\omega t}{2} + \hat{S}_y\cos\frac{\omega t}{2}\sin\frac{\omega t}{2} + S_x\sin^2\frac{\omega t}{2}] = i\gamma B_0\hat{S}_z + i\gamma B\hat{S}_x$$

$$i\hbar|\dot{\psi}_2\rangle = i\hbar[-i\omega\hat{S}_z + i\gamma B_0\hat{S}_z + i\gamma B\hat{S}_x]|\psi_2\rangle;$$

$$|\dot{\psi}_2\rangle = [i(\gamma B_0 - \omega)\hat{S}_z + i\gamma B\hat{S}_x]|\psi_2\rangle$$

$$U_2(t) = \exp[i(\gamma B_0 - \omega)\hat{S}_z t + i\gamma B\hat{S}_x t] =$$

$$= \exp[i\omega_2 t(\frac{\gamma B_0 - \omega}{\omega_2}\hat{S}_z + \frac{\gamma B}{\omega_2}\hat{S}_x)] =$$

$$= \cos\frac{\omega_2 t}{2}\hat{I} + 2i(\frac{\gamma B_0 - \omega}{\omega_2}\hat{S}_z + \frac{\gamma B}{\omega_2}\hat{S}_x)\sin\frac{\omega_2 t}{2} =$$

$$= \cos\frac{\omega_2 t}{2}\hat{I} + 2i(\frac{\omega_0 - \omega}{\omega_2}\hat{S}_z + \frac{\gamma B}{\omega_2}\hat{S}_x)\sin\frac{\omega_2 t}{2}.$$

$$|\psi(t)\rangle = \begin{pmatrix} e^{+i\omega t/2} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{pmatrix}|\psi_2(t)\rangle =$$

HW 2-4

$$= \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \begin{bmatrix} \cos\frac{\omega_2 t}{2} + i\frac{\omega_0-\omega}{\omega_2}\sin\frac{\omega_2 t}{2} & i\frac{\gamma B}{\omega_2}\sin\frac{\omega_2 t}{2} \\ i\frac{\gamma B}{\omega_2}\sin\frac{\omega_2 t}{2} & \cos\frac{\omega_2 t}{2} - i\frac{\omega_0-\omega}{\omega_2}\sin\frac{\omega_2 t}{2} \end{bmatrix}$$

$$\times |\psi(0)\rangle = \begin{bmatrix} e^{i\omega t/2}\left[\cos\frac{\omega_2 t}{2} + i\frac{\omega_0-\omega}{\omega_2}\sin\frac{\omega_2 t}{2}\right] \\ e^{-\frac{i\omega t}{2}}\frac{i\gamma B}{\omega_2}\sin\frac{\omega_2 t}{2} \end{bmatrix}$$

At $\omega_0 = \omega$

$$\psi(t) = \begin{bmatrix} e^{i\omega t/2}\cos\frac{\omega_2 t}{2} \\ e^{-\frac{i\omega t}{2}}\frac{i\gamma B}{\omega_2}\sin\frac{\omega_2 t}{2} \end{bmatrix}$$

$$\omega_2 = \gamma\sqrt{B^2 + \left(\frac{\omega_0}{\gamma} - \frac{\omega}{\gamma}\right)^2} = \gamma B \Rightarrow$$

$$\psi(t) = \begin{bmatrix} e^{i\omega t/2}\cos\frac{\omega_2 t}{2} \\ ie^{-i\omega t/2}\sin\frac{\omega_2 t}{2} \end{bmatrix} = \sqrt{i}\begin{bmatrix} e^{\frac{i\omega t}{2} - \frac{\pi i}{4}}\cos\frac{\omega_2 t}{2} \\ e^{-\frac{i\omega t}{2} + \frac{i\pi}{4}}\sin\frac{\omega_2 t}{2} \end{bmatrix}$$

$|\vec{n}+\rangle$ state

$$\mu_z = \frac{1}{2}\left[|\psi_\uparrow(t)|^2 - |\psi_\downarrow(t)|^2\right] =$$

$$= \frac{1}{2}\left[\cos^2\frac{\omega_2 t}{2} + \left(\frac{\omega_0-\omega}{\omega_2}\right)^2\sin^2\frac{\omega_2 t}{2} - \frac{\gamma^2 B^2}{\omega_2^2}\sin^2\frac{\omega_2 t}{2}\right] =$$

$$= \frac{1}{2}\left[\left(\frac{\omega_0-\omega}{\omega_2}\right)^2 - \frac{\gamma^2 B^2}{\omega_2^2}\sin^2\frac{\omega_2 t}{2} + \cos^2\frac{\omega_2 t}{2}\left(1 - \frac{(\omega_0-\omega)^2}{\gamma^2 B^2 + (\omega_0-\omega)^2}\right)\right] =$$

$$= \mu_z(0)\left[\frac{(\omega_0-\omega)^2}{(\omega_0-\omega)^2 + \gamma^2 B^2} + \frac{\gamma^2 B^2\left(\cos^2\frac{\omega_2 t}{2} - \sin^2\frac{\omega_2 t}{2}\right)}{(\omega_0-\omega)^2 + \gamma^2 B^2}\right] =$$

$$= \mu_z(0)\left[\frac{(\omega_0-\omega)^2}{(\omega_0-\omega)^2 + \gamma^2 B^2} + \frac{\gamma^2 B^2\cos\omega_2 t}{(\omega_0-\omega)^2 + \gamma^2 B^2}\right]$$

14.4.4. $\omega = \gamma B = \frac{eB}{mc}$. Flip means $\Delta\theta = \pi$.

$$\Delta\theta = \omega t \Rightarrow t = \frac{\pi mc}{eB} \approx \frac{3\cdot 10^{-27} g\ 3\cdot 10^{10} \frac{cm}{s}}{5\cdot 10^{-10} cgs\ 10^{2}} \approx$$

$$\approx 2\cdot\frac{10^{-17}}{10^{-8}} s = \underline{\underline{2\cdot 10^{-9} s}}.$$

3.4. $\hat{S}_z(\hat{S}_z+1)(\hat{S}_z-1)|\psi\rangle = \hat{S}_z(\hat{S}_z+1)(\hat{S}_z-1)\sum_{i=-1,0,1}\alpha_i|\psi_i\rangle$

where $\hat{S}_z|\psi_k\rangle = k|\psi_k\rangle$.

$\hat{S}_z|\psi_0\rangle = 0$; $(\hat{S}_z+1)|\psi_{-1}\rangle = 0$; $(\hat{S}_z-1)|\psi_1\rangle = 0$.

Hence, $\hat{S}_z(\hat{S}_z+1)(\hat{S}_z-1) = 0$.

$\hat{S}_z$ and $\hat{S}_x$ are equivalent.

Hence, $\hat{S}_z(\hat{S}_x+1)(\hat{S}_x-1) = 0$.