# Multi-Directional Width-Bounded Geometric Separator and Protein Folding*

Bin Fu[1,2], Sorinel A Oprisan[3], and Lizhe Xu[1,2]

[1] Dept. of Computer Science, University of New Orleans, LA 70148, USA.
[2] Research Institute for Children, 200 Henry Clay Avenue, LA 70118.
[3] Dept. of Psychology, University of New Orleans, LA 70148.
Email: fu@cs.uno.edu,soprisan@uno.edu,lxu@chnola-research.org

**Abstract.** We introduce the concept of multi-directional width-bounded geometric separator and get improved separator for the grid graph, which improves exact algorithm for the protein folding problem in the HP-model. For a grid graph $G$ with $n$ grid points $P$, there exists a separator $A \subseteq P$ such that $A$ has less than or equal to $1.02074\sqrt{n}$ points, and $G - A$ has two disconnected subgraphs with less than or equal to $\frac{2}{3}n$ nodes on each of them. We also derive $0.7555\sqrt{n}$ lower bound for such a separator on grid graph. The previous upper bound record for the grid graph $\frac{2}{3}$-separator is $1.129\sqrt{n}$ [6].

## 1 Introduction

Lipton and Tarjan [11] showed that every $n$ vertices planar graph has at most $\sqrt{8n}$ vertices whose removal separates the graph into two disconnected parts of size at most $\frac{2}{3}n$. Their $\frac{2}{3}$-separator was improved by a series of papers [4, 8, 1, 5] with best record $1.97\sqrt{n}$ by Djidjev and Venkatesan [5]. Spielman and Teng [14] found a $\frac{3}{4}$-separator with size $1.82\sqrt{n}$ for planar graphs. Some other forms of the geometric separators were studied by Miller, Teng, Thurston, and Vavasis [12] and by Smith and Wormald [13]. If each of $n$ input points is covered by at most $k$ regular geometric object such as circles, rectangles, etc, then there exist $O(\sqrt{k \cdot n})$ size separators [12, 13]. In particular, Smith and Wormald obtained the separator of size $4\sqrt{n}$ for the case $k = 1$. The lower bounds $1.555\sqrt{n}$ and $1.581\sqrt{n}$ for the $\frac{2}{3}$-separator for the planar graph were proven by Djidev [5], and by Smith and Wormald [13], respectively.

Each edge in a grid graph connects two grid points of distance 1 in the set of vertices. Thus a grid graph is a special planar graph. Fu and Wang [6] developed a method for deriving sharper upper bound separator for grid graphs by controlling the distance to the separator line. Their separator is determined by a straight line on the plane and the set of grid points with distance less than or equal to $\frac{1}{2}$ to the line. They proved that for an $n$ vertices grid graph on the plane, there is a separator that has less than or equal to $1.129\sqrt{n}$ grid points and each

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of two disconnected subgraphs has at most $\frac{2}{3}n$ grid points. Using this separator and their approximation to the separator line, they obtained the first $n^{O(n^{1-\frac{1}{d}})}$-time exact algorithm for the $d$-dimensional protein folding problem of the HP-model. The method of Fu and Wang [6] was further developed and generalized by Fu [7]. The notion of width-bounded geometric separator was introduced by Fu [7]. For a positive constant $a$ and a set of points $Q$ on the plane, an $a$-wide separator is the region between two parallel lines of distance $a$ that partitions $Q$ into $Q_1$ (on the left side of the separator's region), $S$ (inside the separator's region), and $Q_2$ (on the right side of the separator's region). The width-bounded geometric separators were applied to many other problems, which include the disk covering problem on the plane, the maximum independent set problem, the vertex covering problem, and dominating set problem on disk graphs. Fu [7] derived $2^{O(\sqrt{n})}$-time exact algorithms for all of them, whose previous algorithms need $n^{O(\sqrt{n})}$ time.

This paper introduces the concept of a multi-directional width-bounded separator. For a set of points $P$ on the plane and two vectors $v_1$ and $v_2$, the $(a, b)$-wide separator (along the directions $v_1$ and $v_2$) is the region of points that have distance less than or equal to $a$ to $L$ along $v_1$ or distance less than or equal to $b$ to $L$ along $v_2$, where $L$ is a straight line (separator line) on the plane. The separator size is measured by the number of points from $P$ in the region and the line $L$ partitions the set $P$ into two balanced subsets. In this paper we use this new method to improve the separator for the grid graph. The multi-directional width approach is different from that used in [6, 7], which only controls the regular distance to the middle line in the separator area. Pursuing smaller and more balanced separator is an interesting problem in combinatorics and also gives more efficient algorithms for the divide and conquer applications. For a grid graph $G$ with $n$ grid points $P$, there exists a separator subset $A \subseteq P$ such that $A$ has up to $1.02074\sqrt{n}$ points, and $G - A$ has two disconnected subgraphs with up to $\frac{2}{3}n$ nodes on each of them. This improves the previous $1.129\sqrt{n}$ size separator for the grid graph [6]. We also prove a $0.7555\sqrt{n}$ lower bound for the size of the separators for grid graphs. Our lower bound is based on a result that the shortest curve partitioning an unit circle into two areas with ratio $1 : 2$ is a circle arc. Its length is less than that of the straight line partitioning the circle with the same ratio.

Protein structure prediction with computational technology is one of the most significant problems in bioinformatics. A simplified representation of proteins is a lattice conformation, which is a self-avoiding sequence in $Z^3$. An important representative of lattice models is the HP-model, which was introduced by Lau and Dill [9, 10]. In this model, the 20 amino acids are reduced to a two letter alphabet by H and P, where H represents hydrophobic amino acids, and P, polar or hydrophilic amino acids. Two monomers form a contact in some specific conformation if they are not consecutive, but occupy neighboring positions in the conformation (i.e., the distance vector between their positions in the conformation is a unit vector). A conformation with minimal energy is a conformation with the maximal number of contacts between non-consecutive H-monomers (see

Figure 3). The folding problem in the HP-model is to find the conformation for any HP-sequence with minimal energy. This problem was proved to be NP-hard in both 2D and 3D [2, 3]. We will apply our new separator to the protein folding problem in the 2D HP model to get an $O(n^{5.563\sqrt{n}})$ time exact algorithm, improving the previous $O(n^{6.145\sqrt{n}})$-time algorithm [6].

## 2 Separators upper bound for grid graphs

We first give a series of notations. For a set $A$, $|A|$ denotes the number of elements in $A$. For two points $p_1, p_2$ in the $d$-dimensional space ($R^d$), $\text{dist}(p_1, p_2)$ is the Euclidean distance. For a set $A \subseteq R^d$, $\text{dist}(p_1, A) = \min_{q \in A} \text{dist}(p_1, q)$. The integer set is represented by $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. For integers $x_1$ and $x_2$, $(x_1, x_2)$ is a *grid point*. A *grid square* is an $1 \times 1$ square that has four grid points as its four corner points. For a set $V$ of grid points on the plane, let $E_V$ be the set of edges $(v_i, v_j)$ (straight line segments) such that $v_i, v_j \in V$ and $\text{dist}(v_i, v_j) = 1$. Define $G = (V, E_V)$ as the *grid graph*. For $0 < \alpha < 1$, an $\alpha$-separator for a grid graph $G = (V, E_V)$ is a subset $A \subseteq V$ such that $G - A$ has two disconnected areas $G_1 = (V_1, E_{V_1})$ and $G_2 = (V_2, E_{V_2})$ with $|V_1|, |V_2| \le \alpha|V|$. Define $C(o, r) = \{(x, y)|\text{dist}((x, y), o) \le r\}$, which is the disc area with center at point $o$ and radius $r$. For $r > 0$, define $D(r)$ to be the union region of 4 discs $C((0, -r), r) \cup C((0, r), r) \cup C((-r, 0), r) \cup C((r, 0), r)$ (see the left of Figure 1). For a region $R$ on the plane, define $G(R)$ to be the set of all grid points in the region $R$. For a 2D vector $v$, a *line* $L$ in $R^2$ through a fixed point $p_0 \in R^2$ along the direction $v$ corresponds to the equation $p = p_0 + tv$ that characterizes all the points $p$ on $L$, where the parameter $t \in (-\infty, +\infty)$. For a point $p_0$ and a line $L$, the distance of $p_0$ to $L$ along direction $v$ is $\text{dist}(p_0, q)$, where $q$ is the intersection between $p = p_0 + tv$ and $L$. Let $v_1, v_2, \cdots, v_k$ be $k$ fixed vectors. A point $p$ has distance $\le (a_1, \cdots, a_k)$ to $L$ along directions $v_1, v_2, \cdots, v_k$ if $p$ has distance $\le a_i$ along direction $v_i$ for some $i = 1, \cdots, k$.

**Definition 1.** *Let $P$ be a set of points in $R^2$, $v_1, \cdots, v_m$ be 2D vectors, and $w_1, \cdots, w_m$ be positive real numbers. A $(w_1, \cdots, w_m)$-wide separator for the set $P$ along the directions $v_1, \cdots, v_m$ is a region $R$ determined by a line $L$. The region $R$ consists all points with distance $\le (w_1, \cdots, w_m)$ along $v_1, \cdots, v_m$. The separator size is measured by the number of points of $P$ in the region $R$. Its balance number is the least number $\alpha$ such that each side of $L$ has at most $\alpha|P|$ points from $P$.*

In the rest of this paper, we use two vectors $v_1 = (1, 0)$ an $v_2 = (0, 1)$ to represent the horizontal and vertical directions, respectively. If a point $p$ has distance $\le (a, a)$ from a line $L$, it means that the point $p$ has distance $\le a$ from $L$ along either direction $(1, 0)$ or $(0, 1)$ in the rest of this paper.

**Lemma 1.** *([15]) For an $n$-element set $P$ in a $d$-dimensional space, there is a point $q$ with the property that any half-space that does not contain $q$ covers at most $\frac{d}{d+1}n$ elements of $P$. (Such a point $q$ is called a centerpoint of $P$).*

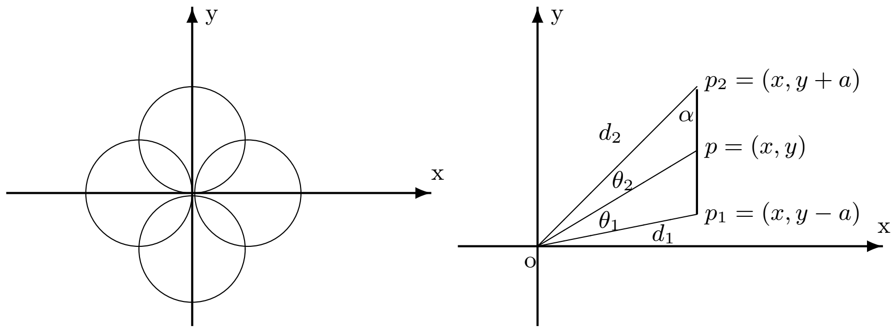


**Fig. 1.** Left: Area of grid points with maximal expectation. Right: Probability analysis

**Lemma 2.** *Let $P$ be a set of grid points on the plane and $(0,0) \notin P$. The sum $\sum_{p=(x,y)\in P} \max(\frac{|x|}{x^2+y^2}, \frac{|y|}{x^2+y^2})$ is maximal when $P \subseteq G(D(R))$, where $R$ is the least radius with $|G(D(R))| \geq |P|$.*

*Proof.* Let $L$ be the line segment connecting $o = (0,0)$ and $p = (x,y)$. If $p' = (x', y')$ is another point between $o$ and $p$ on the line $L$, we have $\frac{\max(|x|,|y|)}{\mathrm{dist}(o,p)} = \frac{\max(|x'|,|y'|)}{\mathrm{dist}(o,p')}$. Since $\mathrm{dist}(o,p) > \mathrm{dist}(o,p')$, we have $\frac{\max(|x|,|y|)}{\mathrm{dist}(o,p)^2} < \frac{\max(|x'|,|y'|)}{\mathrm{dist}(o,p')^2}$. For the constant $c$, let $\frac{|x|}{x^2+y^2} = c$ or $\frac{|y|}{x^2+y^2} = c$. We have $x^2 + y^2 - \frac{1}{c}|x| = 0$ or $x^2 + y^2 - \frac{1}{c}|y| = 0$. The two equations characterize the four circles of $D(\frac{1}{2c})$. All points on the external boundary $D(r)$ have the same value $\frac{\max(|x|,|y|)}{\mathrm{dist}(o,p)^2}$. $\square$

Let $a$ be a constant $> 0$, $p$ and $o$ be two points on the plane, and $P$ be a set of points on the plane. We define the function $f_{p,o,a}(L) = 1$ if $p$ has $\leq (a,a)$ distance to the line $L$ and $L$ is through $o$; and 0 otherwise. Define $F_{P,o,a}(L) = \sum_{p\in P} f_{p,o,a}(L)$, which is the number of points of $P$ with $\leq (a,a)$ distance to $L$ for the line $L$ through $o$. The expectation $\mathrm{E}(F_{P,o,a})$ is the expected number of points in $P$ with distance $\leq (a,a)$ to the random line $L$ through $o$.

**Lemma 3.** *Let $a > 0$ be a constant and $\delta > 0$ be a small constant. Let $P$ be a set of $n$ grid points on the plane and $o$ be a point on the plane. Then $\mathrm{E}(F_{P,o,a}) \leq \frac{(4\pi+8)(1+\delta)a\sqrt{n}}{\pi\sqrt{4+2\pi}}$.*

*Proof.* Without loss generality, we assume that $o = (0,0)$ (Notice that $F_{P,o,a}$ is invariant under translation). Let $\epsilon > 0$ be a small constant that will be fixed later. Let us consider a grid point $p = (x,y) \in P$ on the plane and let $p_1 = (x, y-a)$ and $p_2 = (x, y+a)$. The angle between the two lines $op_1$ and $op_2$ will be estimated (Figure 1). Let $d = \mathrm{dist}(o,p)$, $d_1 = \mathrm{dist}(o,p_1)$ and $d_2 = \mathrm{dist}(o,p_2)$. Define the angles $\theta_1 = \angle pop_1, \theta_2 = \angle pop_2$ and $\alpha = \angle op_2p$.

From $\frac{a}{\sin\theta_2} = \frac{d}{\sin\alpha}$, we have $\sin\theta_2 = \frac{a}{d} \cdot \sin\alpha = \frac{a}{d} \cdot \frac{|x|}{d_2} = \frac{a|x|}{dd_2}$. Similarly, $\sin\theta_1 = \frac{a|x|}{dd_1}$. If $d > a$, then $\frac{a|x|}{d(d+a)} \leq \sin\theta_1, \sin\theta_2 \leq \frac{a|x|}{d(d-a)}$.

Let $\beta_1 = \angle poq_1$ ($\beta_2 = \angle poq_2$) be the angle between the line segments $op$ and $oq_1$ ($oq_2$ respectively), where $q_1 = (x-a, y)$ and $q_2 = (x+a, y)$. If $d > a$, then we also have $\frac{a|y|}{d(d+a)} \leq \sin\beta_1, \sin\beta_2 \leq \frac{a|y|}{d(d-a)}$. There is a constant $d_0$ such that if $d > d_0$, then we have the following inequalities: (i) $\frac{a|y|}{d^2}(1-\epsilon) \leq \beta_1, \beta_2 \leq \frac{a|y|}{d^2}(1+\epsilon)$, (ii) $\frac{a|x|}{d^2}(1-\epsilon) \leq \theta_1, \theta_2 \leq \frac{a|x|}{d^2}(1+\epsilon)$, and (iii) $\frac{(1-\epsilon)a\max(|x'|,|y'|)}{d'^2} < \frac{a\max(|x|,|y|)}{d^2} < \frac{(1+\epsilon)a\max(|x'|,|y'|)}{d'^2}$ for any $(x', y')$ with $\mathrm{dist}((x,y),(x',y')) \leq \sqrt{2}$, where $d' = \mathrm{dist}((x',y'),o)$.

Let $Pr(o,p,a)$ be the probability that the point $p$ has distance $\leq (a,a)$ to a random line $L$ through $o$. If $d \leq d_0$, then $Pr(o,p,a) \leq 1$. Otherwise, $Pr(o,p,a) \leq \frac{\max(2\max(\beta_1,\beta_2), 2\max(\theta_1,\theta_2))}{\pi} \leq \frac{2}{\pi}\max\left(\frac{a|y|}{d^2}, \frac{a|x|}{d^2}\right)(1+\epsilon)$. The number of grid points with distance $\leq d_0$ to $o$ is $\leq \pi(d_0+\sqrt{2})^2$. Then $\mathrm{E}(F_{P,o,a}) = \sum_{p\in P} E(f_{o,p,a}) = \sum_{p\in P} Pr(o,p,a) \leq \sum_{p\in P \text{ and } \mathrm{dist}(p,o)>d_0} Pr(o,p,a) + \sum_{p\in P \text{ and } \mathrm{dist}(p,o)\leq d_0} Pr(o,p,a) \leq \frac{2(1+\epsilon)}{\pi}\sum_{p\in P \text{ and } \mathrm{dist}(p,o)>d_0} \max\left(\frac{|x|}{d^2}, \frac{|y|}{d^2}\right) + \pi(d_0+\sqrt{2})^2$.

We only consider the case to make $\sum_{p\in P \text{ and } d>d_0} \max(\frac{|x|}{d^2}, \frac{|y|}{d^2})$ maximal. By Lemma 2, it is maximal when the points of $P$ are in the area $D(R)$ with the smallest $R$.

For a grid point $p=(i,j)$, define $\mathrm{grid}_1(p) = \{(x,y) | i-\frac{1}{2} < x < i+\frac{1}{2} \text{ and } j-\frac{1}{2} < y < j+\frac{1}{2}\}$, and $\mathrm{grid}_2(p) = \{(x,y) | i-\frac{1}{2} \leq x \leq i+\frac{1}{2} \text{ and } j-\frac{1}{2} \leq y \leq j+\frac{1}{2}\}$. If the grid point $p \notin D(R)$, then $\mathrm{grid}_1(p) \cap D(R-\frac{\sqrt{2}}{2}) = \emptyset$. The area size of $D(R)$ is $2\pi R^2 + 4R^2$. Assume $R$ is the minimal radius such that $D(R)$ contains at least $n$ grid points. The region $D(R-\epsilon)$ contains $< n$ grid points for every $\epsilon > 0$. This implies $D(R-\epsilon-\frac{\sqrt{2}}{2}) \subseteq \cup_{\text{grid point } p\in D(R-\epsilon)} \mathrm{grid}_2(p)$. Therefore, $2\pi(R-\frac{\sqrt{2}}{2}-\epsilon)^2 + 4(R-\frac{\sqrt{2}}{2}-\epsilon)^2 \leq n$. Hence, $R \leq \frac{\sqrt{n}}{\sqrt{4+2\pi}} + \frac{\sqrt{2}}{2} + \epsilon < \frac{\sqrt{n}}{\sqrt{4+2\pi}} + \sqrt{2}$ (the constant $\epsilon$ will be $\leq \frac{\sqrt{2}}{2}$).

Let $A_1 = \{p = (x,y) \in D(R) | \text{the angle between } op \text{ and } x\text{-axis is in } [0, \frac{\pi}{4}]\}$, which is the $\frac{1}{8}$ area of $D(R)$. The probability that a point $p (= (x,y))$ has distance $\leq (a,a)$ to the random line $L$ is $\leq \frac{2(1+\epsilon)ax}{\pi d^2}$ for $p$ in $A_1$ with $\mathrm{dist}(p,o) > d_0$. The expectation of the number of points (with distance $\leq (a,a)$ to $L$ and distance $> d_0$ to $o$) of $P$ in the area $A_1$ is $\sum_{p\in A_1\cap P \text{ and } \mathrm{dist}(p,o)>d_0} Pr(o,p,a) \leq \sum_{p\in A_1\cap P \text{ and } \mathrm{dist}(p,o)>d_0} \frac{2(1+\epsilon)ax}{\pi d^2} \leq \int\int_{A_1} \frac{2(1+\epsilon)^2 ax}{\pi d^2} d_x d_y = \frac{2(1+\epsilon)^2 a}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{2R\cos\theta} \frac{r\cos\theta}{r^2}\cdot r d_r d_\theta = \frac{2(1+\epsilon)^2 a}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{2R\cos\theta} \cos\theta d_r d_\theta = \frac{2(1+\epsilon)^2 aR}{\pi}\int_0^{\frac{\pi}{4}} 2(\cos\theta)^2 d_\theta = \frac{2(1+\epsilon)^2 aR}{\pi}\cdot(\frac{\pi}{4}+\frac{1}{2}) = \frac{(1+\epsilon)^2 aR}{\pi}\cdot(\frac{\pi}{2}+1)$.

Since $R \leq \frac{\sqrt{n}}{\sqrt{4+2\pi}} + \sqrt{2}$, the total expectation is

$\mathrm{E}(F_{P,o,a}) \leq 8\sum_{p\in A_1\cap P \text{ and } \mathrm{dist}(p,o)>d_0} Pr(o,p,a) + \pi(d_0+\sqrt{2})^2 \leq \frac{8(1+\epsilon)^2 aR}{\pi}\cdot(\frac{\pi}{2}+1) + \pi(d_0+\sqrt{2})^2 \leq \frac{(4\pi+8)(1+3\epsilon)a\sqrt{n}}{\pi\sqrt{4+2\pi}} \leq \frac{(4\pi+8)(1+\delta)a\sqrt{n}}{\pi\sqrt{4+2\pi}}$ for all large $n$. We assign to the constant $\epsilon$ the value $\min(\frac{\delta}{3}, \frac{\sqrt{2}}{2})$. □

**Theorem 1.** *Let $a > 0$ be a constant and $P$ be a set of $n$ grid points on the plane. Let $\delta > 0$ be a small constant. There is a line $L$ such that the number of* *points in $P$ with $\leq (a,a)$ distance to $L$ is $\leq \frac{(4\pi+8)(1+\delta)a\sqrt{n}}{\pi\sqrt{4+2\pi}}$, and each half plane has $\leq \frac{2n}{3}$ points from $P$ for all large $n$.*

*Proof.* Let $o$ be the center point of set $P$ (by Lemma 1). The theorem follows from Lemma 3. □

The following corollary shows that for each grid graph of $n$ nodes, its $\frac{2}{3}$-separator size is bounded by $1.02074\sqrt{n}$. For two grid points of distance 1, if they stay on different sides of separator line $L$, one of them has $\leq (\frac{1}{2}, \frac{1}{2})$ distance to $L$.

**Corollary 1.** *Let $P$ be a set of $n$ grid points on the plane. There is a line $L$ such that the number of points in $P$ with $\leq (1/2, 1/2)$ distance to $L$ is $\leq 1.02074\sqrt{n}$, and each half plane has $\leq \frac{2n}{3}$ points from $P$.*

*Proof.* By Theorem 1 with $a = \frac{1}{2}$, we have, $\frac{8(1+\epsilon)}{\pi}\frac{1}{2} \cdot (\frac{\pi}{2}+1) \cdot \frac{1}{\sqrt{4+2\pi}} < 1.02074$ when $\epsilon$ is small enough. □

## 3 Separator lower bound for grid graphs

In this section we prove the existence of a lower bound of $0.7555\sqrt{n}$ for the grid graph separator. We calculate the length of the shortest curve partitioning the unit circle into two areas with ratio $1:2$ (Theorem 2). A simple closed curve in the plane does not cross itself. Jordan's theorem states that every simple closed curve divides the plane into two compartments, one inside the curve and one outside of it, and that it is impossible to pass continuously from one to the other without crossing the curve.

**Theorem 2.** *The shortest curve that partitions an unit circle into two regions with ratio $1:2$ has length $> 1.8937$.*

*Proof.* (Sketch) Using the standard method of variational calculus, the shortest curve partitioning the unit circle with the fixed area ratio between two regions is a circle arc. Using the numerical method, we can calculate the length of the circle arc. □

**Definition 2.** *A graph is connected if there is a path between every two nodes in the graph. For a connected grid graph $G = (V, E_V)$, a contour of $G$ is a circular path $C = v_1v_2\cdots v_kv_1$ such that 1) $(v_i, v_{i+1}) \in E_V (i = 1, 2, \cdots, k-1)$ and $(v_k, v_1) \in E_V$; 2) all points of $V$ are in the one side of $C$; and 3) for any $i \leq j$, $v_1 \cdots v_{i-1}v_{j+1} \cdots v_kv_1$ does not satisfy both 1) and 2). A point $v \in V$ is a boundary point if $d(v, u) = 1$ for some grid point $u \notin V$. A contour $C$ separates $w$ from all grid points $V$ if every path from $w$ to a node in $V$ intersects $C$.*

Example: Let $V$ be the set of all dotted grid points in Figure 2.

$C = v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_1$ is a contour for $V$. The condition 3) prevents $C' = v_1v_2v_{15}v_2v_3v_4v_5v_6v_7v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_1$ from being a contour.

**Lemma 4.** *Let $G = (V, E_V)$ be a connected grid graph. If the grid point $v \in V$ and grid point $w \notin V$ have the distance $\mathrm{dist}(v, w) = 1$, then there is a contour $C$ such that $C$ contains $v$ and separates $w$ from all grid points of $V$.*

*Proof.* Imagine that a region starting from the grid point $w$ grows until it touches all of the reachable edges of $G$ (but never crosses any of them). Since $G$ is a connected grid graph, the boundary forms a contour that consists of edges of $G$. As $\mathrm{dist}(w, v) = 1$, the vertex $v$ should appear in the contour. □

**Lemma 5.** *Let $G = (V, E_V)$ be a grid graph and $C$ be a contour of $G$. Let $U = \{u | u$ is a grid point not in $V$ with $\mathrm{dist}(u, v) = 1$ for some $v \in V$ and $C$ separates $u$ from $V\}$. Then there is a list of grid points $u_1, u_2, \cdots, u_{m+1}$ in $U$ such that $u_{m+1} = u_1$, $dist(u_i, u_{i+1}) \le \sqrt{2}$ for $i = 1, 2, \cdots, m$ and all points of $P$ are on one side of the circle path $u_1 u_2 \cdots u_{m+1}$ (the edge connecting every two consecutive points $u_1, u_2$ is straight line).*

*Proof.* Walking along the contour $C = v_1 \cdots v_k v_1$, we assume that only the left side has the points from $V$. A point $v_i$ on $C$ is called *special point* if $v_{i-1} = v_{i+1}$. The point $v_9$ is a special point at the contour $v_1 v_2 \cdots v_{14} v_1$ in Figure 2. For each edge $(v_i, v_{i+1})$ in $C$, the grid square, which is on the right side of $(v_i, v_{i+1})$ and contains $(v_i, v_{i+1})$ as one of the four boundary edges, has at least one point not in $V$. Let $S_1, S_2, \cdots, S_k$ be those grid squares for $(v_1, v_2), (v_2, v_3), \cdots, (v_k, v_1)$, respectively. For each special point $v_i$ on $C$, it has two special grid squares $S'_i$ and $S''_i$ that share the edge $(v_i, u)$ for some $u \in U$ with $\mathrm{dist}(u, v_i) = 1$ and $\mathrm{dist}(u, v_{i-1}) = 2$ (for example, $S'_9$ and $S''_9$ on Figure 2). Insert $S'_i$ and $S''_i$ between $S_i$ and $S_{i+1}$. We get a new list of grid squares $H_1, H_2, \cdots, H_m$. We claim that for every two consecutive $H_i$ and $H_{i+1}$, there are grid points $u_i \in H_i \cap U$ and $u_{i+1} \in H_{i+1} \cap U$ with $\mathrm{dist}(u_i, u_{i+1}) \le \sqrt{2}$. The lemma is verified by checking the following cases:

Case 1. $H_i = S_j$ and $H_{i+1} = S_{j+1}$ for some $j < k$.

Subcase 1.1. $S_j$ and $S_{j+1}$ share one edge $v_{j+1} u$. An example of this subcase is the grid squares $S_1$ and $S_2$ on Figure 2. It is easy to see that $u \in U$ since $u$ is on the right side when walking along the cycle path $C$.

Subcase 1.2. $S_j = S_{j+1}$. An example of this subcase is the grid squares $S_5$ and $S_6$ on Figure 2. This is a trivial case.

Subcase 1.3. $S_j$ and $S_{j+1}$ only share the point $v_{j+1}$. An example of this subcase is the grid squares $S_{11}$ and $S_{12}$ on Figure 2. We have grid points $u_1 \in U$ and $u_2 \in U$ such that $\mathrm{dist}(u_1, v_{j+1}) = 1$, $\mathrm{dist}(u_2, v_{j+1}) = 1$. Furthermore, $\mathrm{dist}(u_1, u_2) = \sqrt{2}$.

Case 2. $H_i = S''_j$ and $H_{i+1} = S_j$ for some $j < m$. An example of this subcase is the grid squares $S''_9$ and $S_9$ on Figure 2. The two squares share the edge $v_j u$ for some $u \in U$.

Case 3. $H_i = S'_j$ and $H_{i+1} = S''_j$. An example of this subcase is the grid squares $S'_9$ and $S''_9$ on Figure 2. The two squares share the edge $u_j u$ for some $u \in U$.

Case 4. $H_i = S_{j-1}$ and $H_i = S'_j$. An example of this subcase is the grid squares $S_8$ and $S'_9$ on Figure 2. The two squares share $v_j u$ for some $u \in U$. □

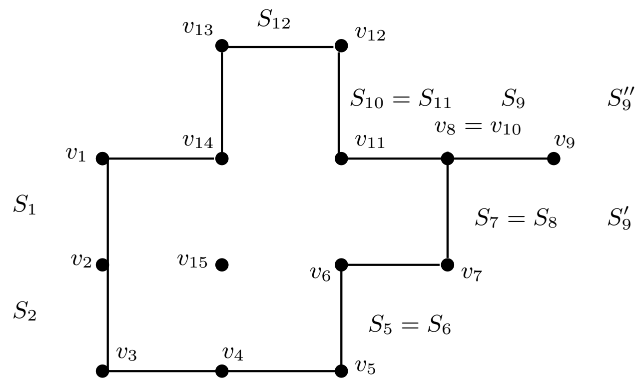


**Fig. 2.** Contour $C = v_1v_2 \cdots v_{14}v_1$. The node $v_9$ is a special point. When walking along $v_1 \cdots v_{14}v_1$, we see that each $S_i$ is the grid square on the right of $v_iv_{i+1}$

**Definition 3.** *For a region $R$ on the plane, define $A(R)$ to be the area size of $R$. An unit circle has radius 1. For a region $R$ in the unit circle, $L(R)$ is the length of the boundary of $R$ inside the internal area of the unit circle. A region $R$ inside a unit circle is type 1 region if part of its boundary is from the unit circle boundary. Otherwise, it is called type 2 region, which does not share any boundary with the unit circle.*

**Lemma 6.** *Assume $s > 0$ is a constant and $p_1, p_2$ are two points on the plane. We have 1) the area with the shortest boundary and area size $s$ on the plane is a circle with radius $\sqrt{\frac{s}{\pi}}$; and 2) the shortest curve that is through both $p_1$ and $p_2$, and forms an area of size $s$ with the line segment $p_1p_2$ is a circle arc.*

The proof of Lemma 6 can be found in regular variational calculus textbooks (e.g. [16]). Let $R$ be a type 1 region of area size $s$. Let $C$ be the part of R boundary that is an unit circle arc with $p_1$ and $p_2$ as two end points. Let $C'$ be the rest of the boundary of $R$. Let $R'$ be the region with the boundary $C$ and line segment $p_1p_2$. Assume the length of $C'$ is minimal. If $A(R) = A(R')$, then $C'$ is the same as the line segment $p_1p_2$. If $A(R) < A(R')$, then $C'$ is a circle arc inside $R'$ (between $C$ and $p_1p_2$). If $A(R) > A(R')$, then $C'$ is also a circle arc outside $R'$. Those facts above follow from Lemma 6.

**Lemma 7.** *Let $s \leq \pi$ be a constant. Let $R_1, R_2, \cdots, R_k$ be $k$ regions inside an unit circle (they may have overlaps), $\sum_{i=1}^{k} A(R_i) = s$ and $\sum_{i=1}^{k} L(R_i)$ is minimal. Then $k = 1$ and $R_1$ is a type 1 region.*

*Proof.* We consider the regions $R_1, \cdots, R_k$ that satisfy $\sum_{i=1}^{k} A(R_i) = s$ and $\sum_{i=1}^{k} L(R_i)$ is minimal for $k \geq 1$. Each $R_i (i = 1, \cdots, k)$ is either type 1 or type 2 region. The part of boundary of $R_i$ that is also the boundary of the unit circle is called *old boundary*. Otherwise it is called type *new boundary*.

A type 2 region has to be a circle (by Lemma 6). For a type 1 region, its new boundary inside the unit circle is also a circle arc (otherwise, its length is not minimal by part 2 of Lemma 6). If we have both type 1 region $R_1$ and type 2 region $R_2$. Move $R_1$ to $R_1^*$ and $R_2$ to $R_2^*$ on the plane so that $R_1^*$ and $R_2^*$ have some intersection (not a circle) at their new boundaries. Let $R_2'$ be the circle with the same area size as $R_1^* \cap R_2^*$. The boundary length of $R_2'$ is less than that of $R_1^* \cap R_2^*$. So, $L(R_1) + L(R_2)$ reduces to $L(R_1^* \cup R_2^*) + L(R_2')$ if $R_1$ and $R_2$ are replaced by $R_1^* \cup R_2^*$ and $R_2'$ (Notice that $A(R_1) + A(R_2) = A(R_1^* \cup R_2^*) + A(R_1^* \cap R_2^*) = A(R_1^* \cup R_2^*) + A(R_2'))$. This contradicts that $\sum_{i=1}^{k} L(R_i)$ is minimal. Therefore, there is no type 2 region. We only have type 1 regions left. Assume that $R_1$ and $R_2$ are two type 1 regions. Let $R_1$ and $R_2$ have the unit circle arcs $p_1p_2$ and $p_2p_3$ respectively. They can merge into another type 1 region $R$ with the unit circle arc $p_1p_2p_3$ and the same area size $A(R) = A(R_1) + A(R_2)$. Furthermore, $L(R) < L(R_1) + L(R_2)$. A contradiction again. Therefore, $k = 1$ and $R_1$ is a type 1 region. □

**Definition 4.** *Let $q$ be a positive real number. Partition the plane into $q \times q$ squares by the horizontal lines $y = iq$ and vertical lines $x = jq$ ($i, j \in Z$). Each point $(iq, jq)$ is a $(q, q)$-grid point, where $i, j \in Z$.*

**Lemma 8.** *Let $V$ be the set of all $(q, q)$ grid points in the unit circle $C$. Let $G = (V, E_V)$ be the grid graph on $V$, where $E_V = \{(v_i, v_j) | \mathrm{dist}(v_i, v_j) = 1$ and $v_i, v_j \in V\}$. Let $t$ be a constant $\geq 1$. Assume that $l$ is a curve that partitions a unit circle $C$ into two regions $U_1$ and $U_2$ with $\frac{A(U_1)}{A(U_2)} = \frac{1}{t}$. If the minimal length of $l$ is $c_0$, then every $\frac{t}{t+1}$-separator for the grid graph $G$ has a size $\geq \frac{(c_0 - o(1))(\sqrt{n} - \sqrt{2\pi})}{\sqrt{2}\sqrt{\pi}}$.*

*Proof.* Assume that the unit circle $C$ area has $n$ $(q, q)$-grid points. We have $\pi(1 + q\sqrt{2})^2 \geq n \cdot q^2$. It implies $q \leq \frac{1}{\frac{\sqrt{n}}{\sqrt{\pi}} - \sqrt{2}}$. Assume $S \subseteq V$ is the smallest separator for $G = (V, E_V)$ such that $G - S$ has two disconnected subgraphs $G_1 = (V_1, E_{V_1})$ and $G_2 = (V_2, E_{V_2})$, which satisfy $|V_1|, |V_2| \leq \frac{tn}{t+1}$. By Corollary 1, $|S| \leq 1.021\sqrt{n}$. Let $G_1$ have connected components $F_1, \cdots, F_m$. By Lemma 5, each $F_i$ is surrounded by a circular path $H_i$ with grid points not from $G_1$. Actually, the grid points of $H_i$ inside $C$ are from the separator $A$. Let $P_1, \cdots, P_k$ be the parts of $H_1, \cdots, H_m$ inside the $C$. They consist of vertices in $A$ and the distance between every two consecutive vertices in each $P_i$ is $\leq \sqrt{2}q$ (by Lemma 5 and scaling $(q, q)$ grid points to $(1, 1)$ grid points).

The number of $(q, q)$-grid points with distance $\leq 2q$ to the unit circle boundary is $O(\sqrt{n})$. For a $(q, q)$-grid point $p = (iq, jq)$ and $h > 0$, define $\mathrm{grid}_h(p) = \{(x, y) | iq - \frac{h}{2} \leq x \leq iq + \frac{h}{2}$ and $jq - \frac{h}{2} \leq y \leq jq + \frac{h}{2}\}$. Let $V_H$ be the set of all $(q, q)$-grid points in $H_1, \cdots, H_m$ and $V_P$ be the set of all $(q, q)$-grid points in $P_1, \cdots, P_k$. Let $S_1 = \cup_{p \in V_1} \mathrm{grid}_q(p)$. Since $|V_1| + |V_2| + |S| = n$, $|V_1|, |V_2| \leq \frac{tn}{t+1}$, and $|S| \leq 1.021\sqrt{n}$, we have $\frac{tn}{t+1} q^2 \geq A(S_1) \geq (\frac{n}{t+1} - 1.021\sqrt{n})q^2$.

Assume that $P_1, \cdots, P_k$ together with the circle boundary partition the unit circle into two parts $X_1$ and $X_2$, where $X_1$ contains all grid points of $V_1$ and $X_2$ contains all grid points of $V_2$. It is easy to see that $S_1 - \cup_{p \in V_H} \mathrm{grid}_{2q}(p) \subseteq X_1 \subseteq S_1 \cup (\cup_{p \in V_H} \mathrm{grid}_{2q}(p))$. Therefore, $A(S_1) - O(q^2\sqrt{n}) \leq A(X_1) \leq A(S_1) + O(q^2\sqrt{n})$. Thus, $A(S_1) - O(\frac{1}{\sqrt{n}}) \leq A(X_1) \leq A(S_1) + O(\frac{1}{\sqrt{n}})$.

For the variable $x \leq \frac{\pi}{2}$, define the function $g(x)$ to be the length of the shortest curve that partitions the unit circle into regions $Q_1$ and $Q_2$ with $A(Q_1) = x$.

Then $g(x)$ is an increasing continuous function. For a small real number $\delta > 0$, let $D$ be the disk of area size $\delta$. Therefore, $D$ has radius $\sqrt{\frac{\delta}{\pi}}$. Put $D$ into the region $Q_2$ and let $D$ be tangent to the boundary of $Q_1$. The length of the boundary of $Q_1 \cup D$ inside the unit circle is $g(x) + 2\pi\sqrt{\frac{\delta}{\pi}} = g(x) + 2\sqrt{\delta\pi}$. Thus, $g(x+\delta) \le g(x) + 2\sqrt{\delta\pi}$.

Assume that total length of $P_1, \cdots, P_k$ is minimal, then $k = 1$ by Lemma 7. The length of $P_1$ is at least $g(\frac{1}{3}) - o(1) = c_0 - o(1)$ by the analysis in the last paragraph. Since every two consecutive grid points in $P_1$ has distance $\le \sqrt{2}q$, there are at least $\frac{c_0 - o(1)}{q\sqrt{2}} \ge \frac{(c_0 - o(1))(\frac{\sqrt{n}}{\sqrt{\pi}} - \sqrt{2})}{\sqrt{2}} = \frac{(c_0 - o(1))(\sqrt{n} - \sqrt{2\pi})}{\sqrt{2}\sqrt{\pi}}$ grid points of $A$ along $P_1$. □

**Theorem 3.** *There exists a grid graph $G = (V, E_V)$ such that for any $A \subseteq V$ if $G - A$ has two disconnected graphs $G_1$ and $G_2$, and $G_i (i = 1, 2)$ has $\le \frac{2|V|}{3}$ nodes, then $|A| \ge 0.7555\sqrt{n}$ when $n$ is large.*

*Proof.* By Theorem 2, the length of the shortest curve partitioning the unit circle into 1 : 2 ratio is $\ge 1.8937$. By Lemma 8 with $c_0 = 1.8937$ and $k = 1$, we have $|A| \ge 0.7555\sqrt{n}$. □

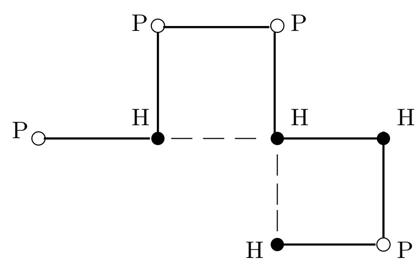


**Fig. 3.** The sequence PHPPHHPH is put on the 2 dimensional grid. There are 2 H-H contacts marked by the dotted lines.

## 4 Application to protein folding in the HP-model

We have shown that there is a size $O(\sqrt{n})$ separator line to partition the folding problem of $n$ letters into 2 problems in a balanced way. The 2 smaller problems are recursively solved and their solutions are merged to derive the solution to the original problem. As the separator has only $O(\sqrt{n})$ letters, there are at most $n^{O(\sqrt{n})}$ cases to partition the problem. The major revision from the algorithm in [6] is the approximation of the optimal separator line.

**Theorem 4.** *There is a $O(n^{5.563\sqrt{n}})$ time algorithm for the 2D protein folding problem in the HP-model.*

*Proof.* (Sketch) The algorithm is similar to that in [6]. We still use approximation to the separator line. Lemma 3 shows that we can avoid the separator line that is almost parallel or vertical to the $x$-axis. Let $\delta > 0$ be a small constant. Let $a = \frac{1}{2}, c = \frac{2}{3} + \delta$ and $d = k_0(a + \delta)$, where $k_0 = \frac{(4\pi+8)(1+\delta)}{\pi\sqrt{4+2\pi}}$ such that $k_0 a\sqrt{n}$ is the upper bound for the number of grid points with $\leq (a, a)$ distance to the separator line (by Theorem 1). With the reduced separator, its computational time is $(\frac{n}{\delta})^{O(\log n)} 2^{O(\sqrt{n})} n^{d(\frac{1}{1-\sqrt{c}})\sqrt{n}} = O(n^{5.563\sqrt{n}})$. □

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