Theory and Application of Width Bounded Geometric Separator*

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Abstract. We introduce the notion of the width bounded geometric separator and develop the techniques for the existence of the width bounded separator in any d-dimensional Euclidean space. The separator is applied in obtaining $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-complete geometric problems, whose previous algorithms take $n^{O(\sqrt{n})}$ time [2,5,1]. One of those problems is the well known disk covering problem, which seeks to determine the minimal number of fixed size disks to cover n points on a plane [10]. They also include some NP-hard problems on disk graphs such as the maximum independent set problem, the vertex cover problem, and the minimum dominating set problem.

1 Introduction

The geometric separator has applications in many problems . It plays an important role in the divide and conquer algorithms for geometric problems. Lipton and Tarjan [18] showed the well known geometric separator for planar graphs. They proved that every n vertices planar graph has at most $\sqrt{8n}$ vertices whose removal separates the graph into two disconnected parts of size at most $\frac{2}{3}n$. Their $\frac{2}{3}$-separator was improved to $\sqrt{6n}$ by Djidjev [8], $\sqrt{5n}$ by Gazit [13], $\sqrt{4.5n}$ by Alon, Seymour and Thomas [6] and $1.97\sqrt{n}$ by Djidjev and Venkatesan [9]. Spielman and Teng [25] showed a $\frac{3}{4}$-separator with size $1.82\sqrt{n}$ for planar graphs. Separators for more general graphs were derived in (e.g.[14]). Some other forms of the geometric separators were studied by Miller, Teng, Thurston, and Vavasis [22,21] and Smith and Wormald [24]. For a set of points on the plane, assume each point is covered by a regular geometric object such as circle, rectangle, etc. If every point on the plane is covered by at most k objects, it is called k-thick. Some $O(\sqrt{k \cdot n})$ size separators and their algorithms were derived in [22,21,24].

The planar graph separators were applied in deriving some $2^{O(\sqrt{n})}$-time algorithms for certain NP-hard problems on planar graphs by Lipton, Tarjan [19], Ravi and Hunt [23]. Those problems include computing the maximum independence set, minimum vertex covers and three-colorings of a planar graph, and the number of satisfying truth assignments to a planar 3CNF formula [17].

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In [24], their separators were applied in deriving $n^{O(\sqrt{n})}$-time algorithms for some geometric problems such as the planar Traveling Salesman and Steiner Tree problems. The separators were applied to the parameterized independent set problem on planar graphs by Alber, Fernau and Niedermeier [3,4] and disk graphs by Alber and Fiala [5].

We introduce the concept of width bounded separator. For a set of points Q on the plane, an a-wide separator is the region between two parallel lines of distance a. It partitions the set Q into two balanced subsets and its size is measured by the number of points of Q in the strip region. Our width bounded separator concept is geometrically natural and can achieve a much smaller constant c for its size upper bound $c\sqrt{n}$ than the previously approaches. Fu and Wang [11] developed a method for deriving sharper upper bound separator for grid points by controlling the distance to the separator line. They proved that for a set of n grid points on the plane, there is a separator that has $\leq 1.129\sqrt{n}$ points and has $\leq \frac{2}{3}n$ points on each side. That method was used to obtain the first sub-exponential time algorithm for the protein folding problem in the HP model. This paper not only generalizes the results of [11], but also substantially improves the techniques in [11]. We would like to mention our new technical developments in this paper. 1) In order to apply the separator to more general geometric problems with arbitrary input points other than grid points, we use weighted points in Euclid space and the sum of weights to measure the separator size instead of counting the number of points close to it. We introduce the local binding method to merge some points into a nearby grid point. This method is combined with our separator in deriving $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-complete geometric problems, whose previous algorithms take $n^{O(\sqrt{n})}$ time [2,5,1]. One of those problems is the well known disk covering problem, which seeks to determine the minimal number of fixed size disks to cover n points on a plane [10]. They also include some NP-hard problems on disk graphs such as the maximum independent set problem, the vertex cover problem, and minimum dominating set problem. 2) We will handle the case of higher dimension. We develop an area ratio method to replace the previous angle ratio method [11] for obtaining higher dimensional separators. 3) We develop a similar separator theorem for a set of points with distance of at least 1 between any two of them, called 1-separated set, we establish the connection between this problem and the famous fixed size discs packing problem. The discs packing problem in 2D was solved in the combinatorial geometry (see [26]). The 3D case, which is the Kepler conjecture, has a very long proof (see [16]). It is still a very elusive problem at higher dimensions. Our Theorem 2 shows how the separator size depends on packing density. 4) We develop a simple polynomial time algorithm to find the width-bounded separator for a fixed dimensional space. This is a starting point for the algorithms finding the width bounded geometric separator, and it is enough to obtain the $2^{O(\sqrt{n})}$ time exact algorithms for a class of NP-hard geometric problems.

2 Width-Bounded Separators on the *d*-Dimension

Throughout section 2 we assume the dimensional number d to be fixed. We will use the following well known fact that can be easily derived from Helly theorem (see [15,26]) to obtain our width bounded separator.

Lemma 1. *For an n-element set P in d-dimensional space, there is a point q with the property that any half-space that does not contain q, covers at most $\frac{d}{d+1}n$ elements of P. (Such a point q is called a centerpoint of P).*

Definition 1. *For two points p_1, p_2 in the d-dimensional Euclidean space R^d,* $\mathrm{dist}(p_1, p_2)$ *is the Euclidean distance between p_1 and p_2. For a set $A \subseteq R^d$,* $\mathrm{dist}(p_1, A) = \min_{q \in A} \mathrm{dist}(p_1, q)$. *In particular, if L is a hyper-plane in R^d,* $\mathrm{dist}(p, L)$ *is the regular distance from the point p to L. For $a > 0$ and a set A of points on d-dimensional space, if the distance between every two points is at least a, the set A is called a-separated. For $\epsilon > 0$ and a set of points $Q \subseteq R^d$, an ϵ-sketch of Q is a set of points $P \subseteq R^d$ such that each point in Q has distance $\leq \epsilon$ to some point in P. We say that P is a sketch of Q if P is an ϵ-sketch of Q for some positive constant ϵ (ϵ does not depend on the size of Q). A sketch set is usually an 1-separated set such as a set of grid points. A weight function $w : P \rightarrow [0, \infty)$ is often used to measure the point density of Q near each point of P. A hyper-plane in R^d through a fixed point $p_0 \in R^d$ is defined by the equation $(p - p_0) \cdot v = 0$, where v is normal vector of the plane and " $.$" is the regular inner product ($u \cdot v = \sum_{i=1}^{d} u_i v_i$ for $u = (u_1, \cdots, u_d)$ and $v = (v_1, \cdots, v_d)$). For $Q \subseteq R^d$ with sketch $P \subseteq R^d$, the width parameter $a > 0$, and the weight function $w : P \rightarrow [0, \infty)$, a $(2a)$-wide-separator is determined by a hyper-plane L. The separator has two measurements for its quality of separation: (1)* $\mathrm{balance}(L, Q) = \frac{\max(|Q_1|, |Q_2|)}{|Q|}$, *where Q_1 is the set of all points of Q on one side of L and Q_2 is the set of all points of Q on the other side of L (Note: Q_1 or Q_2 does not contain any point on L); and (2)* $\mathrm{measure}(L, P, a, w) = \sum_{p \in P, \mathrm{dist}(p, L) \leq a} w(p)$.

2.1 Volume, Area, Integrations and Probability

We need some integrations in order to compute volume and surface area size at arbitrary dimensional space. Some of the materials can be found in standard calculus books. We will treat the case of any fixed dimension. We recommend the reader understands the cases $d = 2$ and 3 first. We use the standard polar transformation: $x_d = r \cos\theta_{d-1}; x_{d-1} = r \sin\theta_{d-1} \cos\theta_{d-2}; \cdots; x_2 = r \sin\theta_{d-1} \sin\theta_{d-2} \cdots \sin\theta_2 \cos\theta_1$; and $x_1 = r \sin\theta_{d-1} \sin\theta_{d-2} \cdots \sin\theta_2 \sin\theta_1$.

It is a smooth map from $[0, R] \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$ to the d-dimensional ball of radius R with center at the origin. The Jacobian form is

$$J_d(r, \theta_{d-1}, \cdots, \theta_1) = \frac{\partial(x_d, x_{d-1}, \cdots, x_1)}{\partial(r, \theta_{d-1}, \cdots, \theta_1)} = \begin{vmatrix} \frac{\partial x_d}{\partial r} & \frac{\partial x_{d-1}}{\partial r} \cdots \frac{\partial x_1}{\partial r} \\ \frac{\partial x_d}{\partial \theta_{d-1}} & \frac{\partial x_{d-1}}{\partial \theta_{d-1}} \cdots \frac{\partial x_1}{\partial \theta_{d-1}} \\ \cdots & \\ \frac{\partial x_d}{\partial \theta_1} & \frac{\partial x_{d-1}}{\partial \theta_1} \cdots \frac{\partial x_1}{\partial \theta_1} \end{vmatrix}.$$

It has the recursive equation: $J_d(r, \theta_{d-1}, \cdots, \theta_1) = r \cdot (\sin\theta_{d-1})^{d-2} \cdot J_{d-1}(r, \theta_{d-2}, \cdots, \theta_1)$ for $d > 2$, which in turn gives the explicit expression: $J_d(r, \theta_{d-1}, \cdots, \theta_1) = r^{d-1} \cdot (\sin\theta_{d-1})^{d-2} \cdot (\sin\theta_{d-2})^{d-3} \cdots (\sin\theta_2)$. Let $B_d(R, o)$ be the d- dimensional ball of radius R and center o. The volume of the d-dimensional ball of radius R is $V_d(R) = \int_{B_d(R,o)} 1 d_z = \int_0^R \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} |J_d(r, \theta_{d-1}, \cdots, \theta_2, \theta_1)| d_r d_{\theta_{d-1}} \cdots d_{\theta_2} d_{\theta_1} = \frac{2^{(d+1)/2}\pi^{(d-1)/2}}{1\cdot 3\cdots(d-2)\cdot d} R^d$ if d is odd, and $\frac{2^{d/2}\pi^{d/2}}{2\cdot 4\cdots(d-2)\cdot d} R^d$, otherwise.

Let the d-dimensional ball have the center at o. We also need the integration $\int_{B_d(R,o)} \frac{1}{\mathrm{dist}(z,o)} d_z$, which is equal to $\int_0^R \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \frac{|J_d(r,\theta_{d-1},\cdots,\theta_2,\theta_1)|}{r} d_r d_{\theta_{d-1}} \cdots d_{\theta_2} d_{\theta_1} = \frac{d}{(d-1)R} V_d(R)$. Thus,

$$\int_{B_d(R,o)} \frac{1}{\mathrm{dist}(z, o)} d_z = \frac{d}{(d-1)R} V_d(R) \tag{1}$$

Let $V_d(r) = v_d \cdot r^d$, where v_d is constant for fixed dimensional number d. In particular, $v_1 = 2, v_2 = \pi$ and $v_3 = \frac{4\pi}{3}$. Define $A_d(h, R) = \{(x_1, \cdots, x_d) | \sum_{i=1}^d x_i^2 \leq R^2$ and $0 \leq x_1 \leq h\}$, which is a horizontal cross section of d-dimensional half ball of the radius R. The volume of $A_d(h, R)$ in the d-dimensional space is calculated by

$$U_d(h, R) = \int_0^h V_{d-1}(\sqrt{R^2 - x_1^2})\ d_{x_1} = v_{d-1} \int_0^h \left(\sqrt{R^2 - x_1^2}\right)^{d-1} d_{x_1} \tag{2}$$

The surface area size of 3D ball ($4\pi R^2$) is the derivative of its volume ($\frac{4}{3}\pi R^3$). The boundary length of circle ($2\pi R$) is the derivative of its area size (πR^2). This fact can be extended to both a higher dimensional ball and a cross section of a ball. The surface area size of $B_d(R, o)$ is $W_d(R) = \frac{\partial V_d(R)}{\partial R} = d \cdot v_d \cdot R^{d-1}$. The side surface of $A_d(h, R)$ is $\{(x_1, \cdots, x_d) | \sum_{i=1}^d x_i^2 = R^2$ and $0 \leq x_1 \leq h\}$. Its area size is

$$S_d(h, R) = \frac{\partial U_d(h, R)}{\partial R} = (d-1) v_{d-1} R \int_0^h \left(\sqrt{R^2 - x_1^2}\right)^{d-3} d_{x_1}$$

When R is fixed and h is small, we have $S_d(h, R) = v_{d-1} \cdot (d-1) \cdot R^{d-2} \cdot h + O(h^2)$. For a $a > 0$, the probability that a d-dimensional point p has the distance $\leq a$ to a random plane through origin will be determined. This probability at dimension 3 was not well treated in [11].

Lemma 2. *Let a be a real number ≥ 0. Let p and o be the two points on a d-dimensional space. Then the probability that p has distance $\leq a$ to a random plane through o is in* $[\frac{h_d \cdot a}{\mathrm{dist}(p,o)} - \frac{c_0 \cdot a^2}{\mathrm{dist}^2(p,o)}, \frac{h_d \cdot a}{\mathrm{dist}(p,o)} + \frac{c_0 \cdot a^2}{\mathrm{dist}^2(p,o)}]$, *where* $h_d = \frac{2(d-1)v_{d-1}}{d \cdot v_d}$ *and c_o are constants for a fixed d. In particular,* $h_2 = \frac{2}{\pi}$ *and* $h_3 = 1$.

Proof. Without loss of generality, let o be the origin $(0, \cdots, 0)$ (notice that the probability is invariant under translation). The point p can be moved to an axis via rotation that does not change the probability. Let's assume the point

$p=(x_1,0,\cdots,0)$, where $x_1=\mathrm{dist}(p,o)$. For an unit vector $v=(v_1,\cdots,v_d)$ with $v_1\ge 0$ in the d-dimensional space, the plane through the origin with normal vector v is defined as $u\cdot v=0$, where $\cdot$ represents the regular inner product between two vectors. The distance between p to the plane is $|p\cdot v|=x_1v_1$. If $x_1v_1\le a$, then $v_1\le \frac{a}{x_1}$. The area size of $\{(v_1,\cdots,v_d)|\sum_{i=1}^d v_i^2=1 \text{ and } 0\le v_1\le \frac{a}{x_1}\}$ is $S_d(\frac{a}{x_1},1)$. The probability that p has a distance $\le a$ to a random hyper-plane through the origin is $\frac{S_d(\frac{a}{x_1},1)}{\frac{1}{2}\cdot W_d(1)}=h_d\cdot\frac{a}{\mathrm{dist}(p,o)}+O(\frac{a^2}{\mathrm{dist}^2(p,o)})$.

2.2 Width Bounded Separator

Definition 2. *The diameter of a region R is $\sup_{p_1,p_2\in R}\mathrm{dist}(p_1,p_2)$. A (b,c)-partition of a d-dimensional space makes the space as the disjoint unions of regions $P_1,P_2,\cdots$ such that each P_i, called a regular region, has the volume to be equal to b and the diameter of each P_i is $\le c$. A (b,c)-regular point set A is a set of points on a d-dimensional space with (b,c)-partition $P_1,P_2,\cdots$ such that each P_i contains at most one point from A. For two regions A and B, if $A\subseteq B$ $(A\cap B\ne\emptyset)$, we say B contains (intersects resp.) A.*

Lemma 3. *Assume that $P_1,P_2,\cdots$ form a (b,c)-partition on a d-dimensional space. Then (i) every d-dimensional ball of radius r intersects at most $\frac{v_d\cdot(r+c)^d}{b}$ regular regions; (ii) every d-dimensional ball of radius r contains at least $\frac{v_d\cdot(r-c)^d}{b}$ regular regions; (iii) every d-dimensional ball of radius $(nb/v_d)^{\frac{1}{d}}+c$ contains at least n (b,c)-regular regions in it; and (iv) every d-dimensional ball of radius $(nb/v_d)^{\frac{1}{d}}-c$ intersects at most n (b,c)-regular regions.*

Proof. (i) If a (b,c)-regular region P_i intersects a ball C of radius r at center o, the regular region P_i is contained by the ball C' of radius $r+c$ at the same center o. As the volume of each regular region is b, the number of regular regions contained by C' is no more than the volume size of the ball C' divided by b. (ii) If a regular region P_i intersects a ball C'' of radius $r-c$ at center o, P_i is contained in the ball C of radius r at the same center o. The number of those regular regions intersecting C'' is at least the volume size of the ball C'' divided by b. (iii) Apply $r=(\frac{bn}{v_d})^{\frac{1}{d}}+c$ to (ii). (iv) Apply $r=(\frac{bn}{v_d})^{\frac{1}{d}}-c$ to (i).

Definition 3. *Let a be a non-negative real number. Let p and o be two points in a d-dimensional space. Define $Pr_d(a,p_0,p)$ as the probability that the point p has $\le a$ perpendicular distance to a random hyper-plane L through the point p_0. Let L be a hyper-plane. Then define the function $f_{a,p,o}(L)=1$ if p has distance $\le a$ to the hyper-plane L and L is through o; and 0 otherwise.*

The expectation of function $f_{a,p,o}$ is $E(f_{a,p,o})=Pr_d(a,o,p)$. Assume that $P=\{p_1,p_2,\cdots,p_n\}$ is a set of n points in R^d and each p_i has weight $w(p_i)\ge 0$. Define function $F_{a,P,o}(L)=\sum_{p\in P}w(p)f_{a,p,o}(L)$. We give an upper bound for the expectation $E(F_{a,P,o})$ for $F_{a,P,o}$ in the lemma below.

Lemma 4. *Let a be a non-negative real number, b and c be positive constants, and $\delta > 0$ be a small constant. Assume that $P_1, P_2, \cdots$ form a (b, c) partition in R^d. Let $w_1 > w_2 > \cdots > w_k > 0$ be positive weights, and $P = \{p_1, \cdots, p_n\}$ be a set of (b, c)-regular points in R^d. Let w be a mapping from P to $\{w_1, \cdots, w_k\}$ and n_j be the number of points $p \in P$ with $w(p) = w_j$. Let o (the center point) be a fixed point in R^d. Then for a random hyper-plane passing through o, $E(F_{a,P,o}) \le (\frac{d \cdot h_d \cdot v_d}{(d-1) \cdot b} + \delta) \cdot a \cdot \sum_{j=1}^{k} w_j (r_j^{d-1} - r_{j-1}^{d-1}) + c_2 \sum_{j=1}^{k-1} w_{j+1} \cdot r_j^{d-2} + O((a + c_1)^d) \cdot w_1$, where (1) $r_0 = 0$ and $r_i (i > 0)$ is the least radius such that $B_d(r_i, o)$ intersects at least $\sum_{j=1}^{i} n_j$ regular regions, (2) c_1 and c_2 are constants for a fixed d, and (3) h_d and v_d are constants as defined in section 2.1.*

Proof. Assume p is a point of P and L is a random plane passing through the center o. Let C be the ball of radius r and center o such that C covers all the points in P. Let C' be the ball of radius $r' = r + c$ and the same center o. It is easy to see that every regular region with a point in P is inside C'. The probability that the point p has a distance $\le a$ to L is $\le h_d \cdot \frac{a}{\mathrm{dist}(o,p)} + \frac{c_0 \cdot a^2}{\mathrm{dist}(o,p)^2}$ (by Lemma 2).

Let $\epsilon > 0$ be a small constant which will be determined later. Select a large constant $r_0 > 0$ and $R_0 = \frac{c_0 \cdot a}{h_d \cdot \epsilon} + r_0$ such that for every point p with $dist(o, p) \ge R_0$, $\frac{c_0 \cdot a}{h_d \cdot \mathrm{dist}(o,p)^2} < \frac{\epsilon}{\mathrm{dist}(o,p)}$ and for every point p' with $\mathrm{dist}(p', p) \le c$, $\frac{1}{\mathrm{dist}(o,p')} \le \frac{1+\epsilon}{\mathrm{dist}(o,p)}$. Let P_1 be the set of all points p in P such that $\mathrm{dist}(o, p) < R_0$. For each point $p \in P_1$, $Pr_d(a, o, p) \le 1$. For every point $p \in P - P_1$, $Pr_d(a, o, p) \le h_d \cdot \frac{a}{\mathrm{dist}(o,p)} + \frac{c_0 \cdot a^2}{\mathrm{dist}(o,p)^2} < \frac{h_d \cdot a (1+\epsilon)}{\mathrm{dist}(o,p)}$. From the transformation $E(F_{a,P,o}) = E(\sum_{i=1}^{n} w(p_i) \cdot f_{a,p_i,o}) = \sum_{i=1}^{n} w(p_i) \cdot E(f_{a,p_i,o}) = \sum_{j=1}^{k} w_j \sum_{w(p_i)=w_j} E(f_{a,p_i,o}) = \sum_{j=1}^{k} w_j \sum_{w(p_i)=w_j} Pr_d(a, o, p_i)$, we have

$$E(F_{a,P,o}) \le w_1 |P_1| + \sum_{j=1}^{k} w_j \sum_{w(p_i)=w_j} \frac{h_d \cdot a \cdot (1+\epsilon)}{b} \cdot \frac{1}{\mathrm{dist}(o, p_i)} \cdot b \tag{3}$$

The contribution to $E(F_{a,P,o})$ from the points in P_1 is $\le w_1 |P_1| \le = w_1 \cdot \frac{v_d (R_0 + c)^d}{b} = w_1 \cdot O((a + c_1)^d)$ for some constant $c_1 > 0$ (by Lemma 3). Next we only consider those points from $P - P_1$. The sum (3) is at a maximum when $dist(p, o) \le dist(p', o)$ implies $w(p) \ge w(p')$. The ball C' is partitioned into k ring regions such that the j-th area is between $B_d(r_j, o)$ and $B_d(r_{j-1}, o)$ and it is mainly used to hold those points with weight w_j. Notice that each regular region has diameter $\le c$ and holds at most one point in P. It is easy to see that all points of $\{p_i \in P | w(p_i) = w_j\}$ are located between $B_d(r_j, o)$ and $B_d(r_{j-1} - c, o)$ when (3) is maximal.

$$\sum_{w(p_i)=w_j} \frac{h_d \cdot a \cdot (1+\epsilon)}{b} \cdot \frac{1}{\mathrm{dist}(o, p_i)} \cdot b \tag{4}$$

$$\le \frac{h_d \cdot a \cdot (1+\epsilon)^2}{b} \int_{B_d(r_j,o) - B_d(r_{j-1}-c,o)} \frac{1}{\mathrm{dist}(o, z)} d_z \tag{5}$$

$$= \frac{h_d a(1+\epsilon)^2}{b}\int_{r_{j-1}-c}^{r_j}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\frac{J_d(r,\theta_{n-1},\cdots,\theta_1)}{r}d_r d_{\theta_{n-1}}\cdots d_{\theta_1} \quad (6)$$

$$= \frac{h_d\cdot a\cdot(1+\epsilon)^2}{b}\cdot\frac{d}{(d-1)}\cdot\left(\frac{V_d(r_j)}{r_j}-\frac{V_d(r_{j-1}-c)}{r_{j-1}-c}\right) \quad (7)$$

$$< \left(\frac{d\cdot h_d\cdot v_d}{(d-1)\cdot b}+\delta\right)\cdot a\cdot(r_j^{d-1}-r_{j-1}^{d-1})+O(r_{j-1}^{d-2}). \quad (8)$$

Note: (6) → (7) → (8) follows from (1), and selecting ϵ small enough.

Lemma 5. *Assume* $a = O(n^{\frac{d-2}{d^2}})$. *Let* o *be a point on the plane,* b *and* c *be positive constants, and* $\epsilon, \delta > 0$ *be small constants. Assume that* $P_1, P_2, \cdots$ *form a* (b, c) *partition in* R^d. *The weights* $w_1 > \cdots > w_k > 0$ *satisfy* $k\cdot\max_{i=1}^{k}\{w_i\} = O(n^{\epsilon})$. *Let* P *be a set of* n *weighted* (b, c)*-regular points in a* d*-dimensional space with* $w(p) \in \{w_1, \cdots, w_k\}$ *for each* $p \in P$. *Let* n_j *be the number of points* $p \in P$ *with* $w(p) = w_j$ *for* $j = 1, \cdots, k$. *Then* $E(F_{a,P,o}) \le (k_d\cdot(\frac{1}{b})^{\frac{1}{d}}+\delta)\cdot a\cdot\sum_{j=1}^{k} w_j\cdot n_j^{\frac{d-1}{d}} + O(n^{\frac{d-2}{d}+\epsilon})$, *where* $k_d = \frac{d\cdot h_d}{d-1}\cdot v_d^{\frac{1}{d}}$. *In particular,* $k_2 = \frac{4}{\sqrt{\pi}}$ *and* $k_3 = \frac{3}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$.

Proof. Let r_j be the least radius such that the ball of radius r_j intersects at least $\sum_{i=1}^{j} n_i$ regular regions ($j = 1, \cdots, k$). By Lemma 3, $\left(\frac{(\sum_{i=1}^{j} n_i)b}{v_d}\right)^{\frac{1}{d}} - c \le r_j \le \left(\frac{(\sum_{i=1}^{j} n_i)b}{v_d}\right)^{\frac{1}{d}} + c$ for $j = 1, \cdots, k$.

$$r_j^{d-1} - r_{j-1}^{d-1} \le \left(\left(\frac{(\sum_{i=1}^{j} n_i)b}{v_d}\right)^{\frac{1}{d}}+c\right)^{d-1} - \left(\left(\frac{(\sum_{i=1}^{j-1} n_i)b}{v_d}\right)^{\frac{1}{d}}-c\right)^{d-1} \quad (9)$$

$$= \left(\frac{b}{v_d}\right)^{\frac{d-1}{d}}\left((\sum_{i=1}^{j} n_i)^{\frac{d-1}{d}} - (\sum_{i=1}^{j-1} n_i)^{\frac{d-1}{d}}\right) + O((\sum_{i=1}^{j} n_i)^{\frac{d-2}{d}}) \quad (10)$$

$$= \left(\frac{b}{v_d}\right)^{\frac{d-1}{d}} n_j^{\frac{d-1}{d}} + O((\sum_{i=1}^{j} n_i)^{\frac{d-2}{d}}) \quad (11)$$

By Lemma 4, Lemma 5 is proven.

Definition 4. *Let* $a_1, \cdots, a_d > 0$ *be positive constants. A* $(a_1, \cdots, a_d)$*-grid regular partition divides the* d*-dimensional space into disjoint union of* $a_1 \times \cdots \times a_d$ *rectangular regions. A* $(a_1, \cdots, a_d)$*-grid regular point is a corner point of a rectangular region. Under certain translation and rotation, each* $(a_1, \cdots, a_d)$*-grid regular point has coordinates* $(a_1 t_1, \cdots, a_d t_d)$ *for some integers* $t_1, \cdots, t_d$.

Theorem 1. *Let* $a = O(n^{\frac{d-2}{d^2}})$. *Let* $a_1, \cdots, a_d$ *be positive constants and* $\epsilon, \delta > 0$ *be small constants. Let* P *be a set of* n $(a_1, \cdots, a_d)$*-grid points in* R^d, *and* Q *be another set of* m *points in* R^d *with sketch* P. *Let* $w_1 > w_2 \cdots > w_k > 0$ *be* *positive weights with* $k \cdot \max_{i=1}^{k}\{w_i\} = O(n^{\epsilon})$*, and* w *be a mapping from* P *to* $\{w_1, \cdots, w_k\}$*. Then there is a hyper-plane* L *such that (1) each half space has* $\leq \frac{d}{d+1}m$ *points from* Q*, and (2) for the subset* $A \subseteq P$ *containing all points in* P *with* $\leq a$ *distance to* L *has the property* $\sum_{p \in A} w(p) \leq \left(k_d \cdot \left(\prod_{i=1}^{d} a_i\right)^{\frac{-1}{d}} + \delta\right) \cdot a \cdot \sum_{j=1}^{k} w_j \cdot n_j^{\frac{d-1}{d}} + O(n^{\frac{d-2}{d}+\epsilon})$ *for all large* n*, where* $n_j = |\{p \in P | w(p) = w_j\}|$*.*

Proof. Let $b = \prod_{i=1}^{d} a_i$, $c = \sqrt{\sum_{i=1}^{d} a_i^2}$, and the point o be the center point of Q via Lemma 1. Apply Lemma 5.

Corollary 1. *[11] Let* Q *be a set of* n $(1,1)$*-grid points on the plane. Then there is a line* L *such that each half plane has* $\leq \frac{2n}{3}$ *points in* Q *and the number of points in* Q *with* $\leq \frac{1}{2}$ *distance to* L *is* $\leq 1.129\sqrt{n}$*.*

Proof. Let each weight be 1, $k = 1, a = \frac{1}{2}$ and $P = Q$. Apply Theorem 1.

Corollary 2. *Let* Q *be a set of* n $(1,1,1)$*-grid points on the 3D Euclidean space. Then there is a plane* L *such that each half space has* $\leq \frac{3n}{4}$ *points in* Q *and the number of points in* Q *with a* $\leq \frac{1}{2}$ *distance to* L *is* $\leq 1.209n^{\frac{2}{3}}$*.*

Corollaries 1 and 2 are the separators for the 2D grid graphs and, respectively, 3D grid graphs. An edge connecting two neighbor grid points has a distance of 1. If two neighbor grid points are at different sides of the separator, one of them has distance $\leq \frac{1}{2}$ to the separator. We have a separator for the 1-separated set.

Theorem 2. *Assume that the packing density (see [26]) for the* d*-dimensional congruent balls is at most* D_d*. Then for every* 1*-separated set* Q *on the* d*-dimensional Euclidean space, there is a hyper-plane* L *with* $\text{balance}(L, Q) \leq \frac{d}{d+1}$ *and the number of points with distance* $\leq a$ *to* L *is* $\leq (2k_d \cdot (D_d/v_d)^{\frac{1}{d}} + o(1))a \cdot |Q|^{\frac{d-1}{d}}$*.*

We develop a brute force method to find the width-bounded separator. In order to determine the position of the hyper-plane in d-dimensional space. For every integer pair $s_1, s_2 \geq 0$ with $s_1 + s_2 = d$, select all possible s_1 points $p_1, \cdots, p_{s_1}$ from P and all possible s_2 points $q_1, \cdots, q_{s_2}$ from Q. Try all the hyper-planes L that are through $q_1, \cdots, q_{s_2}$ and tangent to $B_d(a + \delta, p_i)$ $(i = 1, \cdots, s_1)$. Then select the one that satisfies the balance condition and has small sum of weights for the points of P close to L. A more involved sub-linear time algorithm for finding width-bounded separator was recently developed by Fu and Chen [12].

Theorem 3. *Assume* $a = O(n^{\frac{d-2}{d^2}})$*. Let* $a_1, \cdots, a_d$ *be positive constants and* $\delta, \epsilon > 0$ *be small constants. Let* P *be a set of* n $(a_1, \cdots, a_d)$*-grid points and* Q *be another set of points on* d*-dimensional space. The weights* $w_1 > \cdots > w_k > 0$ *have* $k \cdot \max_{i=1}^{k}\{w_i\} = o(n^{\epsilon})$*. There is an* $O(n^{d+1})$ *time algorithm that finds a separator such that* $\text{balance}(L, Q) \leq \frac{d-1}{d}$*, and* $\text{measure}(L, P, a, w) \leq \left(\frac{k_d}{(a_1 \cdots a_d)^{1/d}} + \delta\right) a \sum_{i=1}^{k} w_i n_i^{\frac{d-1}{d}} + O(n^{\frac{d-2}{d}+\epsilon})$ *for all large* n*, where* $n_i = |\{p \in P | w(p) = w_i\}|$*.*

3 Application of Width Bounded Separator

In this section, we apply our geometric separator to the well-known disk covering problem: Given a set of points on the plane, find the minimal number of discs with fixed radius to cover all of those points. The d-dimensional ball covering problem is to cover n points on the d-dimensional Euclid space with minimal number of d-dimensional ball of fixed radius.

Before proving Theorem 4, we briefly explain our method. To cover a set of points Q on the plane, select a set P of grid points such that each point in Q is close to at least one point in P. A grid point p is assigned the weight i if there are 2^i to 2^{i+1} points of Q on the 1×1 grid square with p as the center. A balanced separator line for Q also has a small sum of weights ($O(\sqrt{n})$) for the points of P near the line. This gives at most $2^{O(\sqrt{n})}$ ways to cover all points of Q close to the separator line and decompose the problem into two problems Q_1 and Q_2 that can be covered independently. This method takes the total time of $2^{O(\sqrt{n})}$.

Theorem 4. *There is a $2^{O(\sqrt{n})}$-time exact algorithm for the disk covering problem on the $2D$ plane.*

Proof. Assume the diameter of any disk is 1. Assume that Q is the set of n input points on the plane. Let's set up an $(1,1)$-grid regular partition. For a grid point $p = (i,j)$ (i and j are integers) on the plane, define $\text{grid}(p) = \{(x,y) | i - \frac{1}{2} \le x < i + \frac{1}{2}, j - \frac{1}{2} < y \le j + \frac{1}{2}\}$, which is a half close and half open 1×1 square. There is no intersection between $\text{grid}(p)$ and $\text{grid}(q)$ for two different grid points p and q. Our "local binding" method is to merge the points of $Q \cap \text{grid}(p)$ to the grid point p and assign certain weight to p to measure the Q points density in $\text{grid}(p)$. The function Partition(Q) divides the set Q into $Q = Q(p_1) \cup Q(p_2) \cup \cdots \cup Q(p_m)$, where p_i is a grid point for $i = 1, 2, \cdots, m$ and $Q(p_i) = Q \cap \text{grid}(p_i) \neq \emptyset$.

Let n_i be the number of grid points $p_j \in P$ with $g^{i-1} \le |Q(p_j)| < g^i$, where g is a constant > 1 (for example, $g = 2$). From this definition, we have

$$\sum_{i=1}^{\lceil \log_g n \rceil} g^i \cdot n_i \le g \cdot n, \tag{12}$$

where $\lceil x \rceil$ is the least integer $\ge x$. Let $P = \{p_1, \cdots, p_m\}$ be the set grid points derived from partitioning set Q in Partition(Q). Define function $w : P \to \{1, 2, \cdots, \lceil \log_g n \rceil\}$ such that $w(p) = i$ if $g^{i-1} \le |Q(p)| < g^i$.

Select small $\delta > 0$ and $a = \frac{3}{2} + \frac{\sqrt{2}}{2}$. By Theorem 3, we can get a line L on the plane such that $balance(L, Q) \le \frac{2}{3}$ and $\text{measure}(L, P, a, w) \le (k_2 + \delta) \cdot a \cdot (\sum_{i=1}^{\lceil \log_g n \rceil} i \cdot \sqrt{n_i})$. Let $J(L) = \{p | p \in P \text{ and } \text{dist}(q, L) \le \frac{1}{2} \text{ for some } q \in Q(p)\}$. After those points of Q with distance $\le \frac{1}{2}$ to the separator line L are covered, the rest of points of Q on the different sides of L can be covered independently. Therefore, the covering problem is solved by divide and conquer method as described by the algorithm below.

```
Algorithm
    Input a set of points $Q$ on the plane.
    run Partition(Q) to get $P = \{p_1, \cdots, p_m\}$ and $Q(p_1), \cdots, Q(p_m)$
    find a separator line $L$ (by Theorem 3) for $P, Q$ with
        balance$(L, Q) \leq \frac{2}{3}$ and measure$(L, P, a, w) \leq (k_2 + \delta)a \sum_{i=1}^{\lceil \log n \rceil} i\sqrt{n_i}$
    for each covering to the points in $Q$ with $\leq 1/2$ distance to $L$
        let $Q_1 \subseteq Q$ be the those points on the left of $L$ and not covered
        let $Q_2 \subseteq Q$ be the those points on the right of $L$ and not covered
        recursively cover $Q_1$ and $Q_2$
        merge the solutions from $Q_1$ and $Q_2$
    Output the solution with the minimal number of discs covering $Q$
End of Algorithm
```

For each grid area grid(p_i), the number of discs containing the points in $Q(p_i)$ is no more than the number of discs covering the 3×3 area, which needs no more than $c_3 = (\left\lceil \frac{3}{\frac{\sqrt{2}}{2}} \right\rceil)^2 = 25$ discs. Two grid points $p = (i, j)$ and $p' = (i', j')$ are *neighbors* if $\max(|i - i'|, |j - j'|) \leq 1$. For each grid point p, define $m(p)$ to be the neighbor grid point q of p (q may be equal to p) with largest weight $w(q)$. For a grid point $p = (i, j)$, the 3×3 region $\{(x, y) | i - \frac{3}{2} \leq x < i + \frac{3}{2}\ and\ j - \frac{3}{2} < y \leq j + \frac{3}{2}\}$ has $\leq 9 \times g^{w(m(p))}$ points in Q. The number of ways to put one disc covering at least one point in $Q(p)$ is $\leq (9 \times g^{w(m(p))})^2$ (let each disc have two points from Q on its boundary whenever it covers at least two points). The number of ways to arrange $\leq c_3$ discs to cover points in $Q(p)$ is $\leq (9 \times g^{w(m(p))})^{2c_3}$. The total number of cases to cover all points with distance $\leq \frac{1}{2}$ to L in $\cup_{p \in J(L)} Q(p)$ is

$$\leq \prod_{p \in J(L)} (9 \cdot g^{w(m(p))})^{2c_3} = \prod_{p \in J(L)} 2^{(\log_2 9 + w(m(p)) \cdot \log_2 g) 2c_3} \tag{13}$$

$$\leq \prod_{p \in J(L)} 2^{2c_3(\log_2 9 + \log_2 g) w(m(p))} \tag{14}$$

$$= 2^{2c_3(\log_2 9 + \log_2 g) \sum_{p \in J(L)} w(m(p))} \leq 2^{2c_3(\log_2 9 + \log_2 g) 9 \cdot \text{measure}(L, P, a, w)} \tag{15}$$

$$\leq 2^{2c_3(\log_2 9 + \log_2 g) 9 (k_2 \cdot a + \delta)(\sum_{i=1}^{\lceil \log n \rceil} i \cdot \sqrt{n_i})} \tag{16}$$

This is because that for each grid point q, there are at most 9 grid points p with $m(p) = q$. Furthermore, for each $p \in J(L)$, p has a distance $\leq \frac{1}{2} + \frac{\sqrt{2}}{2}$ to L and $m(p)$ has a distance $\leq \frac{3}{2} + \frac{\sqrt{2}}{2} = a$ to L. Let the exponent of (16) be represented by $u = 2c_3(\log_2 9 + \log_2 g) 9 (k_2 + \delta) a (\sum_{i=1}^{\lceil \log n \rceil} i \cdot \sqrt{n_i})$. By Cauchy-Schwarz inequality $(\sum_{i=1}^{m} a_i \cdot b_i)^2 \leq (\sum_{i=1}^{m} a_i^2) \cdot (\sum_{i=1}^{m} b_i^2)$,

$$(\sum_{i=1}^{\lceil \log_g n \rceil} i\sqrt{n_i})^2 = (\sum_{i=1}^{\lceil \log_g n \rceil} \frac{i}{g^{i/2}} \cdot g^{i/2}\sqrt{n_i})^2 \leq (\sum_{i=1}^{\lceil \log_g n \rceil} \frac{i^2}{g^i}) \cdot (\sum_{i=1}^{\lceil \log_g n \rceil} g^i n_i) \tag{17}$$

Using standard calculus,$\sum_{i=1}^{\infty} \frac{i^2}{g^i} = \frac{g(g+1)}{(g-1)^3}$. By (17) and (12), $u \leq e(g)\sqrt{n}$, where $e(g) = 2c_3(\log_2 9 + \log_2 g)(k_2 + \delta) a \sqrt{\frac{g(g+1)}{(g-1)^3}} \cdot \sqrt{g}$. Let $T(n)$ be the maximal computational time of the algorithm for covering n points. The problem $T(n)$ is reduced to two problems $T(\frac{2}{3}n)$. We have $T(n) \le 2 \cdot 2^{e(g)\sqrt{n}} T(\frac{2n}{3}) \le 2^{\log_{3/2} n} 2^{e(g)(1+\alpha+\alpha^2+\cdots)\sqrt{n}} = 2^{e(g)(\frac{1}{1-\alpha})\sqrt{n}+\log_{3/2} n} = 2^{O(\sqrt{n})}$, where $\alpha = \sqrt{\frac{2}{3}}$.

Definition 5. *We consider undirected graphs $G = (V, E)$, where V denotes the vertex set and E denotes the edge set. An independent set I of a graph $G = (V, E)$ is a set of pairwise nonadjacent vertices of a graph. A vertex cover C of a graph $G = (V, E)$ is a subset of vertices such that each edge in E has at least one end point in C. A dominating set D is a set of vertices such that the rest of the vertices in G has at least one neighbor in D. For a point p on the plane and $r > 0$, $C_r(p)$ is the disk with center at p and radius r. For a set of disks $D = \{C_{r_1}(p_1), C_{r_2}(p_2), \cdots, C_{r_n}(p_n)\}$, the disk graph is $G_D = (V_D, E_D)$, where vertices set $V_D = \{p_1, p_2, \cdots, p_n\}$ and $E_D = \{(p_i, p_j) | C_{r_i}(p_i) \cap C_{r_j}(p_j) \ne \emptyset\}$. DG is the class of all disk graphs. DG_σ is the class of all disk graphs G_D such that D is the set of disks $\{C_{r_1}(p_1), C_{r_2}(p_2), \cdots, C_{r_n}(p_n)\}$ with $\frac{\max_{i=1}^n r_i}{\min_{i=1}^n r_i} \le \sigma$.*

Several standard graph theoretic problems for GD_1 are NP-hard [7,10,20,27]. The $n^{O(\sqrt{n})}$-time exact algorithm for the maximum independent set problem for DG_σ with constant σ was derived by Alber and Fiala [5] via parameterized approach, which was further simplified by Agarwal, Overmars and Sharir [1] for DG_1. We obtain $2^{O(\sqrt{n})}$-time algorithms for maximum independent set, minimum vertex cover, and minimum dominating set problems for DG_σ with constant σ. Their algorithms are similar each other. The d-dimensional versions of those problems, including the ball covering problem, have algorithms with computational time $2^{O(n^{1-1/d})}$.

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