Constructing Generalized Universal Traversing Sequences of Polynomial Size for Graphs with Small Diameter [1] (Extended Abstract)

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Abstract. The paper constructs a generalized version of universal traversing sequences. The generalization preserves the features of the universal traversing sequences that make them attractive for applications to derandomizations and space-bounded computation. For every n, a sequence is constructed that is used by a finite-automaton with O(1) states in order to traverse all the n-vertex labeled undirected graphs. The automaton walks on the graph; when it is at a certain vertex, it uses the edge labels and the sequence in order to decide which edge to follow. When walking on an edge, the automaton can see the edge labeling. The generalized sequences have size $2^{O(\partial(n))}$ and traverse all the n-vertex undirected graphs G satisfying

$$Diam(G) * log(\Delta(G)) \leq \partial(n),$$

where $Diam(G)$ is the diameter of G, and $\Delta(G)$ is the maximum degree of G. As a corollary we obtain polynomial size generalized universal traversing sequences construcible in $DSpace(logn)$ for the following classes of graphs, where $\partial(n) = O(logn)$: expanders of constant degree, random graphs, butterfly networks, shuffle-exchange networks, cube-connected-cycles networks, de Bruijn networks, cliques.
For other classes of graphs, the construction gives better traversing bounds than the $n^{O(logn)}$ universal traversing sequences constructed by Nisan [11] for arbitrary undirected graphs; for example in the case of the hypercubes, our sequences have size $n^{O(loglogn)}$.

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The only known class of graphs where universal traversing sequences of polynomial size can be constructed is the class of graphs of maximum degree 2 [9].

The construction of universal traversing sequences (in their standard or generalized form) for arbitrary undirected graphs in $DSpace(logn)$ will have strong consequences in complexity theory. What we may call

The Undirected Graph Connectivity Conjecture:
UNDIRECTED CONNECTIVITY is in $DSpace(logn)$

will be established by such a construction. As UNDIRECTED CONNECTIVITY is complete for the complexity class $SymmetricSpace(logn)$, it will follow that

$$DSpace(logn) = SymmetricSpace(logn)$$

and therefore a variety of fundamental computational problems - such as planarity testing, minimum spanning forests - will be solvable in $DSpace(logn)$.

1 History of the problem and previous constructions

S. Cook introduced the concept of *universal traversing sequence* and asked the question of the existence of short, i.e., polynomial size, such sequences. R. Aleliunas (1978) [1] showed that such sequences of size $O(n^3)$ exist for 2-regular graphs. (The result was re-obtained with a different method by A. Cobham [6]). R. Aleliunas, R. Karp, R. Lipton, L.Lovasz, C. Rackoff (1979) [2] proved that universal traversing sequences of size $O(d^2n^3logn)$ exist for n-vertex d-regular graphs. The above results use the probabilistic method, which is inherently non-constructive. The result in [2] implied that UNDIRECTED CONNECTIVITY is in RandomSpace(log n).

There are few constructions in the literature for universal traversing sequences. For graphs of maximal degree 2, there is a log-space construction of universal sequences of size $O(n^{4.76})$ [9]. For general undirected graphs, Nisan [11] gives a construction of size $n^{O(logn)}$ improving over the previous $2^{2^{O(\sqrt{logn})}}$ construction from [3]. The $n^{O(logn)}$ constructive bound was obtained previously for particular classes of graphs: the maximum degree 2 graphs [5], [4], and the cliques [10].

2 The polynomial construction for the cycles: graph compression and colliding sequences

The idea of the construction used in [9] was to *compress* the labeled cycle w to a another cycle w' while preserving a transfer relationship from the traversing sequences of w' to the ones of w. In order to do so, labeling patterns and walking patterns of a fixed sequence were classified, and *some* repetitions were regarded as *redundancies* and therefore removed. The information contained in the removed patterns can be retrieved by expanding a traversing sequence of w' in a uniform way such that the result is a traversing sequence for w. The recursive construction is dependent on the fact that after removal of the *redundant patterns* what is left is a well defined labeled cycle of (possibly) smaller size. In Figure 1a we present an example of cycle compression. For details of the construction see [9].

Let us consider the case of 3-regular graphs. It will suffice to construct in $DSpace(logn)$ universal traversing sequences (in their standard form, or in generalized form defined in this paper) for 3-regular graphs. These sequences can be transformed into universal traversing sequences for arbitrary undirected graphs.

The analysis of labeling patterns and walking patterns for 3-regular graphs turns out to be more difficult. However, there is a way of interpreting the cycle construction such that the removal of redundancies is avoided. Presented in this way, it appears that the implicit paralellism in the construction, made now explicit, is fully responsible for the speedup in the computation.

Figure 1b gives the alternate view of the construction from Figure 1a. Consider the original labeled cycle w_0 as being a directed graph w'_0 with each undirected edge made out of two directed labeled semi-edges. Every vertex has two semi-edges emerging from it. The three transformations we propose are analogues of the three contractions given in [9]. The analogue of the first contraction from [9] is described as a transformation given by the rules:

$$0 \rightarrow 000, 1 \rightarrow 111$$

We apply the transformation in parallel to all the vertices of w'_0 and obtain the new graph w'_1. For each vertex of w'_0 apply the transformation to the semi-edges of the vertex. Consider vertex 10. Walk in w'_0 from vertex 10 the sequence 000. The walk ends-up in at vertex 3. Then in w'_1 direct the 0-semi-edge of vertex 10 to end-up in vertex 3. This transformation is described by the first rule. Similarly we observe that the 1-semi-edge of vertex 10 ends up at vertex 9 in w'_1. We proceed in the same way for the other contractions from [9].

Let us observe that the intermediate directed graphs in the construction are no longer directed connected. Therefore as a natural notion to capture connectivity we propose the following one. A sequence

u is called a *colliding sequence* for a directed graph G if for every two vertices v_1, v_2 from the same directed connected componnent, the walk of generated by u when started at v_1 meets the walk generated by u when started at v_2 (not necessarily after the same number of steps).

The construction in this paper is a generalization of the above scenario for the graphs having small diameter.

3 Generalized Universal Traversing Sequences

Let us define now how a finite-automaton using a sequence can travel on a labeled graph. We restrict our attention to 3-regular graphs. Let G be such an undirected graph. G is labeled with *threets* on the edges. A threet is a label from the set $T = \{0, 1, 2\}$. (The term *threet* is an analogue of the term *bit*.) For every vertex of G the 3 adjacent edges are labeled with 0, 1, and 2 in some random order. Therefore, every edge gets two labels, one for each endpoint vertex.
Let us define a *walking finite-automaton* A that uses the sequence $u \in T^+$ in order to walk on a labeled 3-regular graph.
Let $A = (Q, T, \delta, q_0)$, where:

1. Q is the set of *states*, $Q = Q_1 \cup (Q_2 \times T) \cup (Q_3 \times T)$, where all Q_i are mutually disjoint.

   Q_1 is the set or regular states, $Q_2 \times T$ is the set of walking states, and $Q_3 \times T$ is the set of storing states. Let $\bar{Q} = Q - (Q_2 \times T)$.

   When A is in a walking state (q, t) at vertex v it will walk to the adjacent vertex of v using the edge labeled t. When A is not in a walking state, at a vertex v, it will stay at vertex v.

2. $q_0 \in Q_1$ is the *initial state*. Initially, the automaton in state q_0 is placed at a vertex in the graph.

3. The *transition function* is

   $$\delta : (\bar{Q} \times T) \cup (Q_2 \times T) \rightarrow Q$$

   If A is in state $q \in \bar{Q}$, at vertex v, then the automaton consumes one threet from the sequence, changes state and remains at vertex v. If A is in state $(q, t) \in Q_2 \times T$ at vertex v, then the automaton moves to the adjacent vertex v' and changes state $\delta((q, t)) = (q', t')$, where $(q', t') \in Q_3 \times T$, and the edge (v, v') is labeled $\bullet v \underline{t\ t'} \bullet v'$. This models the fact that the automaton when is walking on an edge can *see* the edge labels.

Let us define $\delta(q_0, u)$, *the computation of the A using a sequence u*, when A is placed initially at vertex v of G. The computation will produce a walk of the automaton on the graph. When the size of u is 1, $\delta(q_0, u)$ is given in the definition of the A together with the corresponding walk of A. Inductively, if A is at vertex v_1 and $u = t_1 u', t_1 \in T$ we put

$$\delta(q, t_1 u') = \delta(\delta(q, t_1), u')$$

for $q \in \bar{Q}$, with A staying at v_1;

$$\delta((q, t), t_1 u') = \delta((q', t'), t_1 u'),$$

where $(q, t) \in Q_2 \times T$ and $\delta(q, t) = (q', t')$. In this case, A is moving at v_2 such that in G we have the labeled edge $\bullet v_1 \underline{t\ t'} \bullet v_2$.

We say that $< A, u >$ *traverses G when started at v* if the computation $\delta(q_0, u)$ of A produces a walk of A visiting every vertex of G. We say that $< A, u >$ *traverses G* if it traverses G when started at v, for every vertex v of G. We call $< A, u >$ a *generalized n-universal traversing sequence* for a class $\mathcal{G}$ of connected graphs if $< A, u >$ traverses every labeled n-vertex graph $G \in \mathcal{G}$.
The standard universal traversing sequence scenario. Given a labeled 3-regular graph, consider a sequence $u \in T^+$. Starting from a given vertex v, the sequence u defines in a natural way a *walk* on the labeled graph: interpret the threets as instructions for following the corresponding edges. A sequence u is called *n-universal traversing sequence* for a class of graphs $\mathcal{G}$ if for every labeled n-vertex graph $G \in \mathcal{G}$, and for every starting vertex, the walk defined by u visits at least once every vertex in G. A sim-

ilar definition can be done for a class of graphs under some restricted labelings.

The standard definition of universal traversing sequences can be obtained as a particular case of our generalized scenario. We take A such that $Q_1 = \{q_0\}, Q_2 = \{q\}, Q_3 = \{q'\} \times T$ and $\delta(q'', t) = (q, t)$. Then $< A, u >$ is a generalized n-universal traversing sequence for $\mathcal{G}$ iff u is an n-universal traversing sequence.

4 Initial Reductions

We construct for every labeled connected 3-regular graph G_0 a connected 3-regular graph G_1 with a special labeling, called I-labeling, and $Diam(G_1) = Diam(G_0) + 1$. Then we show how a traversing sequence u of G_1 can be used by a walking finite-automaton A to traverse G_0. The automaton A is independent of the structure of the graph G_0. Therefore, an n-universal traversing sequence u for graphs with I-labelings and diamemter $\partial(n)$ can be tranformed into a generalized n-universal traversing sequence $< A, u >$ for (arbitrarily) labeled graphs with diamemter $\partial(n)$.

4.1 Reduction to 3-Regular Graphs

A construction of Cook and Rackoff [7] (see also [4]) reduces the problem of constructing universal sequences for arbitrary undirected graphs to that of constructing universal sequences for graphs with maximum degree 3. The idea of the construction is the following. Let G be a n-vertex graph with maximum degree $\Delta(G) = d$. A vertex of degree $d' \leq d$ is replaced by a full binary tree having at least d' leaves; if two such vertices have an edge between them then just connect the corresponding leaves. Let G' be the resulting graph. Then G' has $O(\Delta(G)n)$ vertices, and $Diam(G') \leq Diam(G) * log(\Delta(G))$.
We can now make the next step reducing the problem to the one of constructing universal traversing sequences for 3-regular graphs. Such a reduction was given in [4], and [5]. Applying the reduction to a n-vertex graph yields a 3-regular graph of size $2n$.

4.2 Reduction to 3-Regular Graphs with I-Labeling

This is the part where we use the generalized scenario. For every labeled n-vertex 3-regular graph G_0 we construct a 3-regular $8n$-vertex graph G_1 which is labeled with a special labeling, called I-labeling. In an I-labeling each edge is labeled with two Identical threets, i.e., $\bullet\underline{0\ 0}\bullet$, $\bullet\underline{1\ 1}\bullet$, $\bullet\underline{2\ 2}\bullet$.
The graph G_1 is obtained from two copies of G_0 by the transformation described in the Figure 2. The transformation is applied to every edge of G_0.

Theorem 1 *Let u be an $8n$-universal traversing sequence for the class of I-labeled 3-regular graphs having diameter $\partial(n)$. Then there is a walking finite-automaton A having $O(1)$ states such that $< A, u >$ is a generalized n-universal traversing sequence for the class of n-vertex 3-regular graphs of diameter $\partial(n)$ under all labelings.*

Proof. Consider a sequence u that traverses G_1. Let (v, v') be an edge of G_0 labelled $\bullet v\underline{0\ 1}\bullet v'$. Let $(v_1, v_1'), (v_2, v_2')$ be the corresponding edges in G_1. Suppose u starts at v_1. We construct a walking finite-automaton A. Suppose A is placed initially at v in G_0, and $u = 0u'$. Then A can be instructed to walk to v' and to store the other label 1 from the edge. Then it generates a 1-move and comes back to v. Now A *knows* the labels on the edge (v, v').
The graph from Figure 2b can be interpreted as a finite-automaton A_{01}. Indeed, regard vertices as states and the semi-edges as labeled directed edges. We define A in state "01", meaning a "0" move was started on an edge labeled "0 1", to act as A_{01}. If u reaches v_1' or v_2' then A moves to v'. If u comes back to v and uses a 1 or a 2 then the process stars again. The 1-edge or the 2-edge is inves-

tigated. In this way A using u simulates in G_0 the walk of u in G_1. All other cases are similar. A can be defined such that the number of moving steps on the graph to be no greater than the size of u. □

5 Constructing Polynomial Size Universal Traversing Sequences for I-labeled Graphs with Small Diameter

We give two constructions of universal traversing sequences for I-labeled 3-regular graphs. In the first subsection we present a simple construction performing exhaustive search in a graph where every walk can be reversed. The remaining subsections are devoted to the second construction. It is developed by generalizing some of the key concepts of the polynomial construction for the 2-regular graphs.

5.1 Exhaustive Search on I-labeled Graphs

For every $u \in T^+$ let u^R be the sequence u in reversed. Let $u' = uu^R$. If G_1 has n vertices and diameter $\partial(n)$ then let $u_1, u_2, ..., u_k$ be all the sequences of size $\partial(n)$ in T^+. Then $u^* = u'_1 u'_2 ... u'_k$ is an n-universal traversing sequence for all n-vertex 3-regular graphs that are I-labeled and have diameter $\partial(n)$. Indeed, let G_1 be such a graph and v a vertex of it. For every $i, 1 \leq i \leq k$ the walk of u'_i ends up at v. By the way we selected the u_i it is clear that we visit all the vertices of G_1. u^* has size $2^{O(\partial(n))}$.

5.2 Colliders and Attractors

We start our second construction of universal traversing sequences by transforming G_1 into a new graph with special properties. We will consider G_1 both a labeled undirected graph, and a labeled directed graph as follows. Each labeled edge of G_1 can be regarded as being made out of two directed *semi-edges* each one labeled by a threet. We will refer to the 0- (1-, 2-) semi-edge of a vertex.

Definition 1 *A* collider $C(A)$ *is a directed graph with the following properties:*

- *Each directed edge is called* semi-edge, *and is labeled with a threet*
- *Every vertex has fan-out 3 and the labels of the three out-going semi-edges are* $0, 1,$ *and* 2
- *The fan-in of a vertex v in a graph with n vertices satisfies $0 \leq$ fan-in$(v) \leq 3n$*
- *A is a special vertex of the graph called* the attractor *and has the property that all of its three out-going semi-edges end up in A (they are directed loops).*

This collider is the intermediate structure that will be subject to our transformations. We start by constructing a collider from the graph G_1.

Definition 2 *The collider $C_0 = C_0(A)$ is obtained as follows. Pick one arbitrary vertex A from G_1 and redirect its three semi-edges to come back to A. This vertex is the* attractor *of C_0.*

Definition 3 *Let $DComp_A(C_0)$ be the subgraph of C_0 consisting of all the vertices connected by a directed path with A and the semi-edges between them.*

The following lemma presents the key property that will remain true for all the graphs in our sequence of graph transformations.

Lemma 1 *The collider $C_0(A)$ has the following property:*

- *The vertex sets of the two graphs G_1 and $DComp_A(C_0)$ are equal.*

Proof. It is easy to see that every path in G_1 between a vertex v and A can be transformed into a directed path in C_0. □

5.3 Collisions

A *collision* is a set of semi-edges having identical threet that end up in the same vertex. If the semi-edges are labeled with 0 (1,2), then we call it a 0- (1-, 2-) collision. An I-labeling has no collisions. When the attractor vertex is created, 3 collisions are introduced: two 0-semi-edges collide, two 1-semi-edges collide, and two 2-semi-edges collide.

5.4 Contractions

A *contraction* for a collider C is a pair $K = (M, \kappa)$ where

- M is a *collider-shrinking* transformation that simultaneously changes of the semi-edges according to a set of rules
- κ is a *sequence-expanding* map that transforms a sequence of threets into another sequence of threets according to a set of rules

A collider C_i under a contraction $K = (M, \kappa)$ will be transformed into a new graph $C_{i+1} = M(C_i)$ having the same vertices, but the semi-edges may change their endpoint.
Consider the contraction $K_0 = (M_0, \kappa_0)$ given by the rules:

- $0 \rightarrow 0, 1 \rightarrow 100, 2 \rightarrow 200$

Let $C_1 = K_0(C_0)$ be the graph obtained from C_0 as follows. The vertices of C_1 are the same as those of C_0.

Informally, M_0 acts as follows. For every vertex v the 3 semi-edges going out from v will change their end-points as according to the rules of K_0. For example, the 1-semi-edge from vertex v in C_1 will end up in the vertex v_1 such that the walk 100 from v in C_0 ends up at v_1. Similarly for the other cases.
As far as the sequence-expanding map κ_0 is concerned the transformation is similar. For example, κ_0 when applied to the sequence 1102 gives as result the sequence 1001000200.

Given two contractions $K_i = (M_i, \kappa_i), i = 1, 2$ their composition is defined as $K_1 \circ K_2 = ((M_1 \circ M_2, \kappa_1 \circ \kappa_2)$.

Lemma 2 *Contractions are closed under composition.*

5.5 Universal Colliding Sequences

A sequence $u \in T^+$ is called *a collision sequence* for the labeled strongly connected graph G if for every two vertices v_1, v_2 from the same connected component of G the walk generated by u when started from v_1 meets the walk generated by u when started from v_2 (not necessarily after the same number of threets). A sequence is called *n-universal colliding sequence* for a class of graphs $\mathcal{G}$ if it is a colliding sequence for every n-vertex graph in $\mathcal{G}$. Similarly, the notion can be defined for graphs under special labelings. Every universal traversing sequence is a universal colliding sequence.

The converse happens to be also true with respect to our particular construction.

6 The Graph Compression Construction

In this section we construct our universal colliding sequences for graphs with I-labelings. It will be shown to they are universal traversing sequences too. The construction starts with G_1 which is transformed into a collider C_0. Then C_0 is transformed by a series of contractions. The resulting sequence of colliders

$$C_0, C_1, \ldots, C_{Diam(G_1)}$$

ends up with a special collider where colliding sequences are easy to construct. The progress towards the special collider will be realized by increasing the number of collisions at the attractor. The next lemma gives us a measure of progress and convergence for the collision process.

Lemma 3 *Let $Diam(G_1)$ be the undirected diameter of G_1. Then for every vertex v_0 there exists a* directed *subgraph T_{v_0} such that*

1. *T_{v_0} is a binary tree with root v_0,*
2. *all edges of T_{v_0} are directed towards v_0, and*

3. the depth of T_{v_0}, viewed as a rooted undirected graph is no greater than $Diam(G_1)$.

Proof. Fix v_0. Consider for every vertex v the shortest undirected path $P(v_0, v)$ viewed as a graph. Its length is no grather than $Diam(G_1)$. Let $H = \cup_v P(v_0, v)$. The graph H contains all the vertices of G_1. Consider a spanning tree for H, say T_H. It has depth no greater than $Diam(G_1)$. Let T_{v_0} be the directed graph obtained from T_H by reading each edge as a semi-edge directed toward the root v_0. T_{v_0} is a binary tree by the 3-regularity of G_1 and has undirected depth no greater than $Diam(G_1)$. □

Theorem 2 *Let $\partial(n)$ be a given function and let $\mathcal{G}$ be the class of I-labeled n-vertex 3-regular graphs having the property that*

$$Diam(G) \leq \partial(n).$$

Then n-universal traversing sequences of size $2^{O(\partial(n))}$ can be constructed for the class $\mathcal{G}$ in $DSpace(\partial(n))$.

The proof of the theorem is contained in the next 3 Lemmas.

The algorithm for constructing the n-universal colliding sequence

1. Given $G_1 \in \mathcal{G}$ a graph with n vertices, construct $C_0(A)$ as described in section 5.3.

2. Consider now the following 3 contractions:

   Contraction K_0

   - $0 \to 0, 1 \to 100, 2 \to 200$

   Contraction K_1

   - $0 \to 011, 1 \to 1, 2 \to 211$

   Contraction K_2

   - $0 \to 022, 1 \to 122, 2 \to 2$

3. Define a macro-step of the contraction process to be $K_{012} = (M_{012}, \kappa_{012})$,

   $$K_{012} = K_0 \circ K_1 \circ K_2$$

   Let C_i be $M_{012}^i(C_0), 1 \leq i \leq \partial(n)$ and $C_{*,n} = M_{012}^{\partial(n)}(C_0)$. (Here M_{012}^i is M_{012} composed with itself i times.)

4. Let
   $$u_{*,n} = \kappa_{012}^{\partial(n)}(0),$$
   where $u_{*,n}$ is the n-universal colliding sequence that we construct for the class $\mathcal{G}$.

Lemma 4 *For every $i, 0 \leq i \leq \partial(n)$ the following hold:*

1. *For every vertex v and every semi-edge s going out from v, the end point of s is either A, or remains the same as in C_0.*
2. *Every vertex v of C_i is connected by a directed path to A.*
3. *Consider T_A given by lemma 3 for G_1 (we choose $v_0 = A$). Let T_A^i be the graph T_A in its occurence in C_i. The directed graph T_A^i has the same vertex set as T_A, and the same semi-edges from each vertex; however, the end-point of a semi-edge may be different from its end-point in T_A. Then all the semi-edges of T_A^i at level $i+1$ collide in A.*
4. *The vertex sets of G_1 and $DComp_A(C_i)$ are equal.*
5. *For every colliding sequence u for C_{i+1} the sequence $\kappa_{012}(u)$ is a colliding sequence for C_i.*

Proof. It is easy to check that the 5 facts are true for the collider C_0. (The case 4 comes from Lemma 1.) Assume by induction that they are true for the collider C_i. We show that they are true for C_{i+1} as well.

1. Consider a vertex v in C_{i+1} and its t-semi-edge. Suppose that the t-semi-edge of v in C_i was $(v, v'), v' \neq A$. Then the contraction K_0 might leave it the same $t \to t$ or transform it $t \to tt't'$. In C_{i+1} walking from v after a first t-step if we reach a completely

labeled edge $\underline{.t'\,t'.}$ then we end up the way we started. If the edge is not completely labeled, by induction hypothesis one of the semi-edges is colliding in A. This will imply that the semi-edge t of v after K_0 end up in A. Similar arguments holds for K_1 and K_2. As a corollary, *1* holds true for C_{i+1}.

2. If v was connected by a directed path to A in C_i then by *1* it is still directed connected in to A in C_{i+1} although by a shorter path.
3. By induction hypothesis, in C_i all the semi-edges of T_A^i at level $i+1$ collide in A while the other are in the same position as is T_A or are collided in A. Consider a semi-edge in T_A^{i+1} at level $i+2$, say t, that is not collided in A. It ends up where a t'-semi-edge at level $i+1$ begins. Then when the turn of the $K_{t'}$ contraction comes in the macro-step $i+1$, the t-semiedge of T_A^{i+1} will collide in A. Certainly this holds for all the semi-edges at the $i+2$ level.
4. By *2* it follows that we do not *disconnect*.
5. For every sequence of threets u and every vertex v consider the walk of u in C_{i+1} starting from v. Let v' be the end point of the walk. Then the walk of $\kappa_{012}(u)$ in C_i starting from v ends up in v'. Indeed, if the first walk does not involve A then the trajectories of the two walks are identical. If the first walk involves A, then the second walk will also reach A. Therefore if u is a colliding sequence in C_{i+1}, κ_{012} will be a colliding sequence in C_i.

□

Figure 3 gives the sequence of contractions for a collider.

Lemma 5 *The collider $C_{*,n}$ has the property that all the semi-edges collide in A.*

Proof. By Lemma 4, $C_{\partial(n)-1}$ has the property that all the semi-edges from $T_A^{\partial(n)-1}$ collided in A. Some of the remaining semi-edges of the collider might still end up at a different vertex than A. The last round however will complete the collision process.

□

Lemma 6 *1. The sequence*

$$u_{*,n} = \kappa^{\partial(n)}(0)$$

is an n-universal colliding sequence for the class $\mathcal{G}$.

2. $u_{*,n}$ is n-universal traversing sequence for the class $\mathcal{G}$, has size $3^{(3*\partial(n))}$ and can be constructed in $DSpace(\partial(n))$.

Proof.

Because 0 is a colliding sequence for $C_{*,n}$ it follows by Lemma 4 that $u_{*,n}$ is a colliding sequence for C_0. Moreover, for every v of C_0 the walk of $u_{*,n}$ from v visits A. As our choice of A was arbitrary, the sequence $u_{*,n}$ will visit an entire connected component of the undirected graph G_1 with n vertices. We have $\mid u_{*,n} \mid = 3^{3*\partial(n)}$.

It is not difficult to see that the sequence can be constructed in $DSpace(\partial(n))$. □

7 Constructing Polynomial Size Generalized Universal Traversing Sequences

We return now to arbitrary graphs having small diameter under arbitrary labelings. We use the universal traversing sequences for the class of *I*-labeled 3-regular graphs with small diameter in order to construct generalized universal traversing sequence. Therefore, as a corollary to Theorems 1 and 2 we obtained the following.

Theorem 3 *Let $\partial(n)$ be a given function and let $\mathcal{G}$ be the class of n-vertex graphs having the property that*

$$Diam(G) * \log(\Delta(G)) \leq \partial(n).$$

Then n-universal traversing sequences of size $3^{3*\partial(n)}$ can be constucted in $DSpace(\partial(n))$. In particular, when $\partial(n) = O(logn)$ the generalized universal traversing sequences have polynomial size.

8 Applications

The following classes of graphs have the property that by choosing a proper $\partial(n) = O(logn)$ every graph G in the class satisfies: $Diam(G) * log(\Delta(G)) \leq \partial(n)$. The classes are: expanders of constant degree, random graphs, butterfly networks, shuffle-exchange networks, cube-connected-cycles networks, de Bruijn networks, cliques. As a corollary, for them we can construct generalized universal traversing sequences of polynomial size.

For other classes of graphs, the construction gives better traversing bounds than the the $O(n^{logn})$ universal traversing sequences of [11]; for example in the case of the hypercubes, we construct traversing sequences of size $O(n^{loglogn})$.

9 Concluding Remarks

We constructed a generalized version of universal traversing sequences. In this scenario, one sequence is used by a walking finite-automaton for traversal of all the n-vertex undirected graphs under every labeling. For a given n, the same sequence generates different sequences of moving steps (as opposed to the universal traversing sequence scenario when all such sequences of moving steps were identical to the universal sequence).

For the class of graphs with small diameter we constructed a log-space traversing algorithm. Using log space a Turing Machine can traverse a graph with diameter $c*logn$ by trying all the sequences of size $c*logn$. Our traversing algorithm is performed without *reseting* is a manner similar to the universal traversing sequences. We believe that graph compression and universal colliding sequences hold the key to the extention of the method presented in [9], and partially generalized in this paper, to capture the solution to the Undirected Graph Connectivity Conjecture.

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Figure 1a Figure 1b

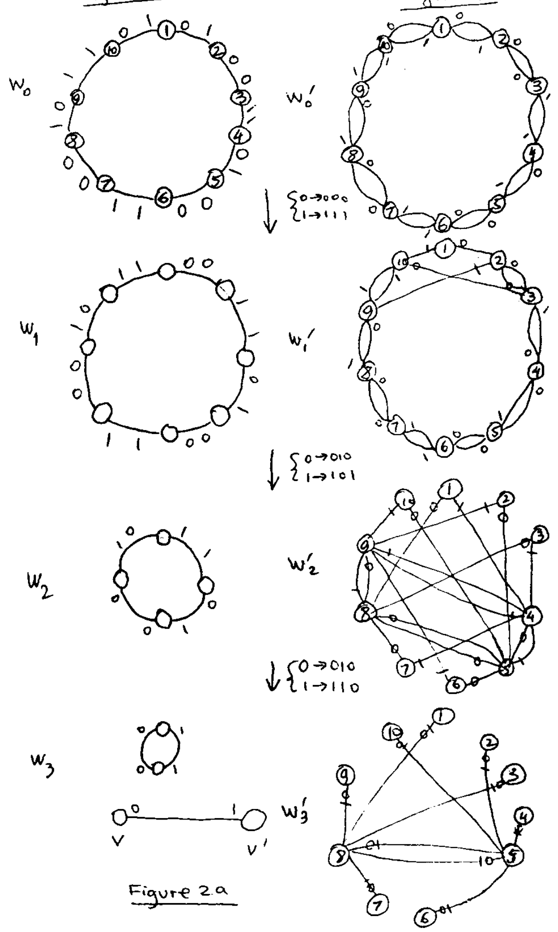
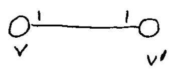


Figure 2a



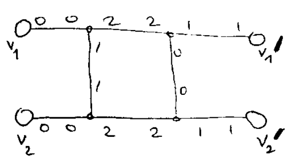
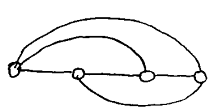


Figure 2c

Figure 2b



G_0

Figure 2e

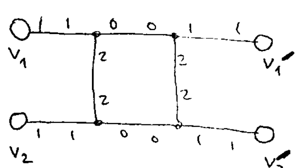
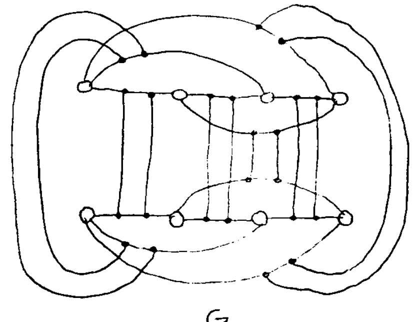


Figure 2d



G_1

Figure 2f

Figure 3