Polynomial Universal Traversing Sequences for Cycles Are Constructible

(Extended Abstract)

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Abstract. The paper constructs the first *polynomial* universal traversing sequences for cycles, solving an open problem of S.Cook and R. Aleliunas, R. Karp, R. Lipton, L.Lovasz, C. Rackoff (1979) [2] in the case of 2-regular graphs. The existence of universal traversing sequences of size $O(d^2n^3 logn)$ for n-vertex d-regular graphs was established in [2] by a probabilistic argument, which was inherently non-constructive. For the cycles, the non-constructive upper bound was improved to $O(n^3)$ by Janowsky (1983) [13] and Cobham (1986) [8]. Previously, the best explicit constructions for cycles were due to Bridgland (1986) and A. Bar-Noy, A. Borodin, M. Karchmer, N. Linial, and M. Werman (1986), and have size $O(n^{logn})$.

Our universal traversing sequence has size $O(n^{4.76})$, and can be constructed in log-space.

1 Introduction

" The study of n-universal sequences is of the utmost importance for frequent museum goers."
Michael Sipser (1985) [17]

Graph connectivity problems are fundamental for complexity theory. One such problem is UNDIRECTED CONNECTIVITY, the problem of determining if two vertices in an undirected graph are connected by a path. Its version for directed graphs is complete for NSPACE($logn$); however the undirected version is not known to be complete for NSPACE($logn$). One approach to studying the problem was proposed by S.Cook, [2] who introduced the notion of *universal traversing sequence*. For a given n, such a sequence is n-universal for all graphs with n vertices, if, regardless of where you start in the graph, it is guaranteed that every vertex in the graph will be visited at least once. Because such a sequence must visit an exponential number of graphs, S.Cook [2] questioned the existence of *short* sequences of this type, i.e., having polynomial size in n. [2] gave a positive answer, establishing by probabilistic arguments that such sequences for d-regular graphs must exist. Their non-constructive upper-bound is $O(d^2n^3logn)$. As a corollary, universal sequences imply that UNDIRECTED CONNECTIVITY is in RandomSPACE($logn$). In the same way, an algorithm for constructing n-universal sequences for general graphs, which needs only $O(logn)$ space, will imply that UNDIRECTED CONNECTIVITY is in DSPACE($logn$). Such a result will have strong consequences for complexity theory because in [15], UNDIRECTED CONNECTIVITY was found to be complete for the class SymSPACE($logn$) of symmetric log space computation. As a consequence we would have the following collapsing :

for every constructible function $f(n) \geq logn$:

DSPACE($f(n)$) = SymSPACE($f(n)$) = CO-SymSPACE($f(n)$).

The relevance of UNDIRECTED CONNECTIVITY is discussed in detail in [5] (where a recent re-classification of $O(logn)$ space bounded complexity classes is given), [17], and [9].

Progress toward constructing universal sequences was made by [6] and [4]. Explicit constructions were given for 2-regular graphs, providing universal sequences of size $O(n^{logn})$. [6] uses $O(log^2 n)$ space. On the other hand, an $\Omega(nlogn)$ lower bound was obtained in [4].

We give the *first polynomial* construction in the literature for universal traversing sequences for 2-regular graphs in $logn$ space having size $O(n^{4.76})$. We consider our result a step towards establishing a similar result for 3-regular graphs, which by results of [10] and [4] will imply the solution in the general d-regular case, and therefore the complexity class collapsing.

As it was pointed out in [4], polynomial universal traversing seqences for cycles might also have applications for distributed algorithms for rings of processors (e.g. anonymous rings).

1.1 Informal description of the construction

The $O(n^{logn})$ construction that was available to us was the one due to Bridgland [6]. Our method builds on his. Bridgland's method is recursive, relating universal sequences for cycles of size n to the ones for cycles of size $n/2$. Informally, the construction is described by an equation

$$f(n) = cnf(\frac{n}{2}) + c'n.$$

(A similar recurrence is accomplished in [4], except that cn is there $c''nlogn$.) Our method is also recursive, and is described by the equation

$$f(n) = 27f(\frac{n}{2}) + 4n + 49.$$

We proceed as follows. There are three steps, called *contractions*. We start with a cycle (called "wheel" in the paper) w that is subject to these contractions. Each contraction provides us with a new wheel w', of size smaller than or equal to the size of w, and a way to transform universal sequences of w' into ones for w. The combined effect of the three contractions will "halve" the wheel, in the sense that a universal for a wheel of half the size of w can be transformed into one that is universal for w. The success of the transformation is that it can only use a locally constant amount of replacement.

The edges of the wheels are divided in two categories: $.\underline{00}.$, $.\underline{11}.$ and $.\underline{01}.$, $.\underline{10}.$, which was an idea used in [6]. One new idea we use classifies the regions of the wheel as multiples or singles for each of the above two categories of edges. The first 2 contractions halve the multiples in each category, while the third contraction "halves", in a different sense, the collection of singles from different categories by "marrying" two singles (from different categories), and then providing a one edge contraction.

In the process of halving the multiples, some multiples can be transformed into singles, and some singles can be transformed into multiples. The contractions are able to take care of the halving even in these cases.

The three contractions, viewed as transformations on sequences, say K_1, K_2, K_3, are defined as replacing a bit by 3 other bits, depending on the value of the bit and the parity of its position. They can be composed as functions, preserving their property. The universal sequence is constructed by applying $K_1 \circ K_2 \circ K_3$, exactly $logn$ times, to an initial 00010 sequence.

The remaining portion of the paper is organized as follows. Section 2 gives some definitions and an important first lemma. Section 3 discusses multiples and singles. Sections 4 and 5 give the first two contractions. The constructions are dual, so the development is similar. Section 5 contains the third contraction, the difficult part of of the construction, while in section 6, we construct the universal traversing sequences of polynomial size. The last section contains conclusions.

2 Definitions and Notations

The first basic object of study in this paper is a *wheel*, which is an undirected graph being a cycle that has at each vertex two edges sharing that vertex labeled 0 and 1.

Definition 1 *A* wheel *is an undirected graph $w = (V_w, E_w)$, where*

- $V_w = \{v_1, ..., v_n\}$
- $E_w = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$
- *for each vertex $v \in V_w$ the two edges sharing v are labeled (near v) with* 0 *and* 1 *arbitrarily.*

The size of a wheel w, denoted $\mid w \mid$ is the number of vertices in V_w, i.e. n.

Example. A region on w might look like

$$...\underline{0\,0}.\underline{1\,1}.\underline{0\,0}.\underline{1\,0}.\underline{1\,0}.\underline{1\,1}.\underline{0\,1}....$$

The second basic object is that of a sequence u of 0's and 1's. Let $\{0,1\}^+$ be, as usual, the set of (non-empty) sequences over 0 and 1. If $u = u(1)u(2)...u(m), u(i) \in \{0,1\}^+$, for all i, $1 \leq i \leq m$ then the size of u, denoted $\mid u \mid$, is m; $u[i]$ for $i \leq m$ denotes $u(1)...u(i)$.

The two objects, w and u, provide the basis for defining universal traversing sequences for cycles. Fix w, u, and a vertex v of w. We say that we *run* the sequence u on w starting from v if we repeat the following step for every $i, 1 \leq i \leq \mid u \mid$: interpret $u(i)$, the ith symbol of u, as the instruction of following the edge labeled $u(i)$ at v; we end up after each step at a neighbour vertex. u visits all the vetices, or *traverses* w starting from v, if each vertex is visited at least once. If for every v on w, u traverses w starting from v, then u is said " universal" for w. If u is universal for every w of size n (there are 2^n such wheels), then u is said to be n-universal. For technical reasons, we will strengthen our notion of universal traversing sequence.

Definition 2 *Given a wheel w, a sequence u is* universal *for w if it satisfies the following condition: for each vertex v, u visits v in an* in-vertex-not-back *manner, i.e. the run of the sequence will visit v (in one of the possible multiple visits of v) coming from the one neighbor vertex of v, and continuing on to the other neighbor vertex of v. A sequence is* n-universal *if it is universal for every wheel w of size n.*

Definition 3 *A* D-arrangement *for wheels w, w' is given by $A = (D, L_w, L_{w'})$ where*

- *i) $D = \{d_1, ..., d_k\}$ is a set of labels;*
- *ii) $L_w : D \to V_w, L_{w'} : D \to V_{w'}$ are two injective functions, arranging $d_1, ..., d_k$ in this order on each wheel, w and w', as labels to some vertices. d is the label assigned to both vertex L_w on w, and vertex $L_{w'}$ on w'.*

A D-arrangement defines on each wheel two types of sets of intervals :

- Type 1: $[d_i, d_{i+1}], 1 \leq i \leq n-1, [d_k, d_1]$.
- Type 2: $[d_1, d_4], [d_3, d_{}], [d_5, d_8], ..., [d_{k-1}, d_2]$.

The type 1 set of intervals partitions the wheel into intervals such that consecutive intervals share an end point, while the type 2 set of intervals partitions the wheel into intervals such that consecutive intervals share two vertices.

Definition 4 *Let $[d, d']$ be an interval and S a mapping, $S : \{0,1\}^+ \to \{0,1\}^+$.*

S is said to be a synchronization function *for $[d, d']$ if for every $\bar{d} \in \{d, d'\}$ and every $u' \in \{0,1\}^+$, a parallel run of u' on w' and $u = S(u')$ on w starting both at $\bar{d}$ will synchronize at the end points d, d'. i.e. $\forall i \forall \tilde{d} \in \{d, d'\}, (u'[i]$ is at $\tilde{d}$ implies $S(u'[i])$ is at $\tilde{d})$. S is said to be a* synchronization function *for D if it is a synchronization function for every interval in D.*

The key notion to our construction of universal sequences is that of *contraction*. Intuitively, such a contraction is given by two parts: a mapping between wheels, and a mapping between sequences. The first captures the actual way of "contracting" a generic wheel w into a new wheel w', potentially having a smaller size. The second component, relates the universal sequences of the (possibly) smaller wheel w' with the universal sequences of w.

Definition 5 *Let $\mathbf{W}_n$ be the set of wheels of size n and $\mathbf{W}_{\leq n}$ the set of wheels of size $\leq n$. A* contraction *is given by*

$$K = (\{M_n \mid n \geq 1\}, \kappa)$$

, where

- $M_n : \mathbf{W}_n \to \mathbf{W}_{\leq n}, n \geq 1$
- $K : \{0,1\}^+ \to \{0,1\}^+$

such that for every n and every w of size n, if the sequence u' is universal for w', then the sequence $K(u')$ is universal for w.

Definition 6 *The* composition *of two contractions K_1 and K_2, denoted $K_1 \circ K_2$, is given by $(\{M_n \mid n \geq 1\}, \kappa)$, where*

- $M_n(w) = M_{2,|M_{1,n}(w)|}(M_{1,n}(w))$
- $\kappa = \kappa_1 \circ \kappa_2$

Lemma 1 *Contractions are closed under composition.*

3 Multiples and Singles

There are 4 types of edges : $.\underline{00}.$, $.\underline{11}.$, having identical bits, and $.\underline{01}.$, $.\underline{10}.$ having different bits. On the wheel, edges with identical bits can occur as *singles*, that is between two edges with different bits : a $.\underline{00}.$ single has an occurence in $.\underline{01}.\underline{00}.\underline{10}$, and a $.\underline{11}.$ single occurs in $.\underline{10}.\underline{11}.\underline{01}$. They can also occur in a region of *multiples* $.\underline{00}.$/ $.\underline{11}.$ edges, that is (at least 2) consecutive edges with identical bits, e.g. ....$.\underline{00}.\underline{11}.$....

Similarly, we have *singles* and *multiples* for the edges with different bits: $.\underline{00}.\underline{10}.\underline{11}.$ has the $.\underline{10}.$ single, $.\underline{11}.\underline{01}.\underline{00}.$ has the $.\underline{01}.$ single; ....$\underline{01}.\underline{01}$.... is a region of *multiples* (at least two) $.\underline{01}.$ (or $.\underline{10}.$) edges.

Notation. Let $q(w)$ be the number of edges of the form $.\underline{01}.$ or $.\underline{10}.$ on w; $\mu(w)$ be the number of multiples of such edges, and $\sigma(w)$ the number of such singles. Similarly, when we consider edges of the form $.\underline{00}.$ and $.\underline{11}.$, the same quantities are denoted by $q'(w), \mu'(w), \sigma'(w)$. Finally, singles($w$), and multiples($w$), represent the number of all singles on w and, respectively, the number of all multiples on w.

We have $\mid w \mid = q(w) + q'(w)$,
$q(w) = \mu(w) + \sigma(w)$, and
$q'(w) = \mu'(w) + \sigma'(w)$.

4 The first contraction

4.1 The construction

Define $K_1 = (\{M_{1,n} \mid n \geq 1\}, \kappa_1)$ as follows: Let w be a wheel of size n. We construct

- a new wheel $w' = M_{1,n}(w)$
- a synchronization function S_1 for w, w'
- the function κ_1

Constructing w'. $M_{1,n}$ "contracts" regions on wheels. In order to construct $w' = M_{1,n}(w)$, we are going to contract regions of multiples $.\underline{01}.$ and $.\underline{10}.$ on the wheel w. Moreover, we intend to halve the number of such multiples.

Such a region of multiples has the form

$$.\underline{11}.\underbrace{\underline{01}.\underline{01}.\dots.\underline{01}}_{m}.\underline{00}.$$

or its reverse. Let m be the number of $.\underline{01}.$ edges in the above region. This region will be transformed into a corresponding one, on the wheel w', according to the value of m. See fig.1.

Now, we perform the transformation described in fig.1 for all the regions of the above form from w. If no such region exists for any $m \geq 0$, then the wheel has only $.\underline{01}.$ edges and we take $w' = w$.

Defining S_1 and κ_1.

Definition 7 *The function $S_1 : \{0,1\}^+ \to \{0,1\}^+$ is defined by*
$S_1(0) = 000$,
$S_1(1) = 111$
and $S_1(u'_1 u'_2) = S_1(u'_1) S_1(u'_2)$, $\forall u'_1, u'_2 \in \{0,1\}^+$.

Definition 8 *The function $\kappa_1 : \{0,1\}^+ \to \{0,1\}^+$ is defined by*

$$\kappa_1(u') = 1^{|w|+1} S_1(u'), \forall u' \in \{0,1\}^+.$$

Properties of S_1 and κ_1

The next Lemma shows that we can construct a D-arrangement of distinguished positions on w and w' such that S_1 synchronizes on D.

Lemma 2 *Let w be a wheel of size n with n even, and $w' = M_{1,n}(w)$. Then:*

1. $\mid w \mid \geq \mid w' \mid$
2. $\mid w' \mid$ *is even*
3. *a D-arrangement can be constructed for w, w' such that S_1 is a synchronization function for D.*

Proof sketch. To construct the D-arrangement, we label with d's all the end points of the $.\underline{00}.$ and $.\underline{11}.$ edges. The arrangement partitions the wheel into intervals of Type 2.

The lines connecting the two wheels in fig.1, keep track of the progress on w as we go on w'. We have two types or progress-reporting lines: straight ones capturing the position as we go from the left end point to the right, and dashed ones indicating the position as we come from right to left.□

Lemma 3 *K_1 is a contraction.*

Proof. Consider a sequence u' which is universal for w'. We have to prove that $\kappa_1(u')$ is universal for w. Consider an arbitrary position p on w, and run $\kappa_1(u')$ from p. $\kappa_1(u')$ starts with $1^{|w|+1}$, at the end of which the run will be at the endpoint of an edge $.\underline{11}.$, if such an edge exists on w: i.e., at an end point d_0 of an interval. If no such egde $.\underline{11}.$ is on w, then we will traverse the wheel entirely. The "+ 1" assures that every vertex on w is visited in the required manner.

Now by Lemma 2, S_1 is a synchronization function on the intervals of w. We use the fact that u' is universal for w'. Start u' at d_0 on w', and $S_1(u')$ at d_0 on w continuing the run of $\kappa_1(u')$ from p. u' will traverse w', and therefore all the interval end points. By synchronization, $S_1(u')$ will visit all the interval end points on w too. By our definition of universality, a vertex on w' is visited by u' at least once in such a way that it is not returning after visiting it. This means that not only every edge is visited, but moreover, every two consecutive edges are visited "straight" in at least one direction. This forces the interior of an interval on w to be traversed entirely when the corresponding interval is traversed on w'.

It is easy to check in fig.1 that a traversal of the in-vertex-not-back type on the intervals of w' will imply the same kind of traversal on intervals from w. □

4.2 Evaluation and convergence

The effect of the first transformation is that on w' the number of multiples $.\underline{01}.$and $.\underline{10}.$is no more than half the same number on w. Also, the quantity of $.\underline{01}.$and $.\underline{10}.$edges decreases by a multiple of 2, from w to w'.

Lemma 4 *1.* $\mu(w') \leq \frac{\mu(w)}{2}$

2. $q(w) \geq q'(w')$ *and* $q(w) \equiv q(w')(mod2)$

5 The second contraction

The construction in this section is similar to the one presented in the previous section. The purpose, this time, is to halve the $.\underline{00}.$and $.\underline{11}.$multiples.

5.1 The construction

Define $K_2 = (\{M_{2,n} \mid n \geq 1\}, \kappa_2)$ as follows. Let w be a wheel of size n. We construct

- a new wheel $w' = M_{2,n}$
- a synchronization function S_2 for w, w'
- the function κ_2

Constructing w'. In order to construct $w' = M_{2,n}$, we are going to contract regions of multiples $.\underline{00}.$and $.\underline{11}.$on the wheel w. Moreover, we intend to halve the number of such multiples. Such a region of multiples has one of the 4 forms

$$.\underline{10}.\underbrace{\underline{11}.\underline{00}.\dots.\underline{11}.\underline{00}.}_{m}\underline{10}.$$

or its reverse, or

$$.\underline{10}.\underbrace{\underline{11}.\underline{00}.\dots.\underline{00}.\underline{11}.}_{m}\underline{01}.$$

or its reverse.

Fig.2 gives the set of rules for obtaining w'.

Defining S_2 and κ_2

Definition 9 *The function* $S_2 : \{0,1\}^+ \to \{0,1\}^+$ *is defined by*
$S_2(0) = 010$,
$S_2(1) = 101$
and $S_2(u_1'u_2') = S_2(u_1')S_2(u_2'), \forall u_1', u_2' \in \{0,1\}^+$.

Definition 10 *The function* $\kappa_2 : \{0,1\}^+ \to \{0,1\}^+$ *is defined by*

$$\kappa_2(u') = (10)^{\lfloor frac|w|2 \rfloor} 1 S_1(u'), \forall u' \in \{0,1\}^+.$$

Properties of S_2 and κ_2

Lemma 5 *Let w be a wheel of size n with n even and $w' = M_{2,n}(w)$. Then :*

1. $\mid w \mid \geq \mid w' \mid$
2. $\mid w' \mid$ *is even*
3. *a D-arrangement can be constructed for w, w' such that S_2 is a synchronization function for D.*

Lemma 6 *κ_2 is a contraction.*

5.2 Evaluation and convergence

Lemma 7 *1.* $\mu'(w') \leq \frac{\mu'(w)}{2}$

2. $q'(w) \geq q'(w')$ *and* $q'(w) \equiv q'(w')(mod2)$

6 The third contraction

The construction described in this section is different from the ones in the previously two sections. The purpose this time, is to "halve" *all the singles of any type* in the same transformation. It seems that $\bullet\underline{0\,1}\bullet$ and $\bullet\underline{1\,0}\bullet$ singles *need* the other $\bullet\underline{0\,0}\bullet$ and $\bullet\underline{1\,1}\bullet$ singles in order to be able to "halve". The process is not as before: instead of halving the total number of singles, we are only forcing a shrinking of the wheel w' with at least half the number of singles from w. We define intervals on w and w', this time of Type 1 according to Definition 2. As before, each interval on w corresponds with one on w'.

6.1 The construction

Define $K_3 = (\{M_{3,n} \mid n \geq 1\}, \kappa_3)$ as follows. Let w be a wheel of size n. We construct

- a new wheel $w' = M_{3,n}(w)$
- a synchronization function S_3 for w, w'
- the function κ_3

<u>Defining S_3 and κ_3</u>

Definition 11 *The function* $S_3 : \{0,1\}^+ \rightarrow \{0,1\}^+$ *is defined by*
$S_3(\text{odd} - 0) = 000$
$S_3(\text{odd} - 1) = 100$
$S_3(\text{even} - 0) = 011$
$S_3(\text{even} - 1) = 111$
and $S_3(u')$ is defined on $u' \in \{0,1\}^+$ by replacing each bit in u', depending on the parity of its occurence with the corresponding triple.

Example.
$S_3(001011) = 000011100011100111$.

Definition 12 *The function* $\kappa_3 : \{0,1\}^+ \rightarrow \{0,1\}^+$ *is defined by*

$$\kappa_3(u') = 00111S_3(u'), \forall u' \in \{0,1\}^+.$$

<u>Constructing w'.</u> We partition w into intervals, each of which shares only one end point. That is, we are constructing a D-arrangement of Type 1.

Algorithm for Interval Construction for w.

let $u = 00, u_0^{-1} = 011, u_1^{-1} = 111$.
fix an arbitrary vertex and an order on w.
mark the vertices in w with positions $1, 2, ..., \mid w \mid$ following the wheel order starting from the fixed vertex.
for $p = 1, 3, 5, ..., \mid w \mid -1$ **do**
begin
run u at p; **let** p' be the resulting position.
let $\bullet\underline{\;z}\overset{p'}{\bullet}\underline{\bar{z}\;}\bullet$ be the bits at p' on w.
run $u_z^{-1} = z11$ at p'; **let** p'_L be the resulting position.
run $u_{\bar{z}}^{-1} = \bar{z}11$; **let** p'_R be the resulting position.
end.

Let $[p'_L, p'_R]$ be the interval defined as that portion of the wheel containing the vertices visited by (one or some of) u, u_0^{-1}, u_1^{-1}. It includes the end points. p' is called the *center*. Note that the center of an interval can be inside or outside the interval:

$$[\bullet\underline{1\,1}\overset{p}{\bullet}\underline{0\,1}\bullet\underline{0\,0}\overset{p'}{\bullet}\underline{1\,1}]$$

or

$$[\bullet\underline{0\,1}\overset{p}{\bullet}\underline{0\,1}\bullet]\underline{0\,1}\overset{p'}{\bullet}\underline{0\,1}\bullet$$

Let Intervals(w) be the set of intervals generated by the algorithm.

Lemma 8 *Intervals(w) partitions w.*

Proof sketch. The Lemma will follow by establishing the following 3 facts:

- i) Intervals do not overlap (except for one end point).
- ii) For consecutive intervals, two end points meet. Moreover, the bits of the centers give the inverses (i.e. u_0^{-1}, u_1^{-1}) that meet. That is, if p'_1 and p'_2 are consecutive centers, and

  $$p'_1\bullet\underline{x\;}\cdots\underline{\;z}\bullet p'_2$$

  then $x11$ from p'_1 and $z11$ from p'_2 will meet.

- iii) Intervals are non-empty.

To give a sample of the proof technique used, we prove only i). Intervals are defined by a p'. From there you run 011 and 111 to find the end points. i) Look at two consecutive p' points, say p'_1, p'_2. They are on odd positions. Suppose we run $x11$ on p'_1 and $y11$ on p'_2 in parallel. After x, respectively y, we are at even positions, and we have left 11 to run on both. It follows that they cannot cross. Indeed, if they approach each other they will be at even distance from each other. If they are at distance 0, the two runs will continue together because they use the same bit. Thus, intervals cannot overlap. $\Box$

Corollary 1 *The "dynamics"-graphs of fig.8 holds for every interval, regardless of whether the center is inside or not.*

That is, from p'_L we reach p by running $x00$, from p we end up at p'_R by running $\bar{z}11$, and similarly for the other two arrows.

Definition 13 *We define $M_{3,n}$ as follows. For a given w of size n, the wheel $w' = M_{3,n}(w)$ is defined as follows.*

For each interval on w, we replace it on w' by following the transformation given in Fig.8.

<u>Properties of S_3 and κ_3</u>

Lemma 9 *Let D be the D-arrangement obtained by placing a d at the interval end points on both wheels. Then S_3, given in Def.10, is a synchronization function on D.*

Proof. By the above Corollary, S_3 is defined as a simulator of the dynamics graph of an interval. Thus, synchronization is achieved. $\Box$.

Lemma 10 *κ_3 is a contraction.*

Proof. Let u' be a universal sequence for w', and v_0 be an arbitrary vertex on w on which we start running $\kappa_3(u')$. We label the vertices with positions starting from v_0. The initial fragment 00111 of $\kappa_3(u')$ has the role of positioning the sequence at an interval end point, say d, before $S_3(u')$ starts. Consider the corresponding d on w', and start running u' from it. Because u' is universal for w', by lemma 9 it follows that $S_3(u')$ will visit all the d's on w. It also follows from our notion of universality that we also visit the interiors of the intervals. Moreover, it is easy to check in fig.4 that each vertex is visited in the required manner. As a result, $S_3(u')$ is universal for w. $\Box$

6.2 Evaluation and Convergence

Evaluating the size of w', we want to show that the singles on w are "halfed" on w', i.e., w' is smaller in size than w by at least (singles$(w)/2$).

Lemma 11 *$|\ w'\ | \leq$ (singles(w) / 2) + multiples(w).*

Proof.

Where can a single (edge) be on w ? It can be in a contracting interval, i.e. any interval of size 4 or 6, or in a non-contracting interval, i.e. any interval of size 2.

Let us call an interval *full of singles* when all the egdes of the interval are singles.

- Case 1: A contracting interval. A full interval of this type can have 4 or 6 singles. The same interval on w' has size 2. Because $2 \leq \frac{4}{2}, 2 \leq \frac{6}{2}$, it follows that w' decreases by an amount that is at least half the number of such singles. Hence, all the singles in a contracting interval are "halfed".
- Case 2: A non-contracting interval.

  The analysis of this case involves several steps:

  - i) Only 4 non-contracting intervals
  - ii) The Domino Graph of fig.5
  - iii) Halving

  There are 4 such intervals of size 2.

  1. $\bullet\underline{11}[\bullet\underline{00}\bullet\underline{11}\bullet]$
  2. $[\bullet\underline{01}\bullet\underline{01}\bullet]\underline{01}\bullet\underline{01}\bullet$
  3. $\bullet\underline{11}[\bullet\underline{01}\bullet\underline{00}\bullet]\underline{11}\bullet$
  4. $[\bullet\underline{11}\bullet\underline{01}\bullet]\underline{01}\bullet\underline{01}\bullet$

  ii) The "domino"-graph from fig.5 gives all the sequences of consecutive non-contracting intervals. The 4 vertices represent the above 4 types of size 2 intervals. Any sequence of consecutive size 2 intervals can be obtained as a path in the graph. Going along the path forces intervals to have size 2. Going out from a vertex, that is, not following any edge emerging from that vertex, implies the continuation on the wheel with an interval of size ≥ 2. The domino-graphs is used in analyzing the contraction. □

The next Lemma shows that the quantity of $\bullet\underline{0\,0}\bullet$ and $\bullet\underline{1\,1}\bullet$ decreases by a multiple of 2. The same holds for the $\bullet\underline{1\,0}\bullet$ and $\bullet\underline{0\,1}\bullet$ edges.

Lemma 12 *1. $q(w) \geq q(w')$ and $q(w) \equiv q(w')(mod 2)$*

2. $q'(w) \geq q'(w')$ and $q'(w) \equiv q'(w')(mod 2)$

7 Constructing the Universal Sequence

We have three contractions K_1, K_2, K_3. By Lemma 1, we know that any composition of them will still be a contraction.

Definition 14 *Let $K_{123} = K_1 \circ K_2 \circ K_3$. The sequence U_n is defined by*

$$U_n = K_{123}^{\lceil \log_2 n \rceil}(00010).$$

Theorem 1 *For any n, n-universal traversing sequences for cycles can be constructed in $\log n$ space. The sequence U_n is such an n-universal sequence, and has size $O(n^{4.76})$.*

Proof. We are going to establish the result in several steps.

- i) $(n+1)$-universal implies n-universal
- ii) the recursive construction
- iii) termination
- iv) evaluating the size
- v) constructing the sequence in log space

i) let u be a $(n+1)$-universal, and w a wheel of size n. Run u on w from an initial vertex v_0. If u is not visiting all the vertices in w, then consider one such vertex v. Replace v by a new edge in the graph and add bits to it. We have now a new wheel of size $n+1$ which is not traversed by u from v_0. This contradicts the $(n+1)$-universality of u.

It follows that if n-universal sequences are constructible for even n, then universal sequences are constructible for every n.

ii) Starting with a wheel w of even length, let

$w_1 = M_{1,|w|}(w)$,

$w_2 = M_{2,|w_1|}(w_1)$,

$w_3 = M_{3,|w_2|}(w_2)$.

We can see that w_3 in the result of applying all three contractions, one after the other, to w.

Claim. $| w_3 | < | w | /2$. Indeed,

$| w_3 | = q(w_3) + q'(w_3)$

$= \mu(w_3) + \sigma(w_3) + \mu'(w_3) + \sigma'(w_3)$

$\leq \mu(w_2) + \mu'(w_2) + \frac{\sigma(w_2)+\sigma'(w_2)}{2}$ (By Lemma 11)

$= \mu(w_2) + \mu'(w_2) + \frac{\sigma(w)+\delta+\sigma'(w)+\delta'}{2}$

where δ, δ' are the number of singles

introduced by K_1, K_2, not existing on w.

We have $\mu(w_2) + \delta \leq \mu(w)/2$ and similarly

for the ''-ones.

$\leq \frac{\mu(w)}{2} + \frac{\mu'(w)}{2} + \frac{\sigma(w)}{2} + \frac{\sigma'(w)}{2}$

$= q(w) + q'(w)$

$= | w | /2$.

iii) For an even wheel w, consider the "progress" as the pair $\pi(w) = (q(w), q'(w))$ composed of even numbers. By the convergence Lemmas 4.(b), 7.(b), and 12.(a,b), it follows that one or both component(s) of the "progress" will decrease after each contraction. This fails to be true when the wheel is $w_0 = \bullet\underline{0\,1}\bullet\cdots\bullet\underline{0\,1}\bullet$, with $\pi(w_1) = (0, q'(w))$, or the wheel $w_1 = \bullet\,\underline{0\,0}\bullet\underline{1\,1}\bullet\cdots\bullet\underline{0\,0}\bullet\underline{1\,1}\bullet$, with $\pi(w_2) = (q(w), 0)$. In these exceptional cases, the wheel is no longer modified by the contractions. So any wheel is eventually tranformed into a w_0 or w_1. In order to see that these are the only two possibilities where "progress" stops, we remark that the $\pi(w)$ starts with two even numbers, and progresses by decreasing one or both of the components by a multiple of 2. As long as both of the componnents are not 0, progress is still possible!

iv) Let the size of U_n be $f(n)$. Then

$f(n) = 3f(n_1) + n + 1$ (accounting for K_1)

$= 3(3f(n_2) + n + 1) + n + 1$ (accounting for K_2)

$= 3(3(3f(\frac{n}{2}) + 5) + n + 1) + n + 1$ (accounting for K_3)

$= 27f(\frac{n}{2}) + 4n + 49$

Therefore, the recurrence relations are

$$f(n) = 27f(\frac{n}{2}) + 4n + 49$$
$$f(1) = 1$$

Then because $\log_2(27) = 3\log_2(3)$ is about 4.76 we have

$$f(n) = O(n^{4.76})$$

v) The above construction can be done in $O(\log n)$. We can reformulate our construction in such a way that an algorithm can be obtained for computing the ith bit of the sequence using only log n space. The details, relatively long, but straightforward, are omitted in this abstract. □

8 Conclusions and Open Problems

The paper presents the first log space construction of universal traversing sequences for 2-regular graphs. The size of the sequence is $O(n^{4.76})$. It seems that there is room for improving the exponent of the polynomial. We leave the task of reducing it to the full paper. Certainly, the open problem is to obtain a result of the same nature for d-regular graphs. Previous attempts at reducing the general case to a particular value of d stopped at $d = 3$ (see [10], [4]). Therefore, the hope is to be able to extend the method used in our construction. Such a result will have outstanding consequences in Complexity Theory. We conjecture that universal traversing sequences for 3-regular graphs are log space constructible.

It is the author's belief, in the light of recent exciting results [11], [5], that UNDIRECTED CONNECTIVITY is complete for NSPACE($logn$), and universal traversing sequences in some generalized form might help in that direction too.

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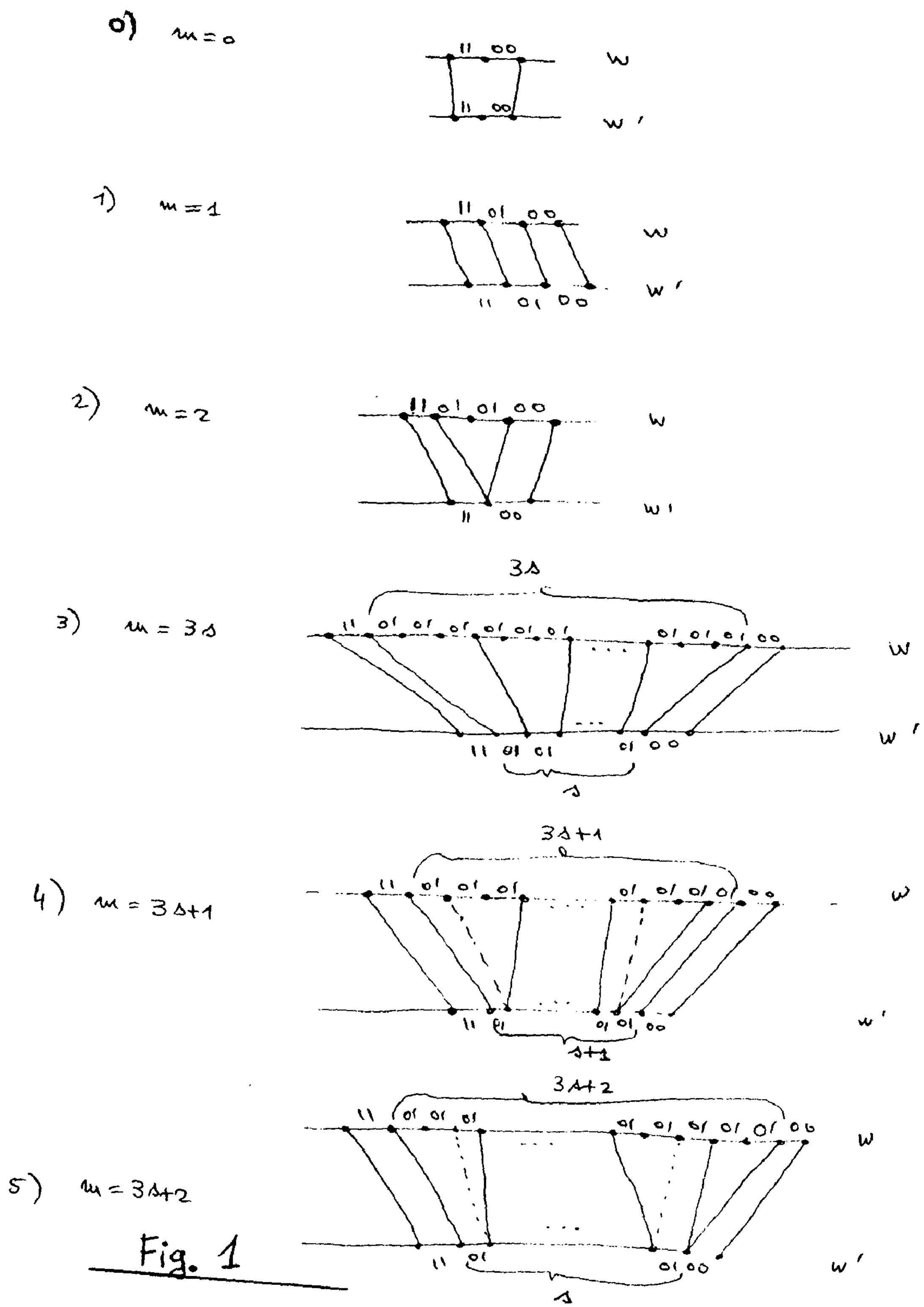

Fig. 1

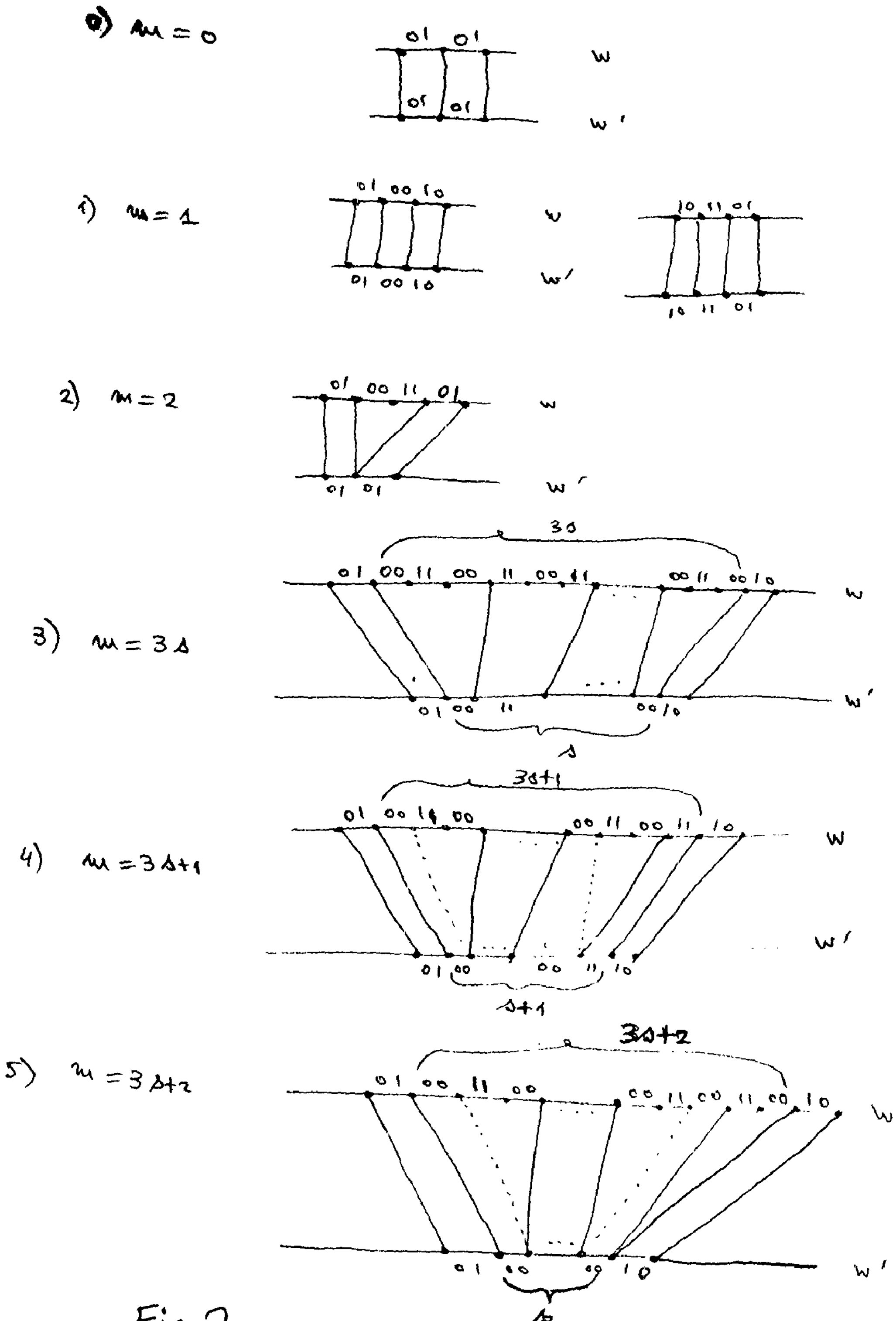

Fig. 2

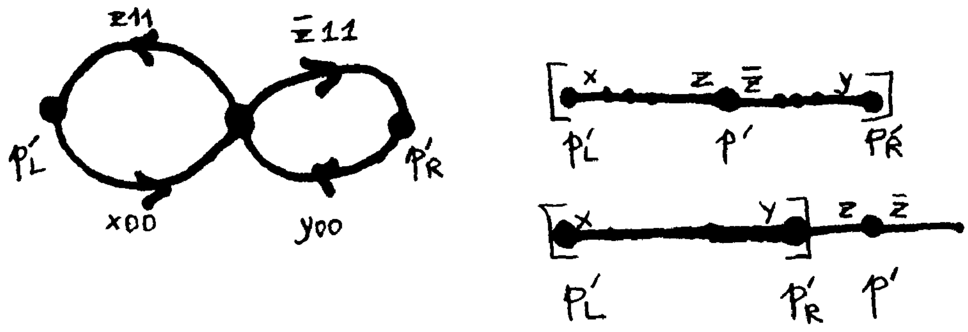

Fig.3

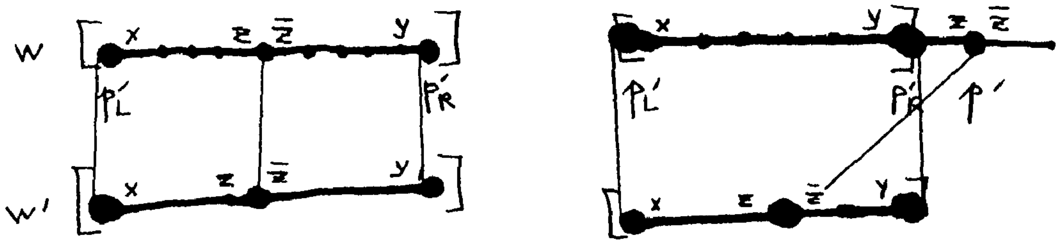

fig.4

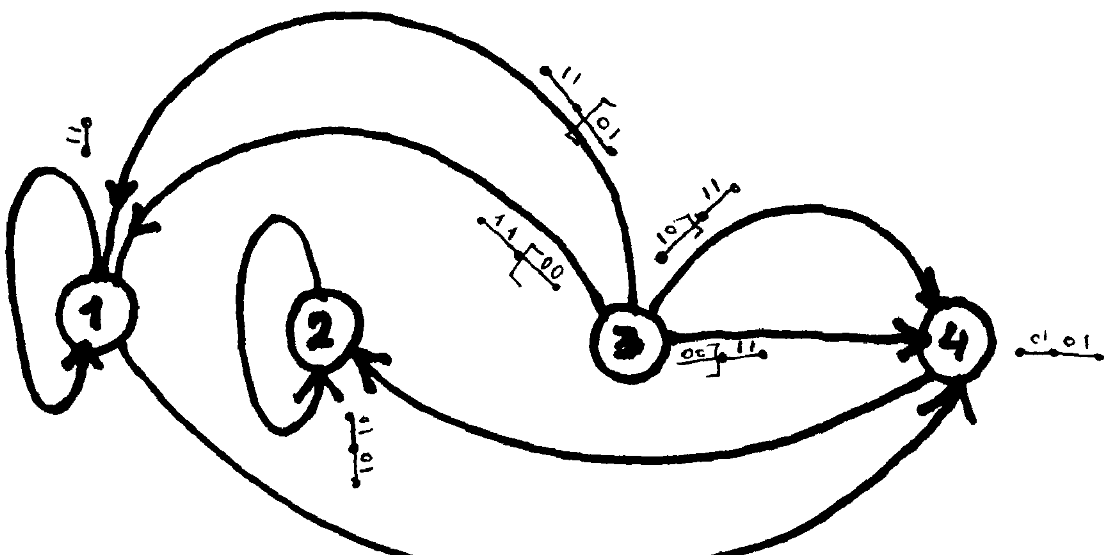

Fig. 5