Can we Make School Choice more Efficient?

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School Choice

Thousands of students in Boston and New York participate in the public school choice system

- Students (strategically) submit a (strict) preference over schools
- Schools (non-strategically) have (weak) priorities over students
- A central mechanism produces the best *stable* matching for students
    - Individually rational
    - Not blocked: no unmatched student-school pair strictly improves by matching to each other

School Choice

- Economists recently redesigned the NYC and Boston school admission systems (Abdulkadiroğlu et al. 2005a,b)
- The current design uses the Deferred Acceptance algorithm, with random (single lottery) tie breaking (DA-STB)
    - Strategy-proof for students
    - Stable
    - Inefficient - *About 1,500 students in NYC annually could be assigned to a more preferred school (Abdulkadiroğlu et al. 2009)*

School Choice

- Student Optimal Stable Matchings (SOSM) can be implemented using the Stable Improvement Cycles mechanism (SIC) (Erdil,Ergin AER2008)
  - Pareto dominates DA-STB
  - *Not strategy-proof for students*
  - Stable (under truth-telling)
  - Efficient (under truth-telling)

School Choice

"Nothing is yet known about what kinds of preferences one could expect to be strategically submitted to such a mechanism, or what their welfare consequences would be. Consequently, there is room for more work to further illuminate the tradeoff between efficiency and strategy-proofness."

(Abdulkadiroğlu, Pathak, Roth AER2009)

This talk

- Two papers
- Equilibrium behavior of the SIC mechanism
- Approximate large school choice problems using a continuum framework
  - Azevedo-Leshno 2010b develop the theory for college admission with a continuum of students, and show when it approximates large discrete problems

This talk

- Using a continuum model, analyze equilibrium of SIC :
  - can *Pareto dominate*, can be *Pareto dominated*, or not be Pareto comparable with DA-STB outcomes
  - assignment be inefficient and unstable with respect to the true preferences
  - simple (intuitive) manipulations, rely only on aggregate information
  - SIC is manipulable in large markets
- Trade changes the valuation
  - students rank schools by their "trade value"
  - results in missallocation
  - trade does not always correct all the missallocation

Literature

DA-STB is inefficient, SOSM is not Strategy-proof: Erdil, Ergin AER 08; Abdulkadiroğlu, Pathak, Roth AER 09

Current Design: Abdulkadiroğlu, Pathak, Roth AER PP 2005 Abdulkadiroğlu, Pathak, Roth, Sönmez AER PP 05

Incentive properties of deferred acceptance: Kojima, Pathak AER 09; Immorlica, Mahdian SODA 05; Dubins, Freedman AMM 81, Roth MOR 82

Experiments: Che, Sönmez JET 06; Featherstone, Niederle 08; Echenique, Wilson, Yariv 09

Alternative mechanisms: Abdulkadiroğlu, Che, Yasuda AER forthcoming; Kesten QJE forthcoming

Continuum matching models: Abdulkadiroğlu, Che, Yasuda 09; Miralles 08

School Choice (with Continuum of Students)

A school choice problem is:

- Finite set of schools $S = \{a, b, ..., c\}$
- Quota q_a for each school a
- A student (type) θ is described by $\theta = (\prec_\theta, e_\theta)$
  - $\prec_\theta$ - a strict preference ordering over $S \cup \{\phi\}$
    (ϕ denotes being unmatched)
  - e_θ - vector of priorities $e_\theta \in \mathbb{R}^{|S|}$

    $e(a)$ is the priority of the student at school a

    θ is acceptable for school $a \in S$ if $e_\theta(a) \geq 0$
- A measure η [m] over the set of student types Θ

School Choice (with Continuum of Students)

A *matching* is a function $\mu : \Theta \to S \cup \{\phi\}$ such that:

- A school a is matched to the set of students $\mu^{-1}(a)$ (a student is matched to a school iff the school is matched to the student)
- School capacity is respected:

$$|\mu(a)| = \eta(\{\theta | \mu(\theta) = a\}) \leq q_a$$

- It is right continuous (technical)

School Choice (with Continuum of Students)

A pair (a, θ) blocks the match μ if:

- The student prefers the school over his current match: $a \succ \mu(\theta)$
- The school is either strictly under capacity $|\mu(a)| < q_a$ and θ is acceptable $e_\theta(a) \geq 0$
  or
  there is θ' such that $\mu(\theta') = a$ but $e'(a) < e(a)$

A matching μ is *stable* if:

- It is individually rational
- There is no pair (a, θ) that blocks

DA-STB (the current mechanism)

Break ties in priorities and run the Gale-Shapley algorithm with students proposing:

stage 0 Single random lottery number for each student. Break ties in favor of students with higher lottery number

$\Longrightarrow$ Strict priorities for all schools

stage 1 Students apply to most preferred school that didn't rejected them.

stage 2 Schools tentatively keep the highest priority students up to capacity, and reject the rest

If students were rejected, go to 1 ; otherwise, end.

The SIC Mechanism

Any efficient mechanism that Pareto dominates DA (SOSM) can be implemented by the SIC algorithm:

- ▷ Run DA-STB
- ▷ Trade: Find a Pareto improvement that won't violate stability (a stable improvement cycle) and implement it.
  Repeat till none exist

The outcome is an efficient stable matching that Pareto dominates the DA-STB match

Example (Erdil-Ergin)

School Capacities: $q_a = q_b = 1, q_z = \infty$

Student preferences:

α	β	ζ
a	b	a
b	a	z
z	z	

Priorities:

	α	β	ζ
a	0	1	0
b	1	0	0
z	0	0	0

Example (Erdil-Ergin): DA-STB

The only tie break that matters is between α and ζ at a.

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- If broken in favor α:

	a	b	z
Step 1	α, ζ	β	
Step 2	α	β	ζ

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The only tie break that matters is between α and ζ at a.

- If broken in favor α:

	a	b	z
Step 1	α, ζ	β	
Step 2	α	β	ζ

- If broken in favor ζ:

	a	b	z
Step 1	ζ, α	β	
Step 2	ζ	α, β	
Step 3	β, ζ	α	
Step 4	β	α	ζ

Example (Erdil-Ergin): DA-STB

The only tie break that matters is between α and ζ at a.

- If broken in favor α:

	a	b	z
Step 1	α, ζ	β	
Step 2	α	β	ζ

Pareto efficient match

- If broken in favor ζ:

	a	b	z
Step 1	ζ, α	β	
Step 2	ζ	α, β	
Step 3	β, ζ	α	
Step 4	β	α	ζ

Pareto inefficient match

Example (Erdil-Ergin): DA-STB

- Outcome of DA-STB is a random matching:

$$\begin{aligned} \mu_{DA-STB} : \alpha &\rightarrow \frac{1}{2}a, \frac{1}{2}b \\ \beta &\rightarrow \frac{1}{2}b, \frac{1}{2}a \\ \zeta &\rightarrow z \end{aligned}$$

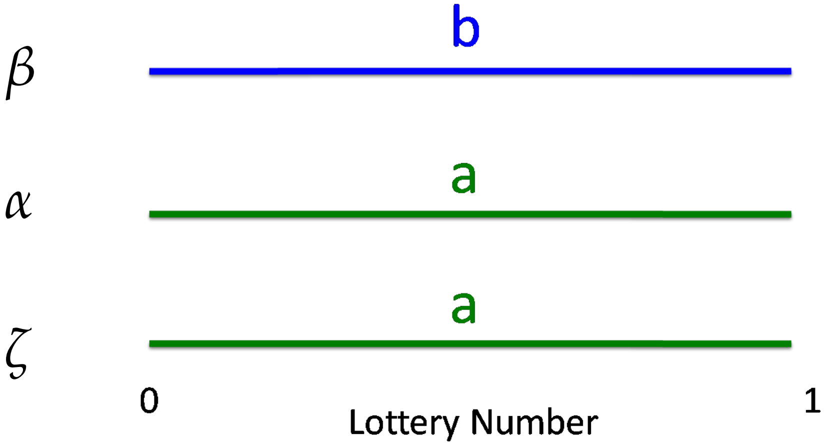
- In the inefficient match α, β form an SIC, i.e. they should trade
- The SOSM outcome (under truth-telling) is:

$$\begin{aligned} \mu_{*} : \alpha &\rightarrow a \\ \beta &\rightarrow b \\ \zeta &\rightarrow z \end{aligned}$$

Example (Erdil-Ergin): Continuum DA-STB

- Unit mass of each type: $m(\alpha) = m(\beta) = m(\zeta) = 1$
- Break ties by drawing a single lottery number.
  The mass of students has lottery numbers uniformly distributed $\ell \sim U[0, 1]$
- School have capacity $q_a = q_b = 1, q_z = \infty$

Example (Erdil-Ergin): Continuum DA-STB



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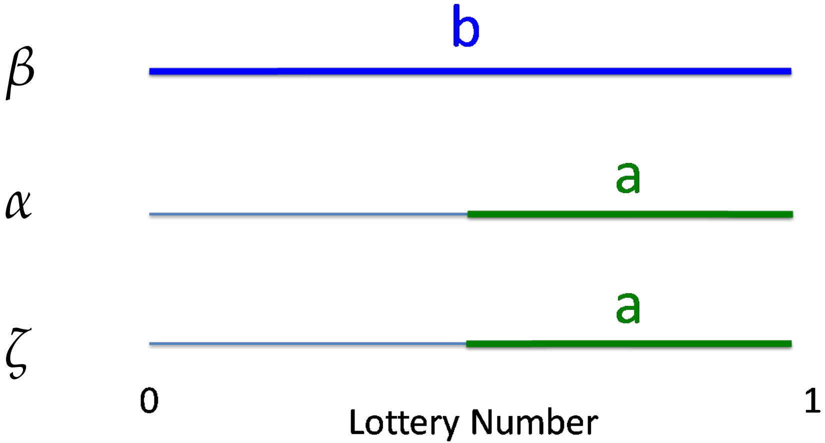

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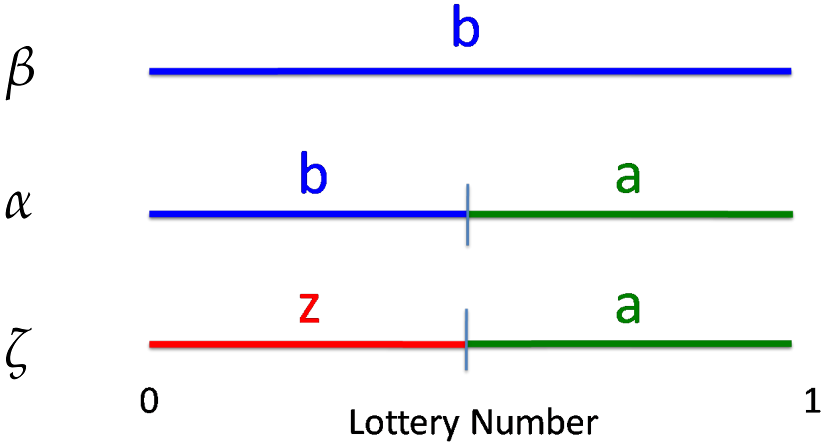

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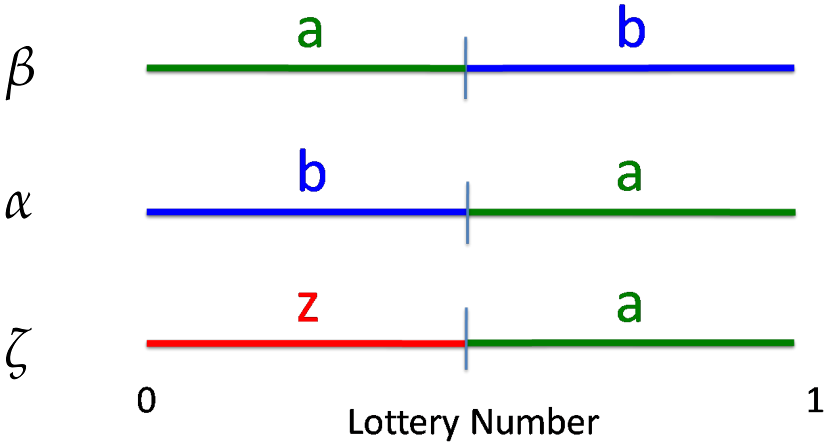

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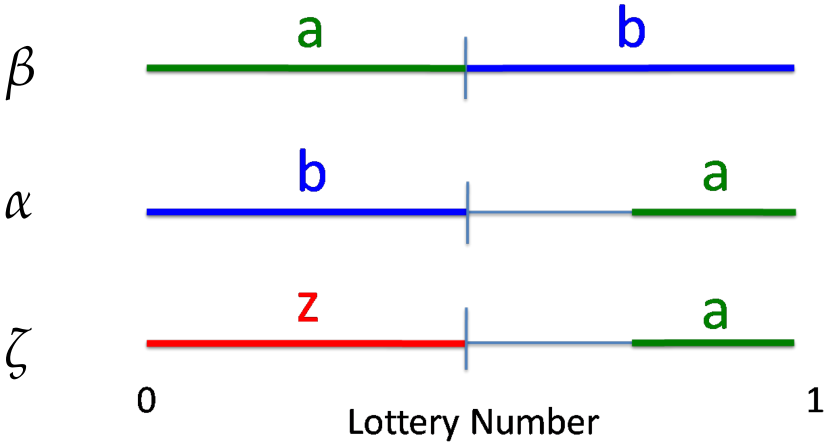

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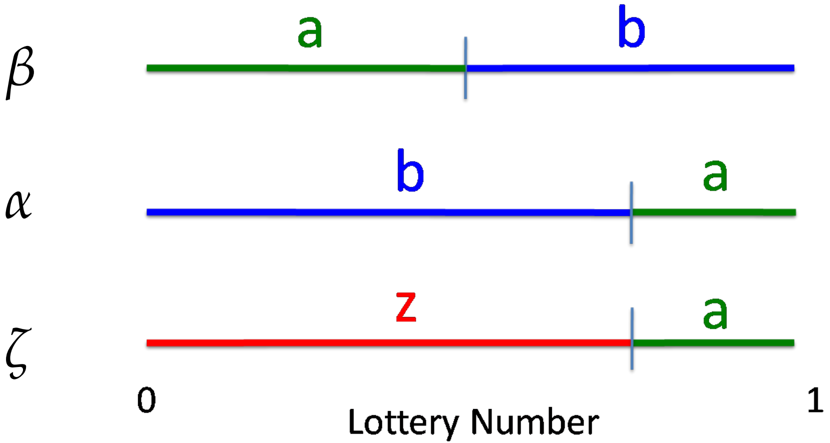

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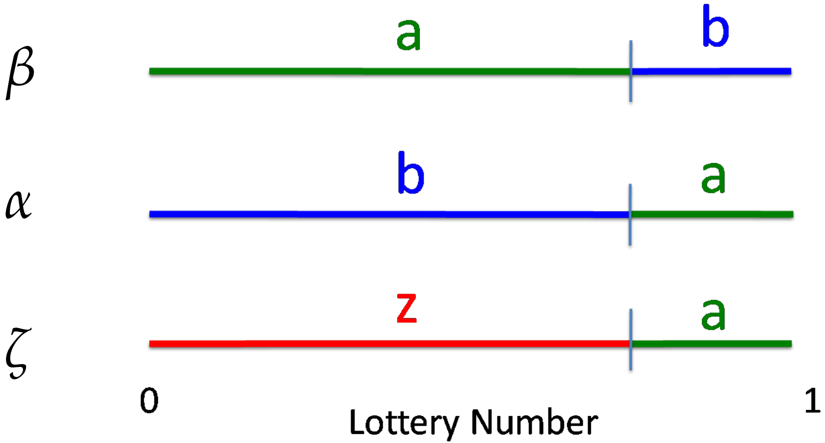

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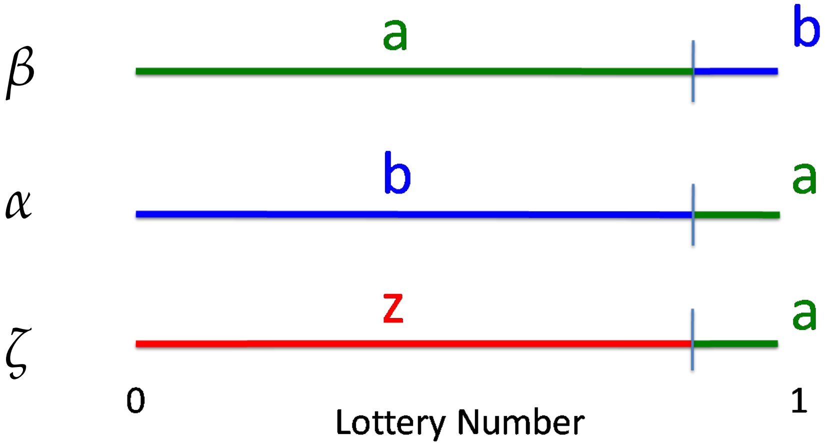

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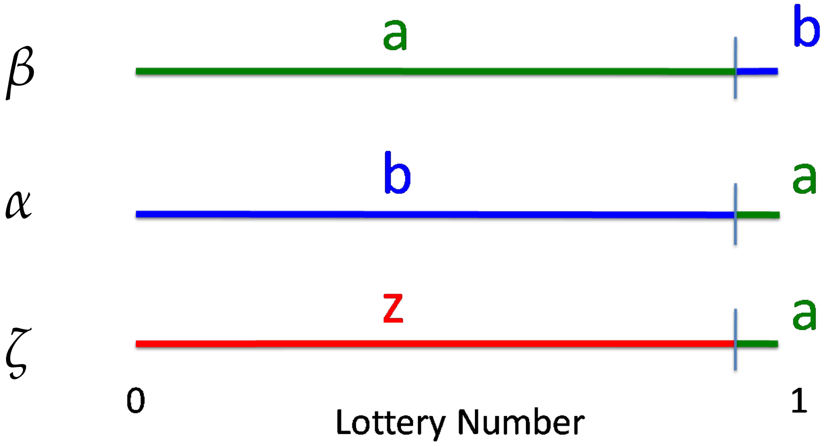

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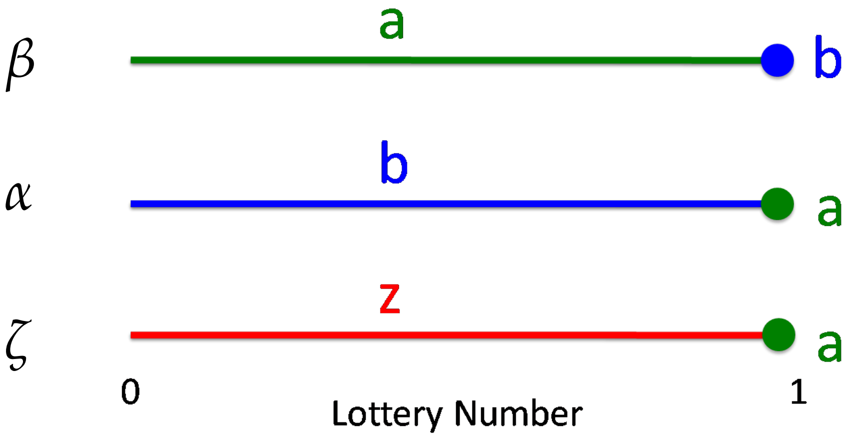


Cutoffs

cutoffs are a useful way to represent stable matches

- A system of *cutoffs* is a vector $p \in \mathbb{R}_+^{|S|}$
- A school a is *attainable* for student θ if $e_\theta(a) \geq p(a)$
- Given cutoffs p, student θ's *demand* is

$$D(p,\theta) = \arg\max_{\prec}(\{a \in S | e(a) \geq p(a)\} \cup \{\phi\})$$

- The aggregate demand for a school $a \in S \cup \{\phi\}$ is:

$$D_a(p) = \eta(\{\theta \in \Theta | D(p,\theta) = a\})$$

Cutoffs

- p is a system of *equilibrium cutoffs* if

$$D_a(p) \leq q_a$$

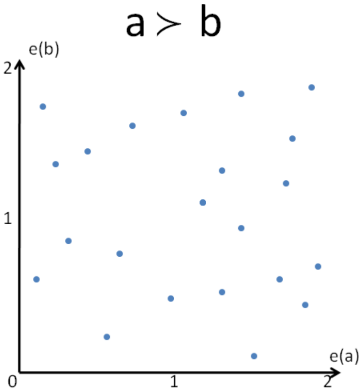
for all $a \in S$, with $p(a) = 0$ if the inequality is strict

Lemma ($\mu \equiv p$)

Equilibrium cutoffs are equivalent to stable matching:

- *The demand of an equilibrium cutoff defines a stable matching.*
- *Every stable matching can be given as the demand of some equilibrium cutoffs.*

Cutoffs



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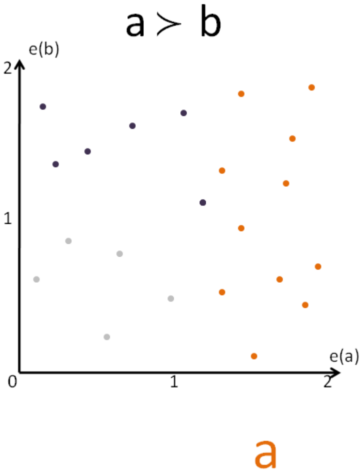

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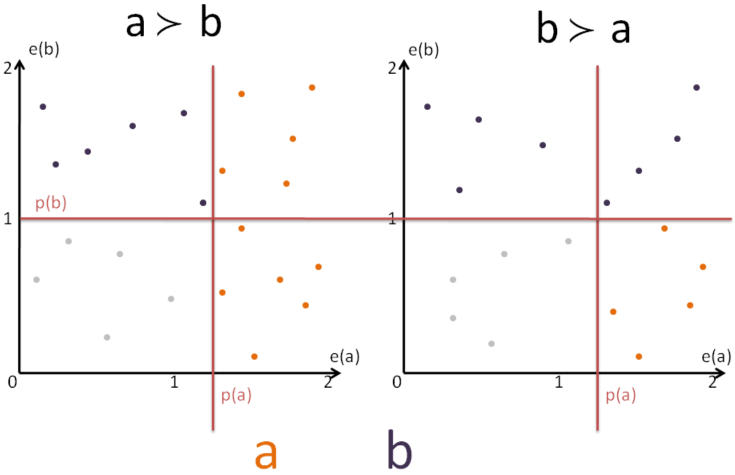

Using the continuum Framework

- we can describe the stable match using cutoffs (p_a, p_b), found by market clearing equations (A-L 2010):

$$\begin{aligned} m(\alpha)(1-p_a) + m(\beta)\cdot p_b \quad + m(\zeta)(1-p_a) &= q_a = 1 \\ m(\alpha)\cdot p_a + m(\beta)(1-p_b) &= q_b = 1 \end{aligned}$$

- Unique solution:

$$p_a = p_b = 1$$

- The continuum matching:

$$\mu_{DA-STB} : \alpha \to b, \beta \to a, \zeta \to z$$

Using the continuum Framework

Theorem

When the continuum has a unique stable matching, the DA-STB outcome of a close large matching problem is close to the continuum matching (A-L 2010)

SIC - manipulations

- Students will misreport a preferences for schools that can be traded
- Over-demanded schools will "enable good trades"
- Congestion, negative externalities
- Students evaluate manipulations ex-ante, considering lotteries over trade.
  Fail to trade ex-post $\Rightarrow$ inefficiency, instability

Ex: Pareto Inefficiency

Two special schools: a = Art , s = Science

Capacities: $q_a = 1, q_s = 2$

Students : $m(\tau) = m(\zeta) = 1$ and $m(\gamma_a) = 2$

School Priorities

0	a	s
τ	1	0
ζ	0	0
γ_a	0	0
γ_{as}	0	0

Student Preferences

τ	ζ	γ_a	γ_{as}
s	s	a	a
a	a	ϕ	s
ϕ	ϕ		ϕ

γ_a is an EU-maximizer with

$$u_{\gamma_a}(a) = 1 > u_{\gamma_a}(\phi) = 0 > u_{\gamma_a}(s) \geq -0.5$$

Ex: Pareto Inefficiency

Lemma

The unique equilibrium outcome of SOSM is Pareto dominated by the DA-STB outcome

Ex: Pareto Inefficiency

- τ and ζ report truthfully in equilibrium
- γ_a can misreport his type to be γ_{as}

Suppose a mass v of γ_a report γ_{as}:

Type	Preferences	Priority at	Mass
τ	$s \succ a$	a	1
ζ	$s \succ a$	–	1
γ_a	a	–	$2 - v$
γ_{as}	$a \succ s$	–	v

Ex: Pareto Inefficiency

We solve for the DA-STB allocation using the market clearing equations:

$$\begin{aligned} m(\tau) \cdot p_s + m(\zeta)(p_s - p_a)^+ + m(\gamma_{as})(1 - p_a) + m(\gamma_a)(1 - p_a) &= q_a = 1 \\ m(\tau)(1 - p_s) + m(\zeta)(1 - p_s) + m(\gamma_{as})(p_a - p_s)^+ &= q_s = 2 \end{aligned}$$

The unique solution is:

$$p_a = \frac{v+2}{v+4}, p_s = \frac{v}{v+4}$$

Ex: Pareto Inefficiency

The DA-STB allocation is:

$$\begin{aligned}
\mu_{DA-STB} : \tau &\rightarrow \frac{4}{v+4}s, \frac{v}{v+4}a \\
\zeta &\rightarrow \frac{4}{v+4}s, \frac{v}{v+4}\phi \\
\gamma_a &\rightarrow \frac{2}{v+4}a, \frac{v+2}{v+4}\phi \\
\gamma_{as} &\rightarrow \frac{2}{v+4}a, \frac{2}{v+4}s, \frac{v}{v+4}\phi
\end{aligned}$$

Ex: Pareto Inefficiency

The unique Pareto dominating efficient allocation is:

$$
\begin{aligned}
\mu_* : \tau &\rightarrow s \\
\zeta &\rightarrow \frac{4}{v+4}s,\ \frac{v}{v+4}\phi \\
\gamma_a &\rightarrow \frac{2}{v+4}a,\ \frac{v+2}{v+4}\phi \\
\gamma_{as} &\rightarrow \frac{3}{v+4}a,\ \frac{1}{v+4}s,\ \frac{v}{v+4}\phi
\end{aligned}
$$

Ex: Pareto Inefficiency

The best response of γ_a under SOSM:

- If reports truthfully:

$$\frac{2}{v+4}u_{\gamma_a}(a) + \frac{v+2}{v+4}u_{\gamma_a}(\phi) = \frac{2}{v+4}$$

- If reports γ_{as}:

$$\frac{3}{v+4}u_{\gamma_a}(a) + \frac{1}{v+4}u_{\gamma_a}(s) + \frac{v}{v+4}u_{\gamma_a}(\phi) \quad = \quad \frac{2.5}{v+4}$$

$\Rightarrow$ reporting γ_{as} is a dominant strategy.

Ex: Pareto Inefficiency

Under DA-STB the unique equilibrium has $m(\gamma_{as}) = 0$

Under SOSM the unique equilibrium has $m(\gamma_{as}) = 2$

$$\mu_{DA-STB}: \qquad \begin{array}{rcl} \tau & \to & s \\ \zeta & \to & s \\ \gamma_a[\gamma_a] & \to & \frac{1}{2}a,\ \frac{1}{2}\phi \end{array}$$

$$\mu_*: \qquad \begin{array}{rcl} \tau & \to & s \\ \zeta & \to & \frac{2}{3}s,\ \frac{1}{3}\phi \\ \gamma_a[\gamma_{as}] & \to & \frac{1}{2}a,\ \frac{1}{6}s,\ \frac{1}{3}\phi \end{array}$$

Ex: Pareto Inefficiency - Comments

Manipulations

- are simple: “grab” a school in order to “trade” it
- work even in a large market, rely only on aggregate information

Students may get “stuck” with a school. Therefore the allocation is

- Pareto inefficient wrt to the true preferences
- unstable and not individually-rational wrt to the true preferences

The General case

DA-STB is strategy proof ⇒ rank schools by their value

SOSM can be implement by DA-STB + trade stage

- DA-STB allocation is a lottery of trade opportunities
- The utility of a DA-STB allocation is the utility of the trade lottery, *not* the utility of a final allocation
  ⇒ misreporting
- Students take lotteries with unacceptable schools
  ⇒ positive probability of instability

Summary

Student optimal stable mechanisms:

1. Can produce unstable outcomes
2. Can:
   - Pareto improve upon DA-STB
   - Be Pareto dominated by DA-STB
   - Be not Pareto comparable to DA-STB
3. Susceptible to (simple) manipulations
4. Manipulable in large markets

- Cutoffs are useful for analyzing school choice problems