Supplementary Appendix (not for publication) for: The Value of Network Information

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This appendix includes the proof of Proposition 11 from the paper "The Value of Network Information." For completeness, we repeat the statement of the Proposition.

Proposition. *Suppose that $\gamma k^{\max} < 1/2$. Then, for any of the price discrimination schemes considered, there exists a unique finite price schedule* $\mathbf{p}$ *that solves the monopoly's profit maximization problem.*

We now prove the Proposition.

Notation and definitions

Recall that $\gamma > 0$ is the network externality coefficient, that out-degrees are indexed by k and in-degrees by l. Recall further that $P(k)$ and $H(l)$ are the fractions of consumers with out-degree k and in-degree l respectively, and that $\hat{k}(l) = E[k|l]$ and $\hat{l}(k) = E[l|k]$ capture the conditional expected out-degree of a consumer with in-degree l and conditional expected in-degree of a consumer with out-degree k.

Let

$$K_{out} = k^{\max}$$
$$K_{in} = k^{\max}$$
$$K_{in/out} = (k^{\max})^2 + 2k^{\max}$$

where $k^{\max}$ is maximal degree (among all in- and out-degrees of all individuals).

Let

$$\varphi_{out} \colon \{0, \ldots, k^{\max}\} \to \{0, \ldots, K_{out}\}$$
$$\varphi_{in} \colon \{0, \ldots, k^{\max}\} \to \{0, \ldots, K_{in}\}$$
$$\varphi_{in/out} \colon \{0, \ldots, k^{\max}\}^2 \to \{0, \ldots, K_{in/out}\}$$

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be the bijections defined by

$$\varphi_{out}(i) = i$$
$$\varphi_{in}(i) = i$$
$$\varphi_{in/out}(x, y) = (k^{\max} + 1)x + y$$

and let $q(k, l) \equiv H(l)P(k|l)$.

The monopolist's profit Π is given by

$$\begin{aligned}
\Pi_{out} &= \sum_k P(k)p(k)x(k) \\
\Pi_{in} &= \sum_l H(l) \sum_k P(k|l)p(l)x(k, l) \\
\Pi_{in/out} &= \sum_l H(l) \sum_k P(k|l)p(k, l)x(k, l)
\end{aligned}$$

and the (negative of) Hessian matrices are

$$\begin{aligned}
\mathcal{H}_{out} &= \left(\left(-\frac{\partial^2 \Pi_{out}}{\partial p(s) \partial p(t)}\right)\right)_{s,t \in \{0,\ldots,k^{\max}\}} \\
\mathcal{H}_{in} &= \left(\left(-\frac{\partial^2 \Pi_{in}}{\partial p(s) \partial p(t)}\right)\right)_{s,t \in \{0,\ldots,k^{\max}\}} \\
\mathcal{H}_{in/out} &= \left(\left(-\frac{\partial^2 \Pi_{in/out}}{\partial p\left(\varphi_{in/out}^{-1}(s)\right) \partial p\left(\varphi_{in/out}^{-1}(t)\right)}\right)\right)_{s,t \in \{0,\ldots,K_{in/out}\}}
\end{aligned}$$

The corresponding first order condition of the monopoly's profit maximization problem are:

$$\begin{aligned}
0 = \frac{\partial \Pi_{out}}{\partial p(s)} &= P(s)\left[1 - 2p(s) + \frac{\gamma}{1 - \gamma \bar{k}}\left((1 - \bar{p}_{out})s - \sum_k P(k)p(k)k \sum_l \frac{lH(l|s)}{\hat{k}}\right)\right] \\
0 = \frac{\partial \Pi_{in}}{\partial p(s)} &= H(s)\left[1 - 2p(s) + \frac{\gamma}{1 - \gamma \bar{k}}\left((1 - \bar{p}_{in})\hat{k}(s) - \frac{s}{\hat{k}} \sum_l H(l)p(l)\hat{k}(l)\right)\right] \\
0 = \frac{\partial \Pi_{in/out}}{\partial p(x, y)} &= H(y)P(x|y)\left[1 - 2p(x, y) + \frac{\gamma(1 - \bar{p}_{in/out})}{1 - \gamma \bar{k}}x\right] - \sum_l H(l) \sum_k P(k|l) \frac{p(k, l)k\gamma}{1 - \gamma \bar{k}} \frac{\partial \bar{p}_{in/out}}{\partial p(x, y)}
\end{aligned}$$

where

$$\begin{aligned}
\bar{p}_{out} &= \sum_l \bar{H}(l) \sum_k P(k|l)p(k) \\
\bar{p}_{in} &= \frac{1}{\hat{k}} \sum_l H(l)lp(l) \\
\bar{p}_{in/out} &= \sum_l \bar{H}(l) \sum_k P(k|l)p(k, l)
\end{aligned}$$

and

$$\frac{\partial \bar{p}_{out}}{\partial p(s)} = P(s) \sum_{l} \frac{l H(l|s)}{\hat{k}}$$
$$\frac{\partial \bar{p}_{in}}{\partial p(s)} = \frac{1}{\hat{k}} H(s) s$$
$$\frac{\partial \bar{p}_{in/out}}{\partial p(x,y)} = \frac{H(y) P(x|y) y}{\hat{k}}$$

Intermediate results

The proof of the Proposition makes use of the following three lemmas.

Lemma 1. *Let* $(K, \Pi, \varphi) \in \{(K_i, \Pi_i, \varphi_i)\}$, *where* $i \in \{out, in, in/out\}$. *Then there exists* $\Gamma \in \mathbb{R}_+$ *and* $(p_j, b_j, c_j)_{j \in \{0,\dots,K\}} \in (\mathbb{R}^3_+)^{K+1}$ *such that for every* $s, t \in \{0, \dots, K\}$,

$$-\frac{\partial^2 \Pi}{\partial p\left(\varphi^{-1}(s)\right) \partial p\left(\varphi^{-1}(t)\right)} = p_s p_t \Gamma \left[b_t c_s + b_s c_t\right] + 2 p_s \mathbf{1}_{\{s=t\}}.$$

Lemma 2. *Let*

$$\mathcal{G} = \left(\left(p_s p_t \Gamma \left[b_t c_s + b_s c_t\right] + 2 p_s \mathbf{1}_{\{s=t\}}\right)\right)_{s,t \in \{0,\dots,K\}}.$$

Then the determinant of $\mathcal{G}$ *is given by*

$$\det(\mathcal{G}) = \left(2^{K-1} \prod_i p_i\right) \left(4 + 4\Gamma \sum_i \left(b_i c_i p_i\right) - \Gamma^2 \sum_{i<j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right)\right).$$

Lemma 3. *Assume that* $\gamma k^{\max} < 1/2$. *Then* $\mathcal{H}_{out}$, $\mathcal{H}_{in}$, *and* $\mathcal{H}_{in/out}$ *are positive definite.*

Proofs

Proof of Proposition. It suffices to show that the first and second order conditions are satisfied. The first order conditions are verified in the Appendix to the paper. To see that the second order conditions hold, note that for each price discrimination scheme, the second order partial derivatives are of the form given in Lemma 1. A necessary and sufficient condition for maximizing profit is that the Hessian matrix is negative definite, or equivalently that the negative of the Hessian matrix is positive definite. The determinant of this matrix is given by the formula in Lemma 2. By Lemma 3, we conclude positive definiteness and hence that the second order condition holds. □

We now complete the proof by proving the three lemmas above.

Lemma 1

Proof. We compute the second partial derivatives. Recall that $C(s) = \sum_l \frac{lH(l|s)}{\hat{k}} = \frac{\hat{l}(s)}{\hat{k}}$ and let $\tilde{C}(t) = \frac{\hat{l}(t)}{\hat{k}}$. Then, for any $t, s, x, y, z, w \in \{0, \ldots, k^{\max}\}$,

$$\begin{aligned}
\frac{\partial^2 \Pi_{out}}{\partial p(t) \partial p(s)} &= \frac{\partial}{\partial p(t)} \left(P(s) \left[1 - 2p(s) + \frac{\gamma(1-\bar{p})}{1-\gamma\bar{k}} s - \frac{\gamma}{1-\gamma\bar{k}} \sum_k P(k)p(k)k \sum_l \frac{lH(l|s)}{\hat{k}} \right] \right) \\
&= P(s) \left(-2 \cdot \mathbf{1}_{\{s=t\}} - \frac{\gamma s}{1-\gamma\bar{k}} \frac{\partial \bar{p}}{\partial p(t)} - \frac{\gamma}{1-\gamma\bar{k}} \frac{\partial}{\partial p(t)} \sum_k P(k)p(k)k \sum_l \frac{lH(l|s)}{\hat{k}} \right) \\
&= P(s) \left(-2 \cdot \mathbf{1}_{\{s=t\}} - \frac{\gamma s}{1-\gamma\bar{k}} P(t) \sum_l \frac{lH(l|t)}{\hat{k}} - \frac{\gamma}{1-\gamma\bar{k}} P(t)t \sum_l \frac{lH(l|s)}{\hat{k}} \right) \\
&= -2P(s)\mathbf{1}_{\{s=t\}} - P(s)P(t) \frac{\gamma}{1-\gamma\bar{k}} \left(s\sum_l \frac{lH(l|t)}{\hat{k}} + t\sum_l \frac{lH(l|s)}{\hat{k}} \right) \\
&= -2P(s)\mathbf{1}_{\{s=t\}} - P(s)P(t) \frac{\gamma}{1-\gamma\bar{k}} \left(sC(t) + tC(s) \right) \\
\frac{\partial^2 \Pi_{in}}{\partial p(t) \partial p(s)} &= \frac{\partial}{\partial p(t)} \left(H(s) \left[1 - 2p(s) + \frac{\gamma}{1-\gamma\bar{k}} \left((1-\bar{p})\hat{k}(s) - \frac{s}{\hat{k}} \sum_l H(l)p(l)\hat{k}(l) \right) \right] \right) \\
&= H(s) \left[-2 \cdot \mathbf{1}_{\{s=t\}} + \frac{\gamma}{1-\gamma\bar{k}} \left(-\hat{k}(s) \frac{\partial \bar{p}}{\partial p(t)} - \frac{s}{\hat{k}} \frac{\partial}{\partial p(t)} \sum_l H(l)p(l)\hat{k}(l) \right) \right] \\
&= H(s) \left[-2 \cdot \mathbf{1}_{\{s=t\}} + \frac{\gamma}{1-\gamma\bar{k}} \left(-\hat{k}(s) \frac{1}{\hat{k}} H(t)t - \frac{s}{\hat{k}} H(t)\hat{k}(t) \right) \right] \\
&= -H(s) \left[-2 \cdot \mathbf{1}_{\{s=t\}} + \frac{\gamma}{1-\gamma\bar{k}} \left(\frac{\hat{k}(s)}{\hat{k}} H(t)t + sH(t) \frac{\hat{k}(t)}{\hat{k}} \right) \right] \\
&= -2H(s)\mathbf{1}_{\{s=t\}} - H(s)H(t) \frac{\gamma}{1-\gamma\bar{k}} \left[t\tilde{C}(s) + s\tilde{C}(t) \right] \\
\frac{\partial^2 \Pi_{in/out}}{\partial p(z,w) \partial p(x,y)} &= \frac{\partial}{\partial p(z,w)} \left(H(y)P(x|y) \left[1 - 2p(x,y) + \frac{\gamma(1-\bar{p})}{1-\gamma\bar{k}} x \right] - \sum_l H(l) \sum_k P(k|l) \frac{p(k,l)k\gamma}{1-\gamma\bar{k}} \frac{\partial \bar{p}}{\partial p(x,y)} \right) \\
&= q(x,y) \left[-2 \cdot \mathbf{1}_{\{(z,w)=(x,y)\}} - \frac{\gamma}{1-\gamma\bar{k}} x \frac{\partial \bar{p}}{\partial p(z,w)} \right] - \frac{\partial}{\partial p(z,w)} \sum_{l,k} q(k,l) \frac{p(k,l)k\gamma}{1-\gamma\bar{k}} \frac{\partial \bar{p}}{\partial p(x,y)} \\
&= q(x,y) \left[-2 \cdot \mathbf{1}_{\{(z,w)=(x,y)\}} - \frac{\gamma}{1-\gamma\bar{k}} x \frac{q(z,w)w}{\hat{k}} \right] - q(z,w) \frac{z\gamma}{1-\gamma\bar{k}} \frac{q(x,y)y}{\hat{k}} \\
&= -2q(x,y)\mathbf{1}_{\{(z,w)=(x,y)\}} - q(x,y)q(z,w) \left(\frac{\gamma}{1-\gamma\bar{k}} \frac{1}{\hat{k}} xw + \frac{\gamma}{1-\gamma\bar{k}} \frac{1}{\hat{k}} zy \right) \\
&= -2q(x,y)\mathbf{1}_{\{(z,w)=(x,y)\}} - q(x,y)q(z,w) \frac{\gamma}{1-\gamma\bar{k}} \frac{1}{\hat{k}} (xw + zy)
\end{aligned}$$

To conclude: for the case of discrimination on out-degree, we take $p_i = P(i)$, $b_i = i$, $c_i = C(i)$, and $\Gamma = \frac{1}{1-\gamma\bar{k}}$; for discrimination on in-degree, we take $p_i = P(i)$, $b_i = i$, $c_i = \tilde{C}(i)$, and $\Gamma = \frac{1}{1-\gamma\bar{k}}$; and for discrimination on in- and out-degrees, we take $p_i = q(i) = q(i_1, i_2)$, $b_i = i_2$, $c_i = i_1$, and $\Gamma = \frac{\gamma}{1-\gamma\bar{k}} \frac{1}{\hat{k}}$. $\square$

Lemma 2

Proof. Fix $k^{\max} = K > 0$. For $\ell \in \{1, \ldots, K\}$, and for any $J \subset \{0, \ldots, K\}$ with $|J| = \ell$, let $L = \{0, \ldots, \ell - 1\}$ and

$$\mathcal{M}_\ell^J = \left((\mathcal{G}_{st})_{s\in L}\right)_{t\in J}.$$

Denote $S = J \cap L$. We will write $J = \{j_0, j_1, \ldots, j_{\ell-1}\}$ with $j_0 < j_1 < \cdots < j_{\ell-1}$. Also denote $Q = J \setminus L = \{m_i\}_{0\le i<|Q|}$ and $R = L \setminus J = \{n_i\}_{0\le i<|R|}$. Note that $J = S \sqcup Q$ and $L = S \sqcup R$.[1] Also, $|Q| = |R|$ since $|J| = \ell = |L|$. Moreover, note that any element of the set Q exceeds any element of the set S. We claim that

$$\left|\mathcal{M}_\ell^J\right| = \kappa(\ell, J)\psi(\ell, J) \tag{$*$}$$

where

$$\kappa(\ell, J) = 2^{|S|} \prod_{i\in J\cup L} p_i \cdot \begin{cases} 1 & \text{if } |R| = 0 \\ (-1)^{n_0+|S|} & \text{if } |R| = 1 \\ (-1)^{n_0+n_1} & \text{if } |R| = 2 \end{cases}$$

and

$$\psi(\ell, J) = \begin{cases} 1 + \Gamma \sum_i b_i c_i p_i + \frac{1}{4}\Gamma^2 \sum_{i\neq j} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) & \text{if } |Q| = 0 \\ \Gamma\left(b_{n_0} c_{m_0} + b_{m_0} c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r\in S} \left(\left(b_{m_0} c_{n_0} + b_{n_0} c_{m_0}\right) b_r c_r p_r - \left(b_{m_0} b_{n_0} p_r c_r^2 + c_{m_0} c_{n_0} p_r b_r^2\right)\right) & \text{if } |Q| = 1 \\ \Gamma^2\left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) & \text{if } |Q| = 2 \\ 0 & \text{if } |Q| \ge 3 \end{cases}.$$

Denote $\Phi = \prod_{i=0}^{\ell-1} p_i$. We will prove the claim by induction on ℓ. First we will show that the formula is valid for $\ell = 1$. Next we will show that, if the formula is valid for $\ell < K$, then it is also valid for $\ell + 1$.

Base case ($\ell = 1$):

$$\begin{aligned} \det \mathcal{G}_{st} &= p_0 p_t \Gamma\left[b_t c_0 + b_0 c_t\right] + 2p_0 \mathbf{1}_{\{0=t\}} \\ &= \begin{cases} p_0^2 \Gamma\left[2 b_0 c_0\right] + 2p_0 & \text{if } t = 0, \text{ i.e. } |Q| = 0 \\ p_0 p_t \Gamma\left[b_t c_0 + b_0 c_t\right] & \text{if } t \neq 0, \text{ i.e. } |Q| = 1 \end{cases} \\ &= \begin{cases} 2p_0\left(1 + \Gamma\left[b_0 c_0 p_0\right]\right) & \text{if } t = 0, \text{ i.e. } |Q| = 0 \\ p_0 p_t \Gamma\left[b_t c_0 + b_0 c_t\right] & \text{if } t \neq 0, \text{ i.e. } |Q| = 1 \end{cases}. \end{aligned}$$

Now assume ($*$) holds for some $\ell < K$. Let $L' = L \cup \{\ell\} = \{0, \ldots, \ell\}$, and let $J' = J \cup \{j\}$ for some $j \notin J$. Accordingly, let $S' = J' \cap L'$, $Q' = J' \setminus L'$, and $R' = L' \setminus J'$. We need to show that ($*$) holds for $\ell + 1$, i.e., that

$$\left|\mathcal{M}_{\ell+1}^{J'}\right| = \kappa(\ell + 1, J')\psi(\ell + 1, J').$$

[1] For two sets X and Y, let $X \sqcup Y$ be the disjoint union of X and Y.

By cofactor expansion along the last row, we have

$$\left|\mathcal{M}_{\ell+1}^{J'}\right| = \sum_{0\leq i\leq \ell}(-1)^{i+\ell}\left|\mathcal{M}_{\ell}^{J'\setminus\{j_i\}}\right|\mathcal{G}_{\ell j_i},$$

so it suffices to show that

$$\sum_{0\leq i\leq \ell}(-1)^{i+\ell}\left|\mathcal{M}_{\ell}^{J'\setminus\{j_i\}}\right|\mathcal{G}_{\ell j_i} = \kappa(\ell+1, J')\psi(\ell+1, J').$$

We consider the following cases.

Case 0. $|R'| = 0$

In this case $Q' = R' = \emptyset$ and $J' = S' = \{0, \dots, \ell\}$, so

$$\begin{aligned}\left|L \setminus \left(J' \setminus \{\ell\}\right)\right| &= \left|\left(L' \setminus \{\ell\}\right) \setminus \left(J' \setminus \{\ell\}\right)\right| \\ &= \left|L' \setminus J'\right| \\ &= 0\end{aligned}$$

Similarly, for $r \in S' \setminus \{\ell\}$,

$$\begin{aligned}\left|L \setminus \left(J' \setminus \{r\}\right)\right| &= \left|L \cap \left(J' \cap \{r\}^C\right)^C\right| \\ &= \left|L \cap \left(\left(J'\right)^C \cup \{r\}\right)\right| \\ &= \left|\left(L \setminus J'\right) \cup \left(L \cap \{r\}\right)\right| \\ &= \left|\emptyset \cup \{r\}\right| \\ &= 1\end{aligned}$$

and in particular $R = \{r\}$. Thus

$$\begin{aligned}\frac{1}{(-1)^{r+(\ell-1)} \cdot 2^{\ell-1}p_\ell\Phi}\left|\mathcal{M}_{\ell}^{J'\setminus\{r\}}\right|\mathcal{G}_{\ell r} &= \left(\Gamma\left(b_r c_\ell + b_\ell c_r\right) + \frac{1}{2}\Gamma^2\sum_{j\neq \ell, r}\left(\left(b_\ell c_r + b_r c_\ell\right)b_j c_j p_j - \left(b_\ell b_r p_j c_j^2 + c_\ell c_r p_j b_j^2\right)\right)\right) \\ &\quad \cdot \left(p_\ell p_r \Gamma\left[b_\ell c_r + b_r c_\ell\right]\right) \\ \frac{1}{2^\ell \Phi}\left|\mathcal{M}_{\ell}^{J'\setminus\{\ell\}}\right|\mathcal{G}_{\ell\ell} &= \left(1 + \Gamma\sum_{i<\ell}b_i c_i p_i + \frac{1}{4}\Gamma^2\sum_{i\neq j}p_i p_j\left(b_i b_j c_i c_j - b_i^2 c_j^2\right)\right)\left(p_\ell^2\Gamma\left[2b_\ell c_\ell\right] + 2p_\ell\right).\end{aligned}$$

We will show that $\frac{1}{2^{|S|}p_\ell\Phi}\left|\mathcal{M}_{\ell+1}^{J'}\right| = 1 + \Gamma\sum_{i\in\{0,\dots,\ell\}}b_i c_i p_i + \frac{1}{4}\Gamma^2\sum_{i\neq j}p_i p_j\left(b_i b_j c_i c_j - b_i^2 c_j^2\right)$. Since $|S| = \ell+1$,

$$\frac{1}{2^{|S|}p_\ell\Phi}\left|\mathcal{M}_{\ell+1}^{J'}\right| = \sum_{r<\ell}(-1)^{r+(\ell-1)}(-1)^{r+\ell}\frac{1}{4}\left(\Gamma\left(b_r c_\ell + b_\ell c_r\right) + \frac{1}{2}\Gamma^2\sum_{j\neq \ell, r}\left(\left(b_\ell c_r + b_r c_\ell\right)b_j c_j p_j - \left(b_\ell b_r p_j c_j^2 + c_\ell c_r p_j b_j^2\right)\right)\right)$$

$$
\begin{aligned}
&\cdot \left(p_\ell p_r \Gamma \left[b_\ell c_r + b_r c_\ell\right]\right) \\
&+ (-1)^{\ell+\ell} \left(1 + \Gamma \sum_{i<\ell} b_i c_i p_i + \frac{1}{4}\Gamma^2 \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right)\right) \left(p_\ell \Gamma \left[b_\ell c_\ell\right] + 1\right) \\
= {}& 1 + A\Gamma + B\Gamma^2 + C\Gamma^3.
\end{aligned}
$$

It suffices to show $A = \sum_{i \in \{0,\ldots,\ell\}} b_i c_i p_i$, $B = \frac{1}{4} \sum_{i \neq j} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right)$, and $C = 0$. Clearly we have $A = p_\ell b_\ell c_\ell + \sum_{i<\ell} b_i c_i p_i$. Next note that

$$
\begin{aligned}
B &= \frac{1}{4} \sum_{r<\ell} -p_\ell p_r \left[b_\ell c_r + b_r c_\ell\right] \left(b_r c_\ell + b_\ell c_r\right) \\
&\quad + p_\ell \left[b_\ell c_\ell\right] \sum_{i<\ell} b_i c_i p_i \\
&\quad + \frac{1}{4} \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \\
&= \frac{1}{4} \sum_{i<\ell} \left(-p_\ell p_i \left[b_\ell c_i + b_i c_\ell\right] \left(b_i c_\ell + b_\ell c_i\right) + 4 b_i c_i p_i b_\ell c_\ell p_\ell\right) \\
&\quad + \frac{1}{4} \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \\
&= \frac{1}{4} \sum_{i<\ell} \left(-p_\ell p_i \left[b_\ell^2 c_i^2 + 2 b_i c_i b_\ell c_\ell + b_i^2 c_\ell^2\right] + 4 b_i c_i p_i b_\ell c_\ell p_\ell\right) \\
&\quad + \frac{1}{4} \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \\
&= \frac{1}{4} \sum_{i<\ell} \left(p_\ell p_i \left[-b_\ell^2 c_i^2 + 2 b_i c_i b_\ell c_\ell - b_i^2 c_\ell^2\right]\right) + \frac{1}{4} \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \\
&= \frac{1}{4} \sum_{i<\ell} p_i p_\ell \left(b_i b_\ell c_i c_\ell - b_i^2 c_\ell^2\right) + \frac{1}{4} \sum_{j<\ell} p_\ell p_j \left(b_\ell b_j c_\ell c_j - b_\ell^2 c_j^2\right) + \frac{1}{4} \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \\
&= \frac{1}{4} \sum_{i \neq j} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right).
\end{aligned}
$$

To see that $C = 0$, note that

$$
\begin{aligned}
\frac{4}{p_\ell} C &= -\frac{1}{2} \sum_{r<\ell} \sum_{j \neq \ell, r} \left(\left(b_\ell c_r + b_r c_\ell\right) b_j c_j p_j - \left(b_\ell b_r p_j c_j^2 + c_\ell c_r p_j b_j^2\right)\right) \left(p_r \left[b_\ell c_r + b_r c_\ell\right]\right) \\
&\quad + \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \left[b_\ell c_\ell\right] \\
&= \sum_{\substack{i \neq j \\ i,j<\ell}} -\frac{1}{2} \left(\left(b_\ell c_i + b_i c_\ell\right) b_j c_j p_j - \left(b_\ell b_i p_j c_j^2 + c_\ell c_i p_j b_j^2\right)\right) \left(p_i \left[b_\ell c_i + b_i c_\ell\right]\right) + p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right) \left[b_\ell c_\ell\right]
\end{aligned}
$$

$$= \sum_{\substack{i \neq j \\ i,j<\ell}} p_i p_j \left[-\frac{1}{2} \left((b_\ell c_i + b_i c_\ell)^2 b_j c_j - \left(b_\ell b_i c_j^2 + c_\ell c_i b_j^2 \right) [b_\ell c_i + b_i c_\ell] \right) + \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) [b_\ell c_\ell] \right]$$

Note that for every i, j, the coefficient of $p_i p_j$ is zero:

$$\begin{aligned}
& - \left[\frac{1}{2} \left((b_\ell c_i + b_i c_\ell)^2 b_j c_j - \left(b_\ell b_i c_j^2 + c_\ell c_i b_j^2 \right) [b_\ell c_i + b_i c_\ell] \right) \right] + \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) [b_\ell c_\ell] \\
& \quad - \left[\frac{1}{2} \left((b_\ell c_j + b_j c_\ell)^2 b_i c_i - \left(b_\ell b_j c_i^2 + c_\ell c_j b_i^2 \right) [b_\ell c_j + b_j c_\ell] \right) \right] + \left(b_i b_j c_i c_j - b_j^2 c_i^2 \right) [b_\ell c_\ell] \\
& = -b_\ell^2 \left[\frac{1}{2} \left(c_i^2 b_j c_j - b_i c_j^2 c_i \right) + \frac{1}{2} \left(c_j^2 b_i c_i - b_j c_i^2 c_j \right) \right] \\
& \quad - c_\ell^2 \left[\frac{1}{2} \left(b_i^2 b_j c_j - c_i b_j^2 b_i \right) + \frac{1}{2} \left(b_j^2 b_i c_i - c_j b_i^2 b_j \right) \right] \\
& \quad - b_\ell c_\ell \left[\frac{1}{2} \left(2 c_i b_i b_j c_j - b_i^2 c_j^2 - c_i^2 b_j^2 \right) - b_i^2 c_j^2 + \frac{1}{2} \left(2 c_j b_j b_i c_i - c_i^2 b_j^2 - b_i^2 c_j^2 \right) - b_j^2 c_i^2 \right] \\
& \quad + b_\ell c_\ell [2 b_i b_j c_i c_j] \\
& = 0.
\end{aligned}$$

Case 1. $|R'| = 1$

Case 1a. $\ell \in S'$

In this case, $|R'| = |L' \setminus J'| = 1$. Note that

$$\begin{aligned}
\left| \mathcal{M}_{\ell+1}^{J'} \right| &= \sum_{j_i \in J'} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} \\
&= \sum_{j_i \in S \cup Q} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} \\
&= (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{\ell\}} \right| \mathcal{G}_{\ell\ell} + \sum_{j_i \in S \setminus \{\ell\}} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} + \sum_{j_i \in Q} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i}.
\end{aligned}$$

First, note analogously to Case 0 that

$$\begin{aligned}
\left| L \setminus \left(J' \setminus \{\ell\} \right) \right| &= \left| L' \setminus J' \right| \\
&= 1.
\end{aligned}$$

Next consider $r \in S' \setminus \{\ell\}$. Note that $r \in J'$ and hence $r \notin (J')^C$. Since $\ell \notin R'$, we have $L' \setminus J' = R' = R' \setminus \{\ell\} = L \setminus J'$, so

$$\begin{aligned}
\left| L \setminus \left(J' \setminus \{r\} \right) \right| &= \left| L \cap \left(J' \cap \{r\}^C \right)^C \right| \\
&= \left| L \cap \left(\left(J' \right)^C \sqcup \{r\} \right) \right| \\
&= \left| \left(L \setminus J' \right) \sqcup \left(L \cap \{r\} \right) \right| \\
&= \left| \left(L \setminus J' \right) \sqcup \{r\} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\left(L' \setminus J'\right) \sqcup \{r\}\right| \\
&= 1 + 1 = 2.
\end{aligned}
$$

Now consider $j \in Q'$. Note that $j \notin \{0, \ldots, \ell\}$. As before $R' = R' \setminus \{\ell\}$, so

$$
\begin{aligned}
\left|L \setminus \left(J' \setminus \{j\}\right)\right| &= \left|L \cap \left(J' \cap \{j\}^C\right)^C\right| \\
&= \left|L \cap \left(\left(J'\right)^C \cup \{j\}\right)\right| \\
&= \left|\left(L' \cap \{\ell\}^C \cap \left(J'\right)^C\right) \cup \left(L \cap \{j\}\right)\right| \\
&= \left|\left(L' \cap \{\ell\}^C \cap \left(J'\right)^C\right) \cup \emptyset\right| \\
&= \left|L \setminus J'\right| \\
&= \left|\left(L' \setminus \{\ell\}\right) \setminus J'\right| \\
&= \left|\left(L' \setminus J'\right) \setminus \{\ell\}\right| \\
&= \left|L' \setminus J'\right| \\
&= 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{(-1)^{n_0+r} 2^{\ell-2} p_{m_0} p_\ell \Phi} \left|\mathcal{M}_\ell^{J' \setminus \{r\}}\right| \mathcal{G}_{\ell r} \\
&= \Gamma^2 \left(b_{n_0} b_{m_0} c_r c_\ell - b_{n_0} b_\ell c_r c_{m_0} - b_r b_{m_0} c_{n_0} c_\ell + b_r b_\ell c_{n_0} c_{m_0}\right) \left(p_\ell p_r \Gamma \left[b_\ell c_r + b_r c_\ell\right]\right) \\
&\frac{1}{(-1)^{n_0+(\ell-1)} 2^{\ell-1} p_{m_0} \Phi} \left|\mathcal{M}_\ell^{J' \setminus \{\ell\}}\right| \mathcal{G}_{\ell\ell} \\
&= \left[\Gamma\left(b_{n_0} c_{m_0} + b_{m_0} c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r \in S' \setminus \{\ell\}} \left(\left(b_{m_0} c_{n_0} + b_{n_0} c_{m_0}\right) b_r c_r p_r - \left(b_{m_0} b_{n_0} p_r c_r^2 + c_{m_0} c_{n_0} p_r b_r^2\right)\right)\right] \left(p_\ell^2 \Gamma \left[2 b_\ell c_\ell\right] + 2 p_\ell\right) \\
&\frac{1}{(-1)^{n_0+(\ell-1)} 2^{\ell-1} p_\ell \Phi} \left|\mathcal{M}_\ell^{J' \setminus \{m_0\}}\right| \mathcal{G}_{\ell m_0} \\
&= \left[\Gamma\left(b_{n_0} c_\ell + b_\ell c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r \in S' \setminus \{\ell\}} \left(\left(b_\ell c_{n_0} + b_{n_0} c_\ell\right) b_r c_r p_r - \left(b_\ell b_{n_0} p_r c_r^2 + c_\ell c_{n_0} p_r b_r^2\right)\right)\right] \left(p_\ell p_{m_0} \Gamma \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right)
\end{aligned}
$$

We will show that

$$
\begin{aligned}
&\frac{1}{2^{|S'|} p_{m_0} p_\ell \Phi} \left|\mathcal{M}_{\ell+1}^{J'}\right| \\
&= \Gamma\left(b_{n_0} c_{m_0} + b_{m_0} c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r \in S'} \left(\left(b_{m_0} c_{n_0} + b_{n_0} c_{m_0}\right) b_r c_r p_r - \left(b_{m_0} b_{n_0} p_r c_r^2 + c_{m_0} c_{n_0} p_r b_r^2\right)\right).
\end{aligned}
$$

First note that

$$
\frac{1}{2^{|S'|} p_{m_0} p_\ell \Phi} \left|\mathcal{M}_{\ell+1}^{J'}\right|
$$

$$
\begin{aligned}
&= \frac{1}{2^{|S'\setminus\{\ell\}|}p_{m_0}p_\ell\Phi}\Bigg(\sum_{j_i\in S'\setminus\{\ell\}}(-1)^{i+\ell}(-1)^{n_0+i}\left|\mathcal{M}_\ell^{J'\setminus\{j_i\}}\right|\mathcal{G}'_{\ell j_i} \\
&\quad + (-1)^{(\ell-1)+\ell}(-1)^{n_0+(\ell-1)}\left|\mathcal{M}_\ell^{J'\setminus\{\ell\}}\right|\mathcal{G}'_{\ell\ell} + (-1)^{\ell+\ell}(-1)^{n_0+(\ell-1)}\left|\mathcal{M}_\ell^{J'\setminus\{m_0\}}\right|\mathcal{G}'_{\ell m_0}\Bigg) \\
&= \frac{(-1)^{n_0+\ell}}{2^{|S'\setminus\{\ell\}|}p_{m_0}p_\ell\Phi}\left(\sum_{j_i\in S'\setminus\{\ell\}}\left|\mathcal{M}_\ell^{J'\setminus\{j_i\}}\right|\mathcal{G}'_{\ell j_i} + \left|\mathcal{M}_\ell^{J'\setminus\{\ell\}}\right|\mathcal{G}'_{\ell\ell} - \left|\mathcal{M}_\ell^{J'\setminus\{m_0\}}\right|\mathcal{G}'_{\ell m_0}\right) \\
&= (-1)^{n_0+\ell}\left(A\Gamma + B\Gamma^2 + C\Gamma^3\right).
\end{aligned}
$$

It suffices to show that $A = (b_{n_0}c_{m_0} + b_{m_0}c_{n_0})$, $B = \frac{1}{2}\sum_{r\in S'}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right)$, and $C = 0$.

First, it is easy to see that

$$
\left(2^{|S'|}p_{m_0}p_\ell\Phi\right)A = \left(2^{|S'|-1}p_{m_0}\Phi\right)2p_\ell\left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right),
$$

and hence $A = (b_{n_0}c_{m_0} + b_{m_0}c_{n_0})$. Next, note that

$$
\begin{aligned}
\left(2^{|S'|}p_{m_0}p_\ell\Phi\right)B &= +\left(2^{|S'|-1}p_{m_0}\Phi\right)2p_\ell\frac{1}{2}\sum_{r\in S'\setminus\{\ell\}}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right) \\
&\quad + \left(2^{|S'|-1}p_{m_0}\Phi\right)p_\ell^2\left[2b_\ell c_\ell\right]\left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right) \\
&\quad - \left(2^{|S'|-1}p_\ell\Phi\right)p_\ell p_{m_0}\left[b_\ell c_{m_0} + b_{m_0}c_\ell\right]\left(b_{n_0}c_\ell + b_\ell c_{n_0}\right) \\
B &= +\frac{1}{2}\sum_{r\in S'\setminus\{\ell\}}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right) \\
&\quad + 2^{-1}p_\ell\left[2b_\ell c_\ell\right]\left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right) \\
&\quad - 2^{-1}p_\ell\left[b_\ell c_{m_0} + b_{m_0}c_\ell\right]\left(b_{n_0}c_\ell + b_\ell c_{n_0}\right) \\
&= +\frac{1}{2}\sum_{r\in S'\setminus\{\ell\}}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right) \\
&\quad + p_\ell\left(b_{n_0}c_{m_0}b_\ell c_\ell + b_{m_0}c_{n_0}b_\ell c_\ell\right) \\
&\quad - \frac{1}{2}p_\ell\left[b_\ell c_{m_0}b_{n_0}c_\ell + b_\ell c_{m_0}b_\ell c_{n_0} + b_{m_0}c_\ell b_{n_0}c_\ell + b_{m_0}c_\ell b_\ell c_{n_0}\right] \\
&= +\frac{1}{2}\sum_{r\in S'\setminus\{\ell\}}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right) \\
&\quad + \frac{1}{2}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_\ell c_\ell p_\ell - \left(b_{m_0}b_{n_0}p_\ell c_\ell^2 + c_{m_0}c_{n_0}p_\ell b_\ell^2\right)\right) \\
&= \frac{1}{2}\sum_{r\in S'}\left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right)b_rc_rp_r - \left(b_{m_0}b_{n_0}p_rc_r^2 + c_{m_0}c_{n_0}p_rb_r^2\right)\right)
\end{aligned}
$$

To see that $C = 0$, note that

$$
2C = \sum_{r\in S'\setminus\{\ell\}}\frac{1}{2}\left(b_{n_0}b_{m_0}c_rc_\ell - b_{n_0}b_\ell c_rc_{m_0} - b_rb_{m_0}c_{n_0}c_\ell + b_rb_\ell c_{n_0}c_{m_0}\right)\left(p_\ell p_r\left[b_\ell c_r + b_rc_\ell\right]\right)
$$

$$- (-1)^{|S'\setminus\{\ell\}|} p_\ell b_\ell c_\ell \sum_{r\in S'\setminus\{\ell\}} \left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right) b_r c_r p_r - \left(b_{m_0}b_{n_0}p_r c_r^2 + c_{m_0}c_{n_0}p_r b_r^2\right)\right)$$
$$+ (-1)^{|S'\setminus\{\ell\}|} p_\ell \left[b_\ell c_{m_0} + b_{m_0}c_\ell\right] \frac{1}{2} \sum_{r\in S'\setminus\{\ell\}} \left(\left(b_\ell c_{n_0} + b_{n_0}c_\ell\right) b_r c_r p_r - \left(b_\ell b_{n_0}p_r c_r^2 + c_\ell c_{n_0}p_r b_r^2\right)\right)$$

Note that each term in the sum is zero: for each r,

$$\begin{aligned}
&\frac{1}{2}\left(b_{n_0}b_{m_0}c_r c_\ell - b_{n_0}b_\ell c_r c_{m_0} - b_r b_{m_0}c_{n_0}c_\ell + b_r b_\ell c_{n_0}c_{m_0}\right)\left[b_\ell c_r + b_r c_\ell\right] \\
&\quad + b_\ell c_\ell \left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right) b_r c_r - \left(b_{m_0}b_{n_0}c_r^2 + c_{m_0}c_{n_0}b_r^2\right)\right) \\
&\quad - \left[b_\ell c_{m_0} + b_{m_0}c_\ell\right]\frac{1}{2}\left(\left(b_\ell c_{n_0} + b_{n_0}c_\ell\right) b_r c_r - \left(b_\ell b_{n_0}c_r^2 + c_\ell c_{n_0}b_r^2\right)\right) \\
&= b_r^2\left(-\frac{1}{2}\left(b_{m_0}c_{n_0}c_\ell - b_\ell c_{n_0}c_{m_0}\right)c_\ell - b_\ell c_\ell c_{m_0}c_{n_0} + \frac{1}{2}\left(b_\ell c_{m_0} + b_{m_0}c_\ell\right)c_\ell c_{n_0}\right) \\
&\quad + c_r^2\left(\frac{1}{2}\left(b_{n_0}b_{m_0}c_\ell - b_{n_0}b_\ell c_{m_0}\right)b_\ell - b_\ell c_\ell b_{m_0}b_{n_0} + \frac{1}{2}\left(b_\ell c_{m_0} + b_{m_0}c_\ell\right)b_\ell b_{n_0}\right) \\
&\quad + b_r c_r\left(\frac{1}{2}\left(c_\ell b_{n_0}b_{m_0}c_\ell - b_{n_0}b_\ell c_{m_0}c_\ell - b_{m_0}c_{n_0}c_\ell b_\ell + b_\ell c_{n_0}c_{m_0}b_\ell\right)\right) \\
&\quad + b_r c_r\left(b_{m_0}c_{n_0}b_\ell c_\ell + b_{n_0}c_{m_0}b_\ell c_\ell\right) \\
&\quad - b_r c_r\frac{1}{2}\left[b_\ell c_{m_0}b_\ell c_{n_0} + b_\ell c_{m_0}b_{n_0}c_\ell + b_{m_0}c_\ell b_\ell c_{n_0} + b_{m_0}c_\ell b_{n_0}c_\ell\right] \\
&= 0.
\end{aligned}$$

Case 1b. $\ell \in R'$

First consider $j \in S'$. Since $\ell \in R' = L' \setminus J'$, we have $\ell \notin J'$. In this case, $1 = |R'| = |Q'| = |J' \setminus L'|$. Since $j \in S'$, we have $j \in L$, so

$$\begin{aligned}
\left|\left(J'\setminus\{j\}\right)\setminus L\right| &= \left|J'\setminus\left(\{j\}\cup\left(L'\setminus\{\ell\}\right)\right)\right| \\
&= \left|J'\setminus\left(L'\setminus\{\ell\}\right)\right| \\
&= \left|\left(J'\setminus\{\ell\}\right)\setminus\left(L'\setminus\{\ell\}\right)\right| \\
&= \left|J'\setminus L'\right| \\
&= 1
\end{aligned}$$

Note that $R' = \{\ell\}$, and hence $S' = \{0,\ldots,\ell-1\}$. This implies that for $j \in S'$, so

$$\frac{1}{(-1)^{j+(\ell-1)}2^{\ell-1}p_{m_0}\Phi}\left|\mathcal{M}_\ell^{J'\setminus\{j\}}\right| = \Gamma\left(b_j c_{m_0} + b_{m_0}c_j\right) + \frac{1}{2}\Gamma^2 \sum_{r\in S'\setminus\{j\}} \left(\left(b_{m_0}c_j + b_j c_{m_0}\right) b_r c_r p_r - \left(b_{m_0}b_j p_r c_r^2 + c_{m_0}c_j p_r b_r^2\right)\right).$$

Now let $j \in Q'$. Recall that in this case, $|L' \setminus J'| = 1$. Using the facts that $j \notin L$ and $\ell \notin J'$, we have

$$\begin{aligned}
\left|L\setminus\left(J'\setminus\{j\}\right)\right| &= \left|L\cap\left(\left(J'\right)^C\cup\{j\}\right)\right| \\
&= \left|\left(L\cap\left(J'\right)^C\right)\cup\left(L\cap\{j\}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\left(L \cap \left(J'\right)^C\right) \cup \emptyset\right| \\
&= \left|L \setminus J'\right| \\
&= \left|\left(L' \setminus \{\ell\}\right) \setminus J'\right| \\
&= \left|\left(L' \setminus J'\right) \setminus \{\ell\}\right| \\
&= \left|R'\right| - 1 \\
&= 1 - 1 = 0.
\end{aligned}
$$

Then for $j \in Q$, we have

$$
\frac{1}{2^\ell \Phi}\left|\mathcal{M}_\ell^{J'\setminus\{j\}}\right| = 1 + \Gamma \sum_{i \in L} b_i c_i p_i + \frac{1}{4}\Gamma^2 \sum_{\substack{i,j \in L \\ i \neq j}} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right).
$$

Thus

$$
\begin{aligned}
&\frac{1}{2^{|S'|} p_{m_0} p_\ell \Phi}\left|\mathcal{M}_\ell^{J'}\right| \\
&= \frac{1}{2^{|S'|} p_{m_0} p_\ell \Phi}\left(\sum_{j_i \in S'} (-1)^{i+\ell}(-1)^{i+(\ell-1)}\left|\mathcal{M}_\ell^{J'\setminus\{j_i\}}\right| \mathcal{G}'_{\ell j_i} + (-1)^{\ell+\ell}\left|\mathcal{M}_\ell^{J'\setminus\{m_0\}}\right|\right) \\
&= -\frac{1}{2}\sum_{j_i \in S'}\left(\Gamma\left(b_{j_i} c_{m_0} + b_{m_0} c_{j_i}\right) + \frac{1}{2}\Gamma^2 \sum_{r \in S'\setminus\{j_i\}}\left(\left(b_{m_0} c_{j_i} + b_{j_i} c_{m_0}\right) b_r c_r p_r - \left(b_{m_0} b_{j_i} p_r c_r^2 + c_{m_0} c_{j_i} p_r b_r^2\right)\right)\right) \\
&\quad \cdot \left(p_{j_i} \Gamma\left[b_\ell c_{j_i} + b_{j_i} c_\ell\right]\right) \\
&\quad + \left(1 + \Gamma \sum_i b_i c_i p_i + \frac{1}{4}\Gamma^2 \sum_{i \neq j} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2\right)\right)\left(\Gamma\left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right) \\
&= (-1)^{\ell+\ell}\left(A\Gamma + B\Gamma^2 + C\Gamma^3\right)
\end{aligned}
$$

It suffices to show that $A = (b_\ell c_{m_0} + b_{m_0} c_\ell)$, $B = \frac{1}{2}\sum_{r \in S'}\left(\left(b_{m_0} c_\ell + b_\ell c_{m_0}\right) b_r c_r p_r - \left(b_{m_0} b_\ell p_r c_r^2 + c_{m_0} c_\ell p_r b_r^2\right)\right)$, and $C = 0$.

First, it is easy to see that $A = b_\ell c_{m_0} + b_{m_0} c_\ell$. Next, note that

$$
\begin{aligned}
B &= -\frac{1}{2}\sum_{j_i \in S'}\left(b_{j_i} c_{m_0} + b_{m_0} c_{j_i}\right) p_{j_i}\left[b_\ell c_{j_i} + b_{j_i} c_\ell\right] + \sum_i b_i c_i p_i \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right] \\
&= \sum_{i < \ell}\left(-\frac{1}{2}\left(b_i c_{m_0} + b_{m_0} c_i\right) p_i \left[b_\ell c_i + b_i c_\ell\right] + b_i c_i p_i \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right) \\
&= \sum_{i < \ell} p_i \left(-\frac{1}{2}\left(b_i c_{m_0} b_\ell c_i + b_i c_{m_0} b_i c_\ell + b_{m_0} c_i b_\ell c_i + b_{m_0} c_i b_i c_\ell\right) + b_i c_i b_\ell c_{m_0} + b_i c_i b_{m_0} c_\ell\right) \\
&= \frac{1}{2}\sum_{i < \ell} p_i \left(b_i c_{m_0} b_\ell c_i + b_{m_0} c_i b_i c_\ell - b_i c_{m_0} b_i c_\ell - b_{m_0} c_i b_\ell c_i\right) \\
&= \frac{1}{2}\sum_{i < \ell} p_i \left(c_{m_0} b_\ell b_i c_i + b_{m_0} c_\ell b_i c_i - c_{m_0} c_\ell b_i^2 - b_{m_0} b_\ell c_i^2\right)
\end{aligned}
$$

$$= \frac{1}{2} \sum_{i<\ell} \left(\left(b_{m_0} c_\ell + b_\ell c_{m_0} \right) b_i c_i p_i - \left(b_{m_0} b_\ell p_i c_i^2 + c_{m_0} c_\ell p_i b_i^2 \right) \right),$$

from which the claim follows since $S' = \{0, \ldots, \ell - 1\}$. To see that $C = 0$, note that

$$\begin{aligned}
&4C \\
&= - \sum_{j_i \in S'} \sum_{r \in S' \setminus \{j_i\}} \left(\left(b_{m_0} c_{j_i} + b_{j_i} c_{m_0} \right) b_r c_r p_r - \left(b_{m_0} b_{j_i} p_r c_r^2 + c_{m_0} c_{j_i} p_r b_r^2 \right) \right) \left(p_{j_i} \left[b_\ell c_{j_i} + b_{j_i} c_\ell \right] \right) \\
&\quad + \sum_{i \neq j} p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \left[b_\ell c_{m_0} + b_{m_0} c_\ell \right] \\
&= \sum_{i \neq j} \left[- \left(\left(b_{m_0} c_i + b_i c_{m_0} \right) b_j c_j p_j - \left(b_{m_0} b_i p_j c_j^2 + c_{m_0} c_i p_j b_j^2 \right) \right) \left(p_i \left[b_\ell c_i + b_i c_\ell \right] \right) + p_i p_j \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \left[b_\ell c_{m_0} + b_{m_0} c_\ell \right] \right] \\
&= \sum_{i \neq j} p_i p_j \left[- \left(\left(b_{m_0} c_i + b_i c_{m_0} \right) b_j c_j - \left(b_{m_0} b_i p_j c_j^2 + c_{m_0} c_i p_j b_j^2 \right) \right) \left[b_\ell c_i + b_i c_\ell \right] + \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \left[b_\ell c_{m_0} + b_{m_0} c_\ell \right] \right]
\end{aligned}$$

We will show that, for each i, j with $i \neq j$, the coefficient of $p_i p_j$ is zero:

$$\begin{aligned}
&- \left(\left(b_{m_0} c_i + b_i c_{m_0} \right) b_j c_j - \left(b_{m_0} b_i c_j^2 + c_{m_0} c_i b_j^2 \right) \right) \left[b_\ell c_i + b_i c_\ell \right] \\
&\quad + \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \left[b_\ell c_{m_0} + b_{m_0} c_\ell \right] \\
&\quad - \left(\left(b_{m_0} c_j + b_j c_{m_0} \right) b_i c_i - \left(b_{m_0} b_j c_i^2 + c_{m_0} c_j b_i^2 \right) \right) \left[b_\ell c_j + b_j c_\ell \right] \\
&\quad + \left(b_i b_j c_i c_j - b_j^2 c_i^2 \right) \left[b_\ell c_{m_0} + b_{m_0} c_\ell \right] \\
&= b_\ell \left[-c_i \left(\left(b_{m_0} c_i + b_i c_{m_0} \right) b_j c_j - \left(b_{m_0} b_i c_j^2 + c_{m_0} c_i b_j^2 \right) \right) + c_{m_0} \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \right] \\
&\quad + c_\ell \left[-b_i \left(\left(b_{m_0} c_i + b_i c_{m_0} \right) b_j c_j - \left(b_{m_0} b_i c_j^2 + c_{m_0} c_i b_j^2 \right) \right) + b_{m_0} \left(b_i b_j c_i c_j - b_i^2 c_j^2 \right) \right] \\
&\quad + b_\ell \left[-c_j \left(\left(b_{m_0} c_j + b_j c_{m_0} \right) b_i c_i - \left(b_{m_0} b_j c_i^2 + c_{m_0} c_j b_i^2 \right) \right) + c_{m_0} \left(b_i b_j c_i c_j - b_j^2 c_i^2 \right) \right] \\
&\quad + c_\ell \left[-b_j \left(\left(b_{m_0} c_j + b_j c_{m_0} \right) b_i c_i - \left(b_{m_0} b_j c_i^2 + c_{m_0} c_j b_i^2 \right) \right) + b_{m_0} \left(b_i b_j c_i c_j - b_j^2 c_i^2 \right) \right] \\
&= 0.
\end{aligned}$$

Hence $C = 0$.

Case 2. $|R'| = 2$

Case 2a. $\ell \in S'$

Recall that

$$\begin{aligned}
\left| \mathcal{M}_{\ell+1}^{J'} \right| &= \sum_{j_i \in J'} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} \\
&= \sum_{j_i \in S' \cup Q'} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} \\
&= (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{\ell\}} \right| \mathcal{G}_{\ell \ell} + \sum_{j_i \in S' \setminus \{\ell\}} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i} + \sum_{j_i \in Q'} (-1)^i \left| \mathcal{M}_\ell^{J' \setminus \{j_i\}} \right| \mathcal{G}_{\ell j_i}.
\end{aligned}$$

Consider $r \in S' \setminus \{\ell\}$. Note as in Case 1a that

$$\begin{aligned}\left|L \setminus \left(J' \setminus \{r\}\right)\right| &= \left|\left(L' \setminus J'\right) \sqcup \{r\}\right| \\ &= 2 + 1 = 3.\end{aligned}$$

This implies $\left|\mathcal{M}_{\ell}^{J' \setminus \{r\}}\right| = 0$ for $r \in S' \setminus \{\ell\}$, so

$$\left|\mathcal{M}_{\ell+1}^{J'}\right| = (-1)^i \left|\mathcal{M}_{\ell}^{J' \setminus \{\ell\}}\right| \mathcal{G}_{\ell\ell} + \sum_{j_i \in Q'} (-1)^i \left|\mathcal{M}_{\ell}^{J' \setminus \{j_i\}}\right| \mathcal{G}_{\ell j_i}$$

Consider $j \in Q' \cup \{\ell\}$. Note also as in Case 1a that

$$\begin{aligned}\left|L \setminus \left(J' \setminus \{j\}\right)\right| &= \left|L' \setminus J'\right| \\ &= 2.\end{aligned}$$

Denote $Q' = \{m_0, m_1\}$ and $R' \setminus \{\ell\} = R' = \{n_0, n_1\}$. By $(*)$,

$$\begin{aligned}&\frac{1}{(-1)^{n_0+n_1} 2^{|S'|-1}} \frac{1}{\Phi} \frac{1}{\prod_{r \in J' \setminus S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_{\ell}^{J' \setminus \{m_1\}}\right| \mathcal{G}_{\ell m_1} \\ &= p_\ell p_{m_0} \left(b_{n_0} b_\ell c_{n_1} c_{m_0} - b_{n_0} b_{m_0} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_0} + b_{n_1} b_{m_0} c_{n_0} c_\ell\right) \left(p_\ell p_{m_1} \Gamma \left[b_\ell c_{m_1} + b_{m_1} c_\ell\right]\right)\end{aligned}$$

$$\begin{aligned}&\frac{1}{(-1)^{n_0+n_1} 2^{|S'|-1}} \frac{1}{\Phi} \frac{1}{\prod_{r \in J' \setminus S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_{\ell}^{J' \setminus \{m_0\}}\right| \mathcal{G}_{\ell m_0} \\ &= p_\ell p_{m_1} \left(b_{n_0} b_\ell c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_\ell\right) \left(p_\ell p_{m_0} \Gamma \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right)\end{aligned}$$

$$\begin{aligned}&\frac{1}{(-1)^{n_0+n_1} 2^{|S'|-1}} \frac{1}{\Phi} \frac{1}{\prod_{r \in J' \setminus S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_{\ell}^{J' \setminus \{\ell\}}\right| \mathcal{G}_{\ell\ell} \\ &= p_{m_0} p_{m_1} \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell^2 \Gamma \left[2 b_\ell c_\ell\right] + 2 p_\ell\right)\end{aligned}$$

The sum of these three terms reduces to

$$\begin{aligned}&\frac{1}{(-1)^{n_0+n_1} 2^{|S'|-1}} \frac{1}{\Phi} \frac{1}{\prod_{r \in J' \setminus S'} p_r} \frac{1}{\Gamma^2} \sum_{j_i \in Q' \cup \{\ell\}} (-1)^{i+\ell} \left|\mathcal{M}_{\ell}^{J' \setminus \{j_i\}}\right| \mathcal{G}_{\ell j_i} \\ &= (-1)^{(\ell-2)+\ell} p_\ell p_{m_0} \left(b_{n_0} b_\ell c_{n_1} c_{m_0} - b_{n_0} b_{m_0} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_0} + b_{n_1} b_{m_0} c_{n_0} c_\ell\right) \left(p_\ell p_{m_1} \Gamma \left[b_\ell c_{m_1} + b_{m_1} c_\ell\right]\right) \\ &\quad + (-1)^{(\ell-1)+\ell} p_\ell p_{m_1} \left(b_{n_0} b_\ell c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_\ell\right) \left(p_\ell p_{m_0} \Gamma \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right) \\ &\quad + (-1)^{\ell+\ell} p_{m_0} p_{m_1} \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell^2 \Gamma \left[2 b_\ell c_\ell\right] + 2 p_\ell\right) \\ &= p_\ell p_{m_0} \left(b_{n_0} b_\ell c_{n_1} c_{m_0} - b_{n_0} b_{m_0} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_0} + b_{n_1} b_{m_0} c_{n_0} c_\ell\right) \left(p_\ell p_{m_1} \Gamma \left[b_\ell c_{m_1} + b_{m_1} c_\ell\right]\right) \\ &\quad - p_\ell p_{m_1} \left(b_{n_0} b_\ell c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_\ell\right) \left(p_\ell p_{m_0} \Gamma \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right) \\ &\quad + p_{m_0} p_{m_1} \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell^2 \Gamma \left[2 b_\ell c_\ell\right] + 2 p_\ell\right) \\ &= b_\ell c_{m_1} \left(b_{n_0} b_\ell c_{n_1} c_{m_0} - b_{n_0} b_{m_0} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_0} + b_{n_1} b_{m_0} c_{n_0} c_\ell\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right)\end{aligned}$$

$$
\begin{aligned}
&- b_\ell c_{m_0} \left(b_{n_0} b_\ell c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_\ell\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right) \\
&+ b_\ell c_\ell \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right) \\
&+ b_{m_1} c_\ell \left(b_{n_0} b_\ell c_{n_1} c_{m_0} - b_{n_0} b_{m_0} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_0} + b_{n_1} b_{m_0} c_{n_0} c_\ell\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right) \\
&- b_{m_0} c_\ell \left(b_{n_0} b_\ell c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_\ell - b_{n_1} b_\ell c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_\ell\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right) \\
&+ b_\ell c_\ell \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell^2 p_{m_0} p_{m_1} \Gamma\right) \\
&+ \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) (2p_\ell) \\
= {}& 2p_\ell \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\mathcal{M}_\ell^{J'}\right| &= \sum_{j_i \in Q' \cup \{\ell\}} (-1)^i \left|\mathcal{M}_\ell^{J' \setminus \{j_i\}}\right| \mathcal{G}_{\ell j_i} \\
&= 2^{|S'|-1} \Phi \prod_{r \in J' \setminus S'} p_r \Gamma^2 \sum_{j_i \in Q' \cup \{\ell\}} (-1)^i \left|\mathcal{M}_\ell^{J' \setminus \{j_i\}}\right| \mathcal{G}_{\ell j_i} \\
&= 2^{|S'|} p_\ell \Phi \prod_{r \in J' \setminus S'} p_r \Gamma^2 \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right).
\end{aligned}
$$

Case 2b. $\ell \in R'$

First consider $j \in S'$. Note as in Case 1b that

$$
\begin{aligned}
\left|\left(J' \setminus \{j\}\right) \setminus L\right| &= \left|J' \setminus L'\right| \\
&= 2.
\end{aligned}
$$

Now let $j \in Q$. As in case 1b, we have

$$
\begin{aligned}
\left|L \setminus \left(J' \setminus \{j\}\right)\right| &= \left|\left(L' \setminus J'\right) \setminus \{\ell\}\right| \\
&= \left|R'\right| - 1 \\
&= 2 - 1 = 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|\mathcal{M}_\ell^{J'}\right| = {}& \sum_{j_i \in J'} (-1)^i \left|\mathcal{M}_\ell^{J' \setminus \{j_i\}}\right| \mathcal{G}_{\ell j_i} \\
&= \left(2^{|S'|-1} \Phi \prod_{r \in J' \setminus S'} p_r\right) \sum_{j_i \in J' \setminus Q'} (-1)^{i+\ell} (-1)^{n_0+i} \Gamma^2 \Big(b_{n_0} b_{m_0} c_{j_i} c_{m_1} \\
&\quad - b_{n_0} b_{m_1} c_{j_i} c_{m_0} - b_{j_i} b_{m_0} c_{n_0} c_{m_1} + b_{j_i} b_{m_1} c_{n_0} c_{m_0}\Big) \mathcal{G}'_{\ell j_i} \\
&\quad + \left(2^{|S'|} \Phi \prod_{r \in (J' \setminus S') \setminus m_0} p_r\right) (-1)^{(\ell-1)+\ell} (-1)^{n_0+(\ell-1)} \Big[\Gamma \left(b_{n_0} c_{m_1} + b_{m_1} c_{n_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\Gamma^2 \sum_{r\in S'} \left(\left(b_{m_1}c_{n_0} + b_{n_0}c_{m_1}\right) b_r c_r p_r - \left(b_{m_1}b_{n_0}p_r c_r^2 + c_{m_1}c_{n_0}p_r b_r^2\right)\right) \Bigg] \mathcal{G}'_{\ell m_0} \\
&+ \left(2^{|S'|}\Phi \prod_{r\in (J'\setminus S')\setminus m_1} p_r \right) (-1)^{\ell+\ell}(-1)^{n_0+(\ell-1)} \Bigg[\Gamma\left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right) \\
&+ \frac{1}{2}\Gamma^2 \sum_{r\in S'} \left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right) b_r c_r p_r - \left(b_{m_0}b_{n_0}p_r c_r^2 + c_{m_0}c_{n_0}p_r b_r^2\right)\right) \Bigg] \mathcal{G}'_{\ell m_1} \\
&= (-1)^{\ell+n_0} \left(2^{|S'|-1}\Phi \prod_{r\in J'\setminus S'} p_r \right) \sum_{j_i\in J'\setminus Q'} \Gamma^2 \Big(b_{n_0}b_{m_0}c_{j_i}c_{m_1} - b_{n_0}b_{m_1}c_{j_i}c_{m_0} \\
&- b_{j_i}b_{m_0}c_{n_0}c_{m_1} + b_{j_i}b_{m_1}c_{n_0}c_{m_0} \Big) \left(p_\ell p_{j_i}\Gamma\left[b_\ell c_{j_i} + b_{j_i}c_\ell\right]\right) \\
&+ (-1)^{\ell+n_0} \left(2^{|S'|}\Phi \prod_{r\in (J'\setminus S')\setminus m_0} p_r \right) \Bigg[\Gamma\left(b_{n_0}c_{m_1} + b_{m_1}c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r\in S'} \Big(\left(b_{m_1}c_{n_0} + b_{n_0}c_{m_1}\right) b_r c_r p_r \\
&- \left(b_{m_1}b_{n_0}p_r c_r^2 + c_{m_1}c_{n_0}p_r b_r^2\right) \Big) \Bigg] \left(p_\ell p_{m_0}\Gamma\left[b_\ell c_{m_0} + b_{m_0}c_\ell\right]\right) \\
&- (-1)^{\ell+n_0} \left(2^{|S'|}\Phi \prod_{r\in (J'\setminus S')\setminus m_1} p_r \right) \Bigg[\Gamma\left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right) + \frac{1}{2}\Gamma^2 \sum_{r\in S'} \Big(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right) b_r c_r p_r \\
&- \left(b_{m_0}b_{n_0}p_r c_r^2 + c_{m_0}c_{n_0}p_r b_r^2\right) \Big) \Bigg] \left(p_\ell p_{m_1}\Gamma\left[b_\ell c_{m_1} + b_{m_1}c_\ell\right]\right) \\
&= A\Gamma^3 + (-1)^{\ell+n_0} 2^{|S'|} p_\ell \Phi \prod_{r\in J'\setminus S'} p_r B\Gamma^2
\end{aligned}
$$

First we will show that $B = b_{n_0}b_{m_0}c_{n_1}c_{m_1} - b_{n_0}b_{m_1}c_{n_1}c_{m_0} - b_{n_1}b_{m_0}c_{n_0}c_{m_1} + b_{n_1}b_{m_1}c_{n_0}c_{m_0}$:

$$
\begin{aligned}
B &= \left(b_{n_0}c_{m_1} + b_{m_1}c_{n_0}\right)\left[b_\ell c_{m_0} + b_{m_0}c_\ell\right] - \left(b_{n_0}c_{m_0} + b_{m_0}c_{n_0}\right)\left[b_\ell c_{m_1} + b_{m_1}c_\ell\right] \\
&= \left(b_{n_0}c_{m_1}b_\ell c_{m_0} + b_{n_0}c_{m_1}b_{m_0}c_\ell + b_{m_1}c_{n_0}b_\ell c_{m_0} + b_{m_1}c_{n_0}b_{m_0}c_\ell\right) \\
&\quad - \left(b_{n_0}c_{m_0}b_\ell c_{m_1} + b_{n_0}c_{m_0}b_{m_1}c_\ell + b_{m_0}c_{n_0}b_\ell c_{m_1} + b_{m_0}c_{n_0}b_{m_1}c_\ell\right) \\
&= b_{n_0}b_{m_0}c_\ell c_{m_1} - b_{n_0}b_{m_1}c_\ell c_{m_0} - b_\ell b_{m_0}c_{n_0}c_{m_1} + b_\ell b_{m_1}c_{n_0}c_{m_0}
\end{aligned}
$$

Now we will show that $A = 0$:

$$
\begin{aligned}
\frac{1}{2^{|S'|-1}p_\ell\Phi\prod_{r\in Q'}p_r}A &= \sum_{r\in S'} \left(b_{n_0}b_{m_0}c_r c_{m_1} - b_{n_0}b_{m_1}c_r c_{m_0} - b_r b_{m_0}c_{n_0}c_{m_1} + b_r b_{m_1}c_{n_0}c_{m_0}\right)\left[b_\ell c_r + b_r c_\ell\right] \\
&- \sum_{r\in S'} \left(\left(b_{m_1}c_{n_0} + b_{n_0}c_{m_1}\right) b_r c_r - \left(b_{m_1}b_{n_0}c_r^2 + c_{m_1}c_{n_0}b_r^2\right)\right)\left[b_\ell c_{m_0} + b_{m_0}c_\ell\right] \\
&+ \sum_{r\in S'} \left(\left(b_{m_0}c_{n_0} + b_{n_0}c_{m_0}\right) b_r c_r - \left(b_{m_0}b_{n_0}c_r^2 + c_{m_0}c_{n_0}b_r^2\right)\right)\left[b_\ell c_{m_1} + b_{m_1}c_\ell\right]
\end{aligned}
$$

It suffices to show that each term is zero. Indeed for every r we have

$$
\left(b_{n_0}b_{m_0}c_r c_{m_1} - b_{n_0}b_{m_1}c_r c_{m_0} - b_r b_{m_0}c_{n_0}c_{m_1} + b_r b_{m_1}c_{n_0}c_{m_0}\right)\left[b_\ell c_r + b_r c_\ell\right]
$$

$$
\begin{aligned}
&\quad - \left((b_{m_1}c_{n_0} + b_{n_0}c_{m_1})\, b_r c_r - \left(b_{m_1}b_{n_0}c_r^2 + c_{m_1}c_{n_0}b_r^2\right)\right) [b_\ell c_{m_0} + b_{m_0}c_\ell] \\
&\quad + \left((b_{m_0}c_{n_0} + b_{n_0}c_{m_0})\, b_r c_r - \left(b_{m_0}b_{n_0}c_r^2 + c_{m_0}c_{n_0}b_r^2\right)\right) [b_\ell c_{m_1} + b_{m_1}c_\ell] \\
&= (b_{n_0}b_{m_0}c_r c_{m_1} - b_{n_0}b_{m_1}c_r c_{m_0} - b_r b_{m_0}c_{n_0}c_{m_1} + b_r b_{m_1}c_{n_0}c_{m_0}) [b_\ell c_r] \\
&\quad + (b_{n_0}b_{m_0}c_r c_{m_1} - b_{n_0}b_{m_1}c_r c_{m_0} - b_r b_{m_0}c_{n_0}c_{m_1} + b_r b_{m_1}c_{n_0}c_{m_0}) [b_r c_\ell] \\
&\quad - (b_{m_1}c_{n_0} + b_{n_0}c_{m_1})\, b_r c_r [b_\ell c_{m_0} + b_{m_0}c_\ell] \\
&\quad + \left(b_{m_1}b_{n_0}c_r^2 + c_{m_1}c_{n_0}b_r^2\right) [b_\ell c_{m_0} + b_{m_0}c_\ell] \\
&\quad + (b_{m_0}c_{n_0} + b_{n_0}c_{m_0})\, b_r c_r [b_\ell c_{m_1} + b_{m_1}c_\ell] \\
&\quad - \left(b_{m_0}b_{n_0}c_r^2 + c_{m_0}c_{n_0}b_r^2\right) [b_\ell c_{m_1} + b_{m_1}c_\ell] \\
&= c_r^2 \left(b_{n_0}b_{m_0}c_{m_1}b_\ell - b_{n_0}b_{m_1}c_{m_0}b_\ell + \underline{b_{m_1}b_{n_0}} \left[b_\ell c_{m_0} + \underline{b_{m_0}c_\ell}\right] - \underline{b_{m_0}b_{n_0}} \left[b_\ell c_{m_1} + \underline{b_{m_1}c_\ell}\right]\right) \\
&+ b_r^2 \left(-b_{m_0}c_{n_0}c_{m_1}c_\ell + b_{m_1}c_{n_0}c_{m_0}c_\ell + \underline{c_{m_1}c_{n_0}} \left[\underline{b_\ell c_{m_0}} + b_{m_0}c_\ell\right] - \underline{c_{m_0}c_{n_0}} \left[\underline{b_\ell c_{m_1}} + b_{m_1}c_\ell\right]\right) \\
&+ b_r c_r \Bigg(- b_{m_0}c_{n_0}c_{m_1}b_\ell + b_{m_1}c_{n_0}c_{m_0}b_\ell + b_{n_0}b_{m_0}c_{m_1}c_\ell - b_{n_0}b_{m_1}c_{m_0}c_\ell \\
&\quad - \left(\underline{b_{m_1}c_{n_0}} + \underline{\underline{b_{n_0}c_{m_1}}}\right) \left[\underline{\underline{b_\ell c_{m_0}}} + \underline{b_{m_0}c_\ell}\right] + \left(\underline{b_{m_0}c_{n_0}} + \underline{\underline{b_{n_0}c_{m_0}}}\right) \left[\underline{\underline{b_\ell c_{m_1}}} + \underline{b_{m_1}c_\ell}\right] \Bigg) \\
&= 0.
\end{aligned}
$$

Case 3. $|R'| = 3$

Case 3a. $\ell \in S'$

For any $j \in J'$, we have

$$
\begin{aligned}
L \setminus (J' \setminus \{j\}) &= L \setminus \left(J' \cap \{j\}^C\right) \\
&= L \cap \left(J' \cap \{j\}^C\right)^C \\
&= L \cap \left((J')^C \sqcup \{j\}\right) \\
&= \left(L \cap (J')^C\right) \sqcup (L \cap \{j\})
\end{aligned}
$$

Since $\ell \in S' \subseteq J'$, we have $L \setminus J' = L' \setminus J'$ and hence

$$
\begin{aligned}
|L \setminus (J' \setminus \{j\})| &\geq \left|L \cap (J')^C\right| \\
&= |L \setminus J'| \\
&= |L' \setminus J'| \\
&= 3.
\end{aligned}
$$

By $(*)$, this implies $\left|\mathcal{M}_\ell^{J' \setminus \{j\}}\right| = 0$ for every $j \in J'$ and hence $\left|\mathcal{M}_{\ell+1}^{J'}\right| = 0$.

Case 3b. $\ell \in R'$

First consider $j \in S'$. Note that $l \in R'$ implies $\ell \notin J'$. Since $j \in S' \subseteq L$, we have

$$\begin{aligned}\left(J' \setminus \{j\}\right) \setminus L &= J' \setminus (\{j\} \cup L) \\ &= J' \setminus L \\ &= (J' \setminus \{\ell\}) \setminus L \\ &= J' \setminus (\{\ell\} \cup L) \\ &= J' \setminus L' \\ &= 3.\end{aligned}$$

This implies $\left|\mathcal{M}_\ell^{J'\setminus\{j\}}\right| = 0$ for $j \in S'$. Hence $\left|\mathcal{M}_{\ell+1}^{J'}\right| = \sum_{j_i \in Q'} (-1)^i \left|\mathcal{M}_\ell^{J'\setminus\{j_i\}}\right| \mathcal{G}_{\ell j_i}$.

Now let $j \in Q'$. As in case 1b, we have

$$\begin{aligned}\left|L \setminus \left(J' \setminus \{j\}\right)\right| &= \left|\left(L' \setminus J'\right) \setminus \{\ell\}\right| \\ &= \left|R'\right| - 1 \\ &= 3 - 1 = 2.\end{aligned}$$

Denote $R' \setminus \{\ell\} = \{n_0, n_1\}$ and $Q' = \{m_0, m_1, m_2\}$ with $n_0 < n_1$ and $m_0 < m_1 < m_2$. By $(*)$, we have

$$\begin{aligned}&\frac{1}{2^{|S|}} \frac{1}{\Phi} \frac{1}{\prod_{r\in S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_\ell^{J'\setminus\{m_0\}}\right| \mathcal{G}_{\ell m_0} \\ &= p_{m_1} p_{m_2} \left(b_{n_0} b_{m_1} c_{n_1} c_{m_2} - b_{n_0} b_{m_2} c_{n_1} c_{m_1} - b_{n_1} b_{m_1} c_{n_0} b_{m_2} + b_{n_1} b_{m_2} c_{n_0} c_{m_1}\right) \left(p_\ell p_{m_0} \Gamma \left[b_\ell c_{m_0} + b_{m_0} c_\ell\right]\right)\end{aligned}$$

$$\begin{aligned}&\frac{1}{2^{|S|}} \frac{1}{\Phi} \frac{1}{\prod_{r\in S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_\ell^{J'\setminus\{m_1\}}\right| \mathcal{G}_{\ell m_1} \\ &= p_{m_0} p_{m_2} \left(b_{n_0} b_{m_0} c_{n_1} c_{m_2} - b_{n_0} b_{m_2} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} b_{m_2} + b_{n_1} b_{m_2} c_{n_0} c_{m_0}\right) \left(p_\ell p_{m_1} \Gamma \left[b_\ell c_{m_1} + b_{m_1} c_\ell\right]\right)\end{aligned}$$

$$\begin{aligned}&\frac{1}{2^{|S|}} \frac{1}{\Phi} \frac{1}{\prod_{r\in S'} p_r} \frac{1}{\Gamma^2} \left|\mathcal{M}_\ell^{J'\setminus\{m_2\}}\right| \mathcal{G}_{\ell m_2} \\ &= p_{m_0} p_{m_1} \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} b_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(p_\ell p_{m_2} \Gamma \left[b_\ell c_{m_2} + b_{m_2} c_\ell\right]\right)\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{p_{m_0} p_{m_1} p_{m_2}} \frac{1}{2^{|S'|}} \frac{1}{\Phi} \frac{1}{\prod_{r\in S'} p_r} \frac{1}{\Gamma^2} \sum_{j_i \in Q} (-1)^{i+\ell} \left|\mathcal{M}_\ell^{J'\setminus\{j_i\}}\right| \mathcal{G}_{\ell j_i} \\ &= \left(b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}\right) \left(b_\ell c_{m_2} + b_{m_2} c_\ell\right) \\ &\quad - \left(b_{n_0} b_{m_0} c_{n_1} c_{m_2} - b_{n_0} b_{m_2} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_2} + b_{n_1} b_{m_2} c_{n_0} c_{m_0}\right) \left(b_\ell c_{m_1} + b_{m_1} c_\ell\right) \\ &\quad + \left(b_{n_0} b_{m_1} c_{n_1} c_{m_2} - b_{n_0} b_{m_2} c_{n_1} c_{m_1} - b_{n_1} b_{m_1} c_{n_0} c_{m_2} + b_{n_1} b_{m_2} c_{n_0} c_{m_1}\right) \left(b_\ell c_{m_0} + b_{m_0} c_\ell\right) \\ &= b_\ell c_{m_2} (b_{n_0} b_{m_0} c_{n_1} c_{m_1} - b_{n_0} b_{m_1} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_1} + b_{n_1} b_{m_1} c_{n_0} c_{m_0}) \\ &\quad - b_\ell c_{m_1} (b_{n_0} b_{m_0} c_{n_1} c_{m_2} - b_{n_0} b_{m_2} c_{n_1} c_{m_0} - b_{n_1} b_{m_0} c_{n_0} c_{m_2} + b_{n_1} b_{m_2} c_{n_0} c_{m_0})\end{aligned}$$

$$
\begin{aligned}
&+ b_\ell c_{m_0}(b_{n_0}b_{m_1}c_{n_1}c_{m_2} - b_{n_0}b_{m_2}c_{n_1}c_{m_1} - b_{n_1}b_{m_1}c_{n_0}c_{m_2} + b_{n_1}b_{m_2}c_{n_0}c_{m_1}) \\
&+ b_{m_2}c_\ell(b_{n_0}b_{m_0}c_{n_1}c_{m_1} - b_{n_0}b_{m_1}c_{n_1}c_{m_0} - b_{n_1}b_{m_0}c_{n_0}c_{m_1} + b_{n_1}b_{m_1}c_{n_0}c_{m_0}) \\
&- b_{m_1}c_\ell(b_{n_0}b_{m_0}c_{n_1}c_{m_2} - b_{n_0}b_{m_2}c_{n_1}c_{m_0} - b_{n_1}b_{m_0}c_{n_0}c_{m_2} + b_{n_1}b_{m_2}c_{n_0}c_{m_0}) \\
&+ b_{m_0}c_\ell(b_{n_0}b_{m_1}c_{n_1}c_{m_2} - b_{n_0}b_{m_2}c_{n_1}c_{m_1} - b_{n_1}b_{m_1}c_{n_0}c_{m_2} + b_{n_1}b_{m_2}c_{n_0}c_{m_1}) \\
= {}& 0,
\end{aligned}
$$

so $\left|\mathcal{M}_{\ell+1}^{J'}\right| = 0$.

Case 4. $|R'| > 3$

In this case, $|R'| = |L' \setminus J'| = \left|L' \cap (J')^C\right| \geq 4$. Since

$$
\begin{aligned}
L \setminus (J' \setminus \{j\}) &= L \setminus (J' \cap \{j\}^C) \\
&= L \cap (J' \cap \{j\}^C)^C \\
&= L \cap ((J')^C \cup \{j\}) \\
&= \left(L \cap (J')^C\right) \cup (L \cap \{j\}),
\end{aligned}
$$

we have

$$
\begin{aligned}
\left|L \setminus (J' \setminus \{j\})\right| &\geq \left|\left(L \cap (J')^C\right)\right| \\
&\geq \left|L' \cap (J')^C\right| - 1 \\
&\geq 4 - 1 = 3.
\end{aligned}
$$

By $(*)$, this implies $\left|\mathcal{M}_\ell^{J'\setminus\{j\}}\right| = 0$ for every $j \in J'$, and hence $\left|\mathcal{M}_{\ell+1}^{J'}\right| = 0$. □

Lemma 3

Proof. First we will consider the case of discrimination on in-degree. The case of discrimination on out-degree is identical. Using the determinant formula given in Lemma 2, it suffices to show that $\Gamma^2 \sum_{i<j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) < 4 + 4\Gamma \sum_i (b_i p_i c_i)$. Recall from the proof of Lemma 1 that $p_i = P(i)$, $b_i = i$, $c_i = C(i)$, and $\Gamma = \frac{1}{1-\gamma\bar{k}}$. Since $\gamma k^{\max} < \frac{1}{2}$ by assumption, we have $\gamma\bar{k} < \frac{1}{2}$. This implies $1 - \gamma\bar{k} > \frac{1}{2}$ and hence $\frac{1}{1-\gamma\bar{k}} < 2$, so

$$
\Gamma = \gamma \cdot \frac{1}{1-\gamma\bar{k}} < \frac{1}{2k^{\max}} \cdot 2 = \frac{1}{k^{\max}}.
$$

Recall that for any random variable X, we have $E[X^2] = \sum x^2 p(x) \leq x^{\max} \sum x p(x) = x^{\max} E(X)$. Thus

$$
\begin{aligned}
\sum_{i<j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) &= \sum_{i<j} \left(p_i p_j \left|i^2 c_j^2 - 2ij c_i c_j + j^2 c_i^2\right|\right) \\
&= \sum_i \sum_{j \neq i} \left(p_i p_j \left|i^2 c_j^2 - ij c_i c_j\right|\right) \\
&\leq \sum_i \sum_j \left(p_i p_j \left|i^2 c_j^2 - ij c_i c_j\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_i \sum_j \left|p_i p_j i^2 c_j^2\right| + \sum_i \sum_j \left|p_i p_j i j c_i c_j\right| \\
&\leq \left(\sum_i i^2 p_i\right)\left(\sum_j p_j c_j^2\right) + \left(\sum_i p_i i c_i\right)^2 \\
&= E\left[k^2\right] E\left[\left(\frac{\hat{l}(k)}{\hat{k}}\right)^2\right] + \left(k^{\max}\right)^2 \left(\sum_i p_i c_i\right)^2 \\
&= \frac{E\left[k^2\right]}{\left(\hat{k}\right)^2} E\left[\left(\hat{l}(k)\right)^2\right] + \left(k^{\max}\right)^2 \left(E[c]\right)^2 \\
&= \frac{E\left[k^2\right]}{\hat{k}} \frac{E\left[\left(\hat{l}(k)\right)^2\right]}{\hat{k}} + \left(k^{\max}\right)^2 \\
&= \frac{E\left[k^2\right]}{\hat{k}} \frac{E\left[\left(\hat{l}(k)\right)^2\right]}{E\left[\hat{l}(k)\right]} + \left(k^{\max}\right)^2 \\
&\leq 2\left(k^{\max}\right)^2.
\end{aligned}
$$

Since $0 < \Gamma < \frac{1}{k^{\max}}$, we have

$$
\begin{aligned}
\Gamma^2 \sum_{i<j} \left[p_i p_j (b_j c_i - b_i c_j)^2\right] &< \left(\frac{1}{k^{\max}}\right)^2 \sum_{i<j} \left[p_i p_j (b_j c_i - b_i c_j)^2\right] \\
&< 2 \\
&< 4 + 4\Gamma \sum_i (b_i p_i c_i)
\end{aligned}
$$

as claimed. Now we will consider the case of discrimination on in- and out-degrees. Again by Lemma 2 it suffices to show that $\Gamma^2 \sum_{i \lhd j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) < 4 + 4\Gamma \sum_i (b_i p_i c_i)$. As before, we have $\frac{1}{1-\gamma\bar{k}} < 2$. This implies

$$
\Gamma = \gamma \cdot \frac{1}{1-\gamma\bar{k}} \frac{1}{\hat{k}} < \frac{1}{2k^{\max}} \cdot 2 \cdot \frac{1}{\hat{k}} = \frac{1}{k^{\max}\hat{k}}.
$$

Recall that $\hat{k} = \sum_{(i_1,i_2)} i_1 q(i_1,i_2)$, $\hat{l} = \sum_{(i_1,i_2)} i_2 q(i_1,i_2)$, and $\hat{l} = \hat{k}$ (due to consistency). Note that

$$
(i_1 j_2 - i_2 j_1)^2 \leq i_1^2 j_2^2 + i_2^2 j_1^2 < i_1 j_2 \left(k^{\max}\right)^2 + i_2 j_1 \left(k^{\max}\right)^2,
$$

Using the parameters $p_i = q(i) = q(i_1,i_2)$, $b_i = i_2$, $c_i = i_1$, and $\Gamma = \frac{\gamma}{1-\gamma\bar{k}} \frac{1}{\hat{k}}$ obtained in the proof of Lemma 1, we have

$$
\begin{aligned}
\sum_{i \lhd j} \left(q(i)q(j)\left[i_1 j_2 - i_2 j_1\right]^2\right) &< \sum_{i \lhd j} \left(q(i)q(j)\left(k^{\max}\right)^2 (i_1 j_2 + j_1 i_2)\right) \\
&< \left(k^{\max}\right)^2 \sum_{i,j} i_1 j_2 q(i) q(j)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(k^{\max}\right)^2 \left(\sum_{(i_1,i_2)} i_1 q(i)\right) \left(\sum_{(j_1,j_2)} j_2 q(j)\right) \\
&= \left(k^{\max}\right)^2 \hat{k}\hat{l} \\
&= \left(k^{\max}\right)^2 \left(\hat{k}\right)^2 .
\end{aligned}
$$

Since $0 < \Gamma < \frac{1}{k^{\max}\hat{k}}$, we have

$$
\begin{aligned}
\Gamma^2 \sum_{i \lhd j} \left(q(i)q(j) \left[i_1 j_2 - i_2 j_1\right]^2\right) &< \frac{4}{\left(k^{\max}\hat{k}\right)^2} \sum_{i \lhd j} \left(q(i)q(j) \left[i_1 j_2 - i_2 j_1\right]^2\right) \\
&< 4 \\
&< 4 + 4\Gamma \sum_{i} \left(i_1 i_2 q(i)\right) .
\end{aligned}
$$

This shows that $\det \mathcal{H} > 0$ (where the size of the matrix K is arbitrary). Note that the $\tilde{K} \times \tilde{K}$ upper-left sub-matrix of $\mathcal{H}$ is defined exactly by the same formula as $\mathcal{H}$. Then, since the result holds for the sub-matrix of any size $\tilde{K}$, we conclude that $\mathcal{H}$ is positive definite. □