Market Design: Theory and Applications

Auction Theory

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Today

- Auction Theory Basics:
  1. Bidding and Equilibria in Independent Private Values (IPV) model
  2. Revenue Equivalence
  3. Common Values
- Internet Auctions: eBay vs. Amazon

The Independent Private Values Model

Basic Auction Environment:

- Bidders $i = 1, ..., n$
- One object to be sold
- Bidder i observes a "signal" $S_i \sim F(\cdot)$, with typical realization $s_i \in [\underline{s}, \overline{s}]$. Assume F is continuous.
- Bidders' signals $S_1, ..., S_n$ are independent.
- Bidder i's value $v_i(s_i) = s_i$.
- A set of auction rules will give rise to a game between the bidders.

The IPV Model

Two important features of the model:

- Bidder i's information (signal) is independent of bidder j's information (signal).
- Bidder i's value is independent of bidder j's information (i.e. *private values*).

Vickrey (Second-Price) Auction

Auction Rules:

- Bidders are asked to submit sealed bids $b_1, \ldots, b_n$.
- Bidder who submits the highest bid wins the object.
- Winner pays the amount of the second highest bid.

Proposition

In a second price auction, it is a (weakly) dominant strategy to bid one's value, $b_i(s_i) = s_i$.

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Proof. Bidding b_i means i will win if and only if the price is below b_i.
Bid $b_i > s_i \Rightarrow$ sometimes win at price above value.
Bid $b_i < s_i \Rightarrow$ sometimes lose at price below value.

Expected Vickrey Auction Revenue

- Seller's revenue equals second highest value.
- Let $S^{i:n}$ denote the i^{th} highest of n draws from distribution F.
- Seller's expected revenue is

$$\mathbb{E}\left[S^{2:n}\right].$$

- If $F(\cdot)$ is the uniform distribution over $[0,1]$, then

$$\mathbb{E}\left[S^{2:n}\right] = \frac{n-1}{n+1}$$

- For example, in an auction with 10 participants, each of whom values the object at a (uniformly) random value between \$0 and \$10, then the expected revenue for the seller is about \$8.18 [*Check for yourself!*].

Open Ascending Auction

Auction Rules:

- Prices rise continuously from zero.
- Bidders have the option to drop out at any point.
- Auction ends when only one bidder remains.
- Winner pays the ending price.

Open Ascending vs Vickery Auctions

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- Why is this the case?
  - In a Vickery auction, the winner pays the bid of the next-highest bidder.
  - In an Open Ascending auction, the winner is crowned when the next highest bidder drops out. When will this occur?

Sealed Bid (First-Price) Auction

Auction Rules:

- Bidders submit sealed bids $b_1, ..., b_n$.
- Bidders who submits the highest bid wins the object.
- Winner pays his own bid.

Under these rules, bidders will want to shade bids below their values. Why is this the case?

Sealed Bid (First-Price) Auction

Example: $F(\cdot)$ is the uniform distribution over $[0,1]$.

- Suppose bidders $j \neq i$ bid $b_j = a \cdot s_j$
- Bidder i's expected payoff:

$$\begin{aligned}
U(b_i, s_i) &= (s_i - b_i) \cdot \Pr\left[b_j = b(S_j) \le b_i,\ \forall j \neq i\right] \\
&= (s_i - b_i) \cdot \Pr\left[a \cdot s_j \le b_i\right]^{n-1} \\
&= (s_i - b_i) \cdot \Pr\left[s_j \le \frac{b_i}{a}\right]^{n-1} \\
&= (s_i - b_i) \cdot \left[\frac{b_i}{a}\right]^{n-1} \\
&= (s_i - b_i) \cdot \frac{(b_i)^{n-1}}{a^{n-1}}
\end{aligned}$$

Sealed Bid (First-Price) Auction - Example

- Suppose bidders $j \neq i$ bid $b_j = a \cdot s_j$
- Bidder i's expected payoff:

$$U(b_i, s_i) = (s_i - b_i) \cdot \frac{(b_i)^{n-1}}{a^{n-1}}$$

- First order condition:

$$0 = (s_i - b_i) \cdot \frac{(n-1)(b_i)^{n-2}}{a^{n-1}} - \frac{(b_i)^{n-1}}{a^{n-1}}$$

and

$$b_i = \frac{n-1}{n} \cdot s_i$$

Expected First Price Auction Revenue

- Expected revenue is

$$\mathbb{E}\left[b(S^{1:n})\right] = \mathbb{E}\left[\frac{n-1}{n} \cdot S^{1:n}\right] = \frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1}.$$

- So *first and second price auction yield same expected revenue.*

Revenue Equivalence Theorem

Theorem (Revenue Equivalence)
Suppose n bidders have values $s_1, ..., s_n$ identically and independently distributed with CDF $F(\cdot)$. Then any equilibrium of any auction game in which *(i)* the bidder with the highest value wins the object, and *(ii)* a bidder with value $\underline{s}$ gets zero profits, generates the same revenue in expectation.

Using the Revenue Equivalence Theorem

Many applications of the revenue equivalence theorem.

If the auction equilibrium satisfies *(i)* and *(ii)*, we know:

- Expected Revenue
- Expected Bidder Profits (payoff equivalence)
- Expected Total Surplus

We can use this to solve for auction equilibria.

All-Pay Auctions

- Bidders 1, .., n
- Values $s_1, ..., s_n$, i.i.d. with CDF F
- Bidders submit bids $b_1, ..., b_n$
- Bidder who submits the highest bid gets the object.
- Every bidder must pay his or her bid.

Solving the All-Pay Auction

Suppose all pay auction has a symmetric equilibrium with an increasing strategy $b^A(s)$. Then it must be that $b^A(\underline{s}) = 0$.

In this equilibrium, i's expected payoff given value s_i will be:

$$U(s_i) = s_i \cdot [F(s_i)]^{n-1} - b^A(s_i)$$

and in our example of uniform distribution of s_i:

$$\begin{aligned} U(s_i) &= s_i \cdot (s_i)^{n-1} - b^A(s_i) \\ &= \left(s_i - \frac{n-1}{n} \cdot s_i\right) \cdot \frac{\left(\frac{n-1}{n} \cdot s_i\right)^{n-1}}{\left(\frac{n-1}{n}\right)^{n-1}} \end{aligned}$$

So

$$b^A(s_i) = (s_i)^n - \frac{1}{n-1}(s_i)^n = \frac{n-1}{n} \cdot (s_i)^n$$

Other Revenue Equivalent Auctions

1. The English (oral ascending) auction. All bidders start in the auction with a price of zero. Price rises continuously; bidders may drop out at any point in time. Once they drop out, they cannot re-enter. Auction ends when only one bidder is left; this bidder pays the price at which the second-to-last bidder dropped out.

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2. The Dutch (descending price) auction. The price starts at a very high level and drops continuously. At any point in time, a bidder can stop the auction, and pay the current price. Then the auction ends.

“Common Value” Auctions

We would like to generalize the model to allow for the possibility that:

1. Learning bidder j’s information could cause bidder i to re-assess his estimate of how much he values the object,
2. The information of i and j is not independent (when j’s estimate is high, i’s is also likely to be high).

Examples:

- Selling natural resources such as oil or timber.
- Selling financial assets such as treasury bills.
- Selling a company.

Second Price Auction with Common Values - Example

- Two bidders.
- $v(s_1, s_2) = s_1 + s_2$
- $s_i \tilde{} U[0,1]$

Observations:

1. $E[v(s_1, s_2) | s_2] = s_1 + \frac{1}{2}$
2. $b(s_1) = s_1 + \frac{1}{2}$ and $b(s_2) = s_2 + \frac{1}{2}$ is not an equilibrium.
3. The *winner's curse*.

Failure of Revenue Equivalence Theorem

- The RET does not hold in this more general environment.
- Basic idea: bidders profits are due to information rents. When signals / values are correlated, information rents can be reduced if "more information" is used in setting the price.
- Example: English auction generates more revenue than a second price auction.