Local discontinuous Galerkin method for the Hunter-Saxton equation and its zero-viscosity and zero-dispersion limit

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Abstract

In this paper, we develop, analyze and test a local discontinuous Galerkin (LDG) method for solving the Hunter-Saxton (HS) equation which contains nonlinear high order derivatives and its zero-viscosity and zero-dispersion limit. The energy stability for general solutions are proved and numerical simulation results for different types of solutions of the nonlinear HS equation are provided to illustrate the accuracy and capability of the LDG method. The zero-viscosity and zero-dispersion properties of the HS equation are shown in numerical simulation.

AMS subject classification: 65M60, 37K10

Key words: local discontinuous Galerkin method, Hunter-Saxton equation, zero-viscosity limit, zero-dispersion limit, stability

1 Introduction

In this paper, we consider numerical approximations to the Hunter-Saxton (HS) equation [12, 13]

\[ u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0, \]  \hspace{1cm} (1.1)
its regularization with viscosity \[15\]
\[
u_{xxt} + 2u_x u_{xx} + uu_{xxx} - \varepsilon_1 u_{xxxx} = 0, \tag{1.2}\]
and its regularization with dispersion \[15\]
\[
u_{xxt} + 2u_x u_{xx} + uu_{xxx} - \varepsilon_2 u_{xxxxx} = 0, \tag{1.3}\]
where \(\varepsilon_1 \geq 0\) and \(\varepsilon_2\) are small constants. This equation arises in two different physical contexts in two nonequivalent variational forms. It describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field, and it is the high frequency limit of the Camassa-Holm equation \[3, 4\], which is an integrable model equation for shallow water waves.

The HS equation is a completely integrable, bi-hamiltonian system. Also it is a scale-invariant, non-dispersive wave equation. This is a very unusual property for known integrable equations. In particular, the HS equation combines the formation of singularities with complete integrability. Shock wave theory and the theory of integrable systems have not overlapped significantly, but both of these theories are important for (1.1). Smooth solutions of (1.1) break down in finite time because the slope \(u_x \to -\infty\) \[12\]. In \[14\], Hunter and Zheng proved that equation (1.1) has continuous global weak solutions. A class of these weak solutions conserves energy even after their derivative blows up. Hunter and Zheng also gave a detailed discussion of the global existence of weak solutions and their zero-viscosity and dispersion limits in \[15\]. It was proved that the solution of equation (1.2) will converge to the dissipative weak solution of (1.1) when \(\varepsilon_1 \to 0^+\).

This equation has attracted a lot of attention in the literature. Bressan and Constantin \[2\] constructed a continuous semigroup of weak, dissipative solutions. In \[23\], Yin proved the local existence of strong solutions of the periodic HS equation and showed that all strong solutions except space-independent solutions blow up in finite time. Tiğlay also discussed the periodic Cauchy problem of the modified HS equation in \[19\]. The associated inverse scattering problem is used to obtain the explicit solutions of the finite dimensional flows in both the compact and non-compact cases in \[1\]. Lenells \[16, 17\] also gave the proof in which the HS equation describes the geodesic flow on a sphere.

Although the HS equation is well studied from a mathematical point of view, there are only a few numerical works in the literature to solve the HS equation. Holden et al. \[11\] proved the convergence of a finite difference method for the HS equation. The lack
of smoothness of $u_x$ introduces high-frequency oscillation errors into the calculation. It is a challenge to design stable and accurate numerical schemes for solving equation (1.1).

In this paper, we discuss a local discontinuous Galerkin (LDG) method for the HS equation and give a rigorous proof for the energy stability. In this method basis functions are used which are discontinuous in space. The LDG discretization also results in an extremely local, element based discretization, which is beneficial for parallel computing, and it maintains high order accuracy on unstructured meshes. In particular, the LDG method is well suited for $hp$-adaptation, which consists of local mesh refinement and/or the adjustment of the polynomial order in individual elements. The main motivation for the algorithm discussed in this article originates from the LDG techniques which have been developed for convection diffusion equations (containing second derivatives) [7], nonlinear one-dimensional and two-dimensional KdV equations [22, 20] and the Camassa-Holm equation [21]. More general information about DG methods for elliptic and hyperbolic partial differential equations can be found in [5, 8, 9, 10].

The nonlinear terms in the HS equation are similar to those in the Camassa-Holm equation. Even though the method developed in [21] gives us the hint how to treat these nonlinear derivative terms, the different solution property of the HS equation gives rise to new difficulties. When $u_x$ becomes discontinuous, a special numerical treatment for the derivative is needed. To our best knowledge, this is the first provably stable finite element method for the HS equation.

The paper is organized as follows. In Section 2, we present and analyze our LDG method for the HS equation (1.1). In Section 2.2, we present the LDG method. Details related to the implementation of the method are described in Section 2.3. We give a proof of the energy stability in Section 3. Section 4 contains numerical results to demonstrate the accuracy and capability of the methods. Concluding remarks are given in Section 5.

2 The LDG method for the HS-type equation

2.1 Notation

We denote the mesh in $[0, L]$ by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \ldots, N$. The center of the cell is $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ and the mesh size is denoted by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max_{1 \leq j \leq N} h_j$ being the maximum mesh size. We assume the mesh is regular, namely the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements. We define the piecewise-polynomial space $V_h$ as the space of
polynomials of the degree up to \( k \) in each cell \( I_j \), i.e.

\[
V_h = \{ v \in L^2(\Omega) : v \in P^k(I_j) \text{ for } x \in I_j, \ j = 1, \ldots, N \}.
\]

Note that functions in \( V_h \) are allowed to have discontinuities across element interfaces.

The solution of the numerical scheme is denoted by \( u_h \), which belongs to the finite element space \( V_h \). We denote by \( (u_h)^+_{j+\frac{1}{2}} \) and \( (u_h)^-_j \) the values of \( u_h \) at \( x_{j+\frac{1}{2}} \), from the right cell \( I_{j+1} \), and from the left cell \( I_j \), respectively. We use the usual notations \( [u_h] = u^+_h - u^-_h \) and \( \bar{u}_h = \frac{1}{2}(u^+_h + u^-_h) \) to denote the jump and the mean of the function \( u_h \) at each element boundary point, respectively.

### 2.2 The LDG method

In this section, we define our LDG method for the HS-type equations (1.1), (1.2) and (1.3), written in the following form

\[
\begin{align*}
\frac{du}{dx} &= q, \\
\frac{dq}{dt} &= \frac{1}{2}(\frac{u_x^2}{x} - \frac{1}{2}(u^2))_{xxx} + \epsilon_1 u_{xxxx} + \epsilon_2 u_{xxxxx}
\end{align*}
\]

with an initial condition

\[
u(x, 0) = u_0(x),
\]

where \( \epsilon_1 \geq 0 \) and \( \epsilon_2 \) are small constants.

- When \( \epsilon_1 = \epsilon_2 = 0 \), (2.1)-(2.2) is the HS equation (1.1). The boundary conditions are

\[
\begin{align*}
u(0, t) &= u_x(0, t) = u_x(L, t) = 0, \\
or
\mu(L, t) &= u_x(0, t) = u_x(L, t) = 0.
\end{align*}
\]

- When \( \epsilon_1 > 0, \epsilon_2 = 0 \), (2.1)-(2.2) is the regularization with viscosity of the HS equation (1.2).

- When \( \epsilon_1 = 0, \epsilon_2 \neq 0 \), (2.1)-(2.2) is the regularization with dispersion of the HS equation (1.3). In the following, we assume \( \epsilon_2 > 0 \).

To define the local discontinuous Galerkin method, we further rewrite the equation (2.1) as a first order system:

\[
q - r_x = 0,
\]
The LDG method for the equations (2.6), where \( q \) is assumed known and we would want

to solve for \( u \), is formulated as follows: find \( u_h, \ r_h \in V_h \) such that, for all test functions \( \rho, \ \vartheta \in V_h \),

\[
\begin{align*}
\int_{I_j} q_h \rho dx + \int_{I_j} r_h \rho_x dx - (\hat{r}_h^\rho^-)_{j+\frac{1}{2}} + (\hat{r}_h^\rho^+)_{j-\frac{1}{2}} &= 0, \\
\int_{I_j} r_h \vartheta dx + \int_{I_j} u_h \vartheta_x dx - (\hat{u}_h^\vartheta^-)_{j+\frac{1}{2}} + (\hat{u}_h^\vartheta^+)_{j-\frac{1}{2}} &= 0.
\end{align*}
\tag{2.7}
\tag{2.8}
\]

The “hat” terms in (2.7)–(2.8) in the cell boundary terms from integration by parts are

the so-called “numerical fluxes”, which are single valued functions defined on the edges

and should be designed based on different guiding principles for different PDEs to ensure

stability. For the standard elliptic equation (2.6), we can take the simple choices such

that

\[
\hat{r}_h = r_h^+, \quad \hat{u}_h = u_h^-,
\tag{2.9}
\]

where we have omitted the half-integer indices \( j+\frac{1}{2} \) as all quantities in (2.9) are computed

at the same points (i.e. the interfaces between the cells).

We remark that the choice for the fluxes (2.9) is not unique. We can for example

also choose the following numerical flux

\[
\hat{r}_h = r_h^-, \quad \hat{u}_h = u_h^+.
\tag{2.10}
\]

For the equation (2.2), we can also rewrite it into a first order system:

\[
\begin{align*}
q_t + p_x - B(r)_x - \varepsilon_1 s_x - \varepsilon_2 v_x &= 0, \\
p - (b(r)u)_x &= 0, \\
v - s_x &= 0, \\
s - w_x &= 0, \\
w - r_x &= 0, \\
r - u_x &= 0,
\end{align*}
\tag{2.11}
\]

where \( B(r) = \frac{1}{2} r^2 \) and \( b(r) = B'(r) = r \). Now we can define a local discontinuous

Galerkin method to equations (2.11), resulting in the following scheme: find \( q_h, \ p_h, \ r_h \in V_h \) such that, for all test functions \( \varphi, \ \psi, \ \phi, \ \xi, \ \zeta, \ \eta \in V_h \),

\[
\begin{align*}
\int_{I_j} (q_h)_t \varphi dx - \int_{I_j} (p_h - B(r_h) - \varepsilon_1 s_h - \varepsilon_2 v_h) \varphi_x dx
\end{align*}
\tag{2.12}
\]
The boundary conditions for the LDG scheme of the HS equation are taken as

\[ \int_I p_h \psi dx + \int_I b(r_h) u_h \psi_x dx - (\hat{b} \hat{u}_h \psi^-)_{j+\frac{1}{2}} + (\hat{b} \hat{u}_h \psi^+)_{j-\frac{1}{2}} = 0, \quad (2.13) \]

\[ \int_I v_h \phi dx + \int_I s_h \phi_x dx - (\hat{s} \hat{w}_h \phi^-)_{j+\frac{1}{2}} + (\hat{s} \hat{w}_h \phi^+)_{j-\frac{1}{2}} = 0, \quad (2.14) \]

\[ \int_I s_h \xi dx + \int_I w_h \xi_x dx - (\hat{w} \hat{r}_h \xi^-)_{j+\frac{1}{2}} + (\hat{w} \hat{r}_h \xi^+)_{j-\frac{1}{2}} = 0, \quad (2.15) \]

\[ \int_I w_h \xi dx + \int_I r_h \xi_x dx - (\hat{r} \hat{w}_h \xi^-)_{j+\frac{1}{2}} + (\hat{r} \hat{w}_h \xi^+)_{j-\frac{1}{2}} = 0, \quad (2.16) \]

\[ \int_I r_h \eta dx + \int_I u_h \eta_x dx - (\hat{u} \hat{r}_h \eta^-)_{j+\frac{1}{2}} + (\hat{u} \hat{r}_h \eta^+)_{j-\frac{1}{2}} = 0. \quad (2.17) \]

The numerical fluxes in equations (2.12)–(2.17) are chosen as

\[ \hat{p}_h = p_h^+, \quad \hat{u}_h = u_h^-, \quad \hat{B}(r_h) = B(r_h^+), \quad \hat{b}(r_h) = \frac{B(r_h^+ - B(r_h^-)}{r_h^+ - r_h^-}, \quad (2.18) \]

\[ \hat{s}_h = s_h^+, \quad \hat{v}_h = v_h^+, \quad \hat{s}_h = s_h^-, \quad \hat{w}_h = w_h^-, \quad \hat{r}_h = r_h^+, \quad \hat{u}_h = u_h^- . \]

The boundary conditions for the LDG scheme of the HS equation are taken as

\[ (u_h)^-_N = 0, \quad (r_h)^-_N = 0, \quad (r_h)^+_N = 0, \quad (p_h)^+_N = 0. \quad (2.19) \]

For the zero-limit case, we impose the following boundary conditions

\[ (s_h)^-_N = 0, \quad (s_h)^+_N = 0, \quad (v_h)^+_N = 0, \quad (w_h)^-_N = 0. \quad (2.20) \]

The definition of the algorithm is now complete. Notice that we have used the same notation \( r_h \) in (2.8) and in (2.17) as the schemes as well as the fluxes chosen are identical.

### 2.3 Algorithm flowchart

In this section, we give a similar process related to the implementation of the method as that in [21].

- First, from the equations (2.7)–(2.9), we obtain \( q_h \) in the following matrix form

\[ q_h = A u_h, \quad (2.21) \]

where \( q_h \) and \( u_h \) are the vectors containing the degrees of freedom for \( q_h \) and \( u_h \), respectively.
• From (2.12)-(2.18), we obtain the LDG discretization of the residual $\frac{1}{2}((u_x)^2)_x - \frac{1}{2}(u^2)_{xxx} + \varepsilon_1 u_{xxxx} + \varepsilon_2 u_{xxxxx}$ in the following vector form

$$ (q_h)_t = \text{res}(u_h). \quad (2.22) $$

• We then combine (2.21) and (2.22) to obtain

$$ A(u_h)_t = \text{res}(u_h). \quad (2.23) $$

• We use a time discretization method to solve

$$ (u_h)_t = A^{-1} \text{res}(u_h). \quad (2.24) $$

This step involves a linear solver with the matrix $A$. We perform a $LU$ decomposition for $A$ at the beginning and use it for all time steps. Any standard ODE solvers can be used here, for example the Runge-Kutta methods.

The LDG matrix $A$ is a sparse block matrix, hence its multiplication with vectors and a linear solver involving it as the coefficient matrix can be implemented efficiently.

3 Energy stability of the LDG method

In this section, we prove the energy stability of the LDG method for the HS equation defined in the previous section.

**Proposition 3.1.** (Energy stability) The solution to the schemes (2.7)-(2.9) and (2.12)-(2.18) satisfies the energy stability

$$ \frac{d}{dt} \int_0^L r_h^2 dx \leq 0. \quad (3.1) $$

**Proof.** For equation (2.7), we first take the time derivative and get

$$ \int_{I_j} (q_h)_t \rho dx + \int_{I_j} (r_h)_t \rho_x dx - ((r_h)_t \rho^-)_{j+\frac{1}{2}} + ((r_h)_t \rho^+)_{j-\frac{1}{2}} = 0. \quad (3.2) $$

Since (3.2), (2.8) and (2.12)-(2.17) hold for any test functions in $V_h$, we can choose

$$ \rho = u_h, \quad \vartheta = (r_h)_t, \quad \varphi = -u_h, \quad \phi = \varepsilon_2 r_h, \quad \xi = \varepsilon_1 r_h - \varepsilon_2 w_h, $$

$$ \zeta = \varepsilon_1 w_h + \varepsilon_2 s_h, \quad \psi = -r_h, \quad \eta = p_h - \varepsilon_1 s_h - \varepsilon_2 v_h. $$

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With these choices of test functions and summing up the five equations in (3.2), (2.8) and (2.12)–(2.17), we obtain

\[
\int_{I_j} (r_h)_t r_h \, dx + \int_{I_j} (p_h u_h)_x \, dx - (\tilde{\omega}_h u_h + \hat{\omega}_h p_h^\ast)_{j+\frac{1}{2}} + (\tilde{\omega}_h u_h^+ + \hat{\omega}_h p_h^\ast)_{j-\frac{1}{2}} \\
- \int_{I_j} (B(r_h) u_h)_x \, dx + (B(r_h) \hat{u}_h)^{-} + b(r_h) \tilde{\omega}_h r_h^{-})_{j+\frac{1}{2}} - (B(r_h) u_h^+ + b(r_h) \tilde{\omega}_h r_h^+))_{j-\frac{1}{2}} \\
- \int_{I_j} \varepsilon_1 (s_h u_h)_x \, dx + \varepsilon_1 (\tilde{s}_h u_h^- + \hat{\omega}_h s_h^-)_{j+\frac{1}{2}} - \varepsilon_1 (s_h u_h^+ + \hat{\omega}_h s_h^+))_{j-\frac{1}{2}} \\
+ \int_{I_j} \varepsilon_1 (r_h w_h)_x \, dx - \varepsilon_1 (\tilde{\omega}_h w_h^- + \hat{\omega}_h r_h^-)_{j+\frac{1}{2}} + \varepsilon_1 (\hat{\omega}_h w_h^+ + \tilde{\omega}_h r_h^+))_{j-\frac{1}{2}} \\
- \int_{I_j} \varepsilon_2 (v_h u_h)_x \, dx + \varepsilon_2 (\hat{\omega}_h u_h^- + \tilde{\omega}_h v_h^-)_{j+\frac{1}{2}} - \varepsilon_2 (\hat{\omega}_h u_h^+ + \tilde{\omega}_h v_h^+))_{j-\frac{1}{2}} \\
+ \int_{I_j} \varepsilon_2 (r_h s_h)_x \, dx - \varepsilon_2 (\tilde{\omega}_h s_h^- + \hat{\omega}_h r_h^-)_{j+\frac{1}{2}} + \varepsilon_2 (\hat{\omega}_h s_h^+ + \tilde{\omega}_h r_h^+))_{j-\frac{1}{2}} \\
+ \varepsilon_1 \int_{I_j} w_h^2 \, dx - \int_{I_j} \varepsilon_2 w_h (w_h)_x \, dx + \varepsilon_2 (\hat{\omega}_h w_h^-)_{j+\frac{1}{2}} - \varepsilon_2 (\tilde{\omega}_h w_h^+))_{j-\frac{1}{2}} \\
+ \int_{I_j} ((r_h)_t u_h)_x \, dx - ((\tilde{\omega}_h)_t u_h^- + \hat{\omega}_h (r_h^-)_t)_{j+\frac{1}{2}} + ((\hat{\omega}_h)_t u_h^+ + \tilde{\omega}_h (r_h^+)_t)_{j-\frac{1}{2}} = 0.
\]

We have

\[
\int_{I_j} (r_h)_t r_h \, dx + \varepsilon_1 \int_{I_j} w_h^2 \, dx + \Psi_{j+\frac{1}{2}} - \Psi_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}} = 0,
\]

where the numerical entropy fluxes are given by

\[
\Psi_{j+\frac{1}{2}} = \left( p_h u_h^- - (\tilde{\omega}_h u_h^- + \hat{\omega}_h p_h^-) + \varepsilon_2 (s_h r_h^- - (\tilde{s}_h r_h^- + \hat{\omega}_h s_h^-)) \right) \\
+ \varepsilon_2 (v_h u_h^- + (\tilde{\omega}_h u_h^- + \hat{\omega}_h v_h^-)) + \varepsilon_2 (\frac{1}{2} w_h w_h^- + \hat{\omega}_h w_h^-) \\
+ \varepsilon_1 (w_h r_h^- - (\hat{\omega}_h r_h^- + \tilde{\omega}_h w_h^-)) + \varepsilon_1 (s_h u_h^- + (\tilde{s}_h u_h^- + \hat{\omega}_h s_h^-)) \\
- B(r_h^-) u_h^- + B(r_h) u_h^- + b(r_h) \tilde{\omega}_h r_h^- + (r_h^-)_t u_h^- - ((r_h)_t u_h^- + \hat{\omega}_h (r_h^-)_t))_{j+\frac{1}{2}},
\]

and the extra term \( \Theta \) is given by

\[
\Theta_{j-\frac{1}{2}} = \left( -[p_h u_h] + \tilde{\omega}_h [u_h] + \hat{\omega}_h [p_h] + \varepsilon_2 (\frac{1}{2} [w_h^2] - \hat{\omega}_h [w_h]) \right) \\
+ \varepsilon_2 ([v_h u_h] - \tilde{\omega}_h [u_h] - \hat{\omega}_h [v_h]) + \varepsilon_2 ([s_h u_h] + \tilde{s}_h [r_h] + \hat{\omega}_h [s_h]) \\
+ \varepsilon_1 ([s_h u_h] - \tilde{s}_h [u_h] - \hat{\omega}_h [s_h]) + \varepsilon_1 ([w_h r_h] + \tilde{\omega}_h [r_h] + \hat{\omega}_h [w_h]) \\
+ [B(r_h) u_h] - b(r_h) \tilde{\omega}_h [r_h] - B(r_h) [u_h] - ((r_h)_t [u_h] + \hat{\omega}_h [r_h])_{j-\frac{1}{2}}.
\]
With the definition (2.9) and (2.18) of the numerical fluxes and after some algebraic manipulation, we easily obtain

\[-[p_h u_h] + \hat{p}_h[u_h] - \hat{u}_h[p_h] = 0,
[sh u_h] - \hat{sh}[u_h] - \hat{u}_h[s_h] = 0,
[vh u_h] - \hat{vh}[u_h] - \hat{u}_h[v_h] = 0,
-\[w_h r_h] + \hat{w}_h[r_h] + \hat{r}_h[w_h] = 0,
-\[s_h r_h] + \hat{s}_h[r_h] + \hat{r}_h[s_h] = 0,
[B(r_h)u_h] - b(r_h)\hat{u}_h[r_h] - B(r_h)[u_h] = 0,
-[(r_h)_t u_h] + (\hat{r}_h)_t[u_h] - \hat{u}_h[(r_h)_t] = 0\]

and hence

\[\Theta_{j-\frac{1}{2}} = \varepsilon_2 \left( \frac{1}{2}[w_h^2] - \hat{w}_h[w_h] \right) = \varepsilon_2 2[w_h]^2.\]

Summing up the cell entropy inequalities and taking care of the boundary conditions (2.19) and (2.20), we obtain

\[\int_I (r_h)_t r_h dx + \varepsilon_1 \int_I w_h^2 dx + \varepsilon_2 \sum_{j=1}^{N} \frac{1}{2}[w_h]_{j-\frac{1}{2}}^2 = 0, \quad (3.4)\]

Then we can get the desired energy stability (3.1) from equation (3.4).

**Remark 3.1.** We give some remarks on this proposition.

- **When** \(\varepsilon_1 = \varepsilon_2 = 0\), \(\frac{d}{dt} \int_0^L r_h^2 dx = 0\). The energy of the LDG scheme for the HS equation (2.1) is conservative.

- **When** \(\varepsilon_1 > 0\), \(\varepsilon_2 = 0\), \(\frac{d}{dt} \int_0^L r_h^2 dx \leq 0\). The energy of the LDG scheme will decay.

- **When** \(\varepsilon_1 = 0\), \(\varepsilon_2 > 0\), \(\frac{d}{dt} \int_0^L r_h^2 dx \leq 0\). The energy of the LDG scheme will decay.

We will demonstrate this property by the numerical simulation in Section 4.

### 4 Numerical results

In this section we provide numerical examples to illustrate the accuracy and capability of the method. Time discretization is by the third order explicit TVD Runge-Kutta method in [18]. This is not the most efficient method for the time discretization to our LDG scheme. However, we will not address the issue of time discretization efficiency in this paper. We have verified with the aid of successive mesh refinements, that in all cases, the results shown are numerically convergent.
4.1 The HS equation

In this section, we consider numerical simulation for the HS equation

\[ u_{xxt} + 2u_xu_{xx} + uu_{xxx} = 0. \] (4.1)

Example 4.1.

In this example, we consider the numerical solutions of the HS equation (4.1) with the initial condition

\[
  u(x, 0) = \begin{cases} 
    0, & \text{if } x \leq 0, \\
    x, & \text{if } 0 < x < 1, \\
    1, & \text{if } 1 \leq x.
  \end{cases}
\] (4.2)

The exact solution of the HS equation (4.1) is

\[
  u(x, t) = \begin{cases} 
    0, & \text{if } x \leq 0, \\
    \frac{x}{(0.5t + 1)}, & \text{if } 0 < x < (0.5t + 1)^2, \\
    (0.5t + 1), & \text{if } x \geq (0.5t + 1)^2.
  \end{cases}
\] (4.3)

The computational domain is $[-6, 6]$. Even though the solution $u$ is continuous, the lack of the smoothness of $u_x$ will introduce high-frequency oscillation into the calculation of the residual. To control the oscillation of $r_h$ (approximation of $u_x$), we use the total variation bounded in the means (TVBM) limiter in [6] to remove the oscillation near the discontinuity of $r_h$. In Figures 4.1 and 4.2, we show the results $P^2$ elements with $N = 80$ cells for the solution $u$ and $r$ (i.e. $u_x$). The left column are the solutions without limiter and the right column are solutions with the TVBM limiter on $r$. We can see that the oscillations of $r$ are eliminated and the solution of $u$ are smoothed out when the TVBM limiter is used. In Figure 4.3, the energy \[ \int_0^L r_h^2 dx \] as a function of time for the numerical solutions is shown. Even when the solution of $r$ is oscillatory without the limiter, the energy is still conserved. When the limiter is used, the energy is decaying.

Example 4.2.

In this example, we consider the numerical solution of the HS equation (4.1) with the initial condition

\[ u(x, 0) = 1 - \tanh \left( \frac{x - \frac{2}{3}}{0.1} \right). \] (4.4)
Figure 4.1: Solution $u$ for the HS equation (4.1) with the initial condition (4.2). Left: no limiter, Right: limiter.

The computational domain is $[-0.2, 1.2]$. In Figures 4.4 and 4.5, we show the results with $P^2$ element with $N = 800$ cells for the solution $u$ and $r$ (i.e. $u_x$). Even though the solution $u$ is bounded, the derivative of $u$ is going to blow up. In Figure 4.6, the energy $\int_0^L r_i^2 dx$ as a function of time for the numerical solutions is shown. The energy is conserved as the theoretical result in Proposition 3.1 predicts.

Example 4.3.
In this example, we consider the numerical solution of the HS equation with the initial condition

\[ u(x, 0) = 2 \exp \left( - \left( \frac{x - 0.5}{0.15} \right)^2 \right). \]  

The computational domain is \([-1, 1.5]\). In Figures 4.7 and 4.8, we show the results with \(P^2\) element with \(N = 800\) cells for the solution \(u\) and \(r\) (i.e. \(u_r\)). In Figure 4.9, the energy \( \int_0^L r_h^2 \, dx \) as a function of time for the numerical solutions is shown. The energy
Energy for the HS equation (4.1) with the initial condition (4.2).

Figure 4.3: Energy for the HS equation (4.1) with the initial condition (4.2).

Solution $u$ for the HS equation (4.1) with the initial condition (4.4).

Figure 4.4: Solution $u$ for the HS equation (4.1) with the initial condition (4.4).

is still conserved as the theoretical result in Proposition 3.1 predicts, even when the derivative of $u$ is going to blow up.

Example 4.4.

In this example, we consider the numerical solution of the HS equation with the
Figure 4.5: Solution \( r \) (i.e. \( u_x \)) for the HS equation (4.1) with the initial condition (4.4).

Figure 4.6: Energy for the HS equation (4.1) with the initial condition (4.4).

The initial condition

\[
u(x, 0) = 2 \left( \frac{x - 0.5}{0.1} \right) \exp \left( - \left( \frac{x - 0.5}{0.1} \right)^2 \right) + 1 - \tanh \left( \frac{x - 0.5}{0.1} \right).
\]  

(4.6)

The computational domain is \([-0.2, 1.2]\). In Figures 4.10 and 4.11, we show the results with \(P^2\) element with \(N = 800\) cells for the solution \(u\) and \(r\) (i.e. \(u_x\)). Even though the solution \(u\) is bounded, the derivative of \(u\) is going to blow up. In Figure 4.12, the
energy $\int_0^L r^2 dx$ as a function of time for the numerical solutions is shown. The energy is conserved as the theoretical result in Proposition 3.1 predicts.

4.2 The zero-viscosity limit of the HS equation

In this section, we consider the regularization with viscosity of the HS equation

$$u_{xxx} + 2u_xu_{xx} + uu_{xxx} - \varepsilon u_{xxxx} = 0,$$  \hspace{1cm} (4.7)

where $\varepsilon \geq 0$ is a constant.

Example 4.5.

In this example, we consider zero-viscosity limit of the equation (4.7) with the initial data

$$u(x, 0) = \begin{cases} 
0, & \text{if } x \leq 0, \\
-x, & \text{if } 0 < x < 1, \\
-1, & \text{if } x \geq 1. 
\end{cases}$$  \hspace{1cm} (4.8)
Figure 4.8: Solution $r$ (i.e. $u_x$) for the HS equation (4.1) with the initial condition (4.5).

The exact solution of the HS equation (4.1) (i.e. zero-viscosity limit $\varepsilon=0$) is

$$
  u(x, t) = \begin{cases} 
    0, & \text{if } x \leq 0, \\
    \frac{x}{0.5t-1}, & \text{if } 0 < x < (1 - 0.5t)^2, \\
    0.5t - 1, & \text{if } x \geq (1 - 0.5t)^2.
  \end{cases}
$$

(4.9)

The computational domain is $[-10, 10]$. We use the $P^2$ element with $N = 160$ cells in our computation of the LDG method. No limiter is used in this test. In Figures 4.13, we...
Figure 4.10: Solution $u$ for the HS equation (4.1) with the initial condition (4.6).

show the results for the solution $u$ with different values of $\varepsilon$ at time $t = 0, 0.4, 0.8, 1.2, 1.6$. Here, we take $\varepsilon = 0.15625, 0.03125, 0.00625, 0.00025, 0.00005$, and $0.0$. From the results, we can see that the solution with successively smaller $\varepsilon$ converges to the zero-viscosity limit of the HS equation. These numerical results are consistent with the theory in [15]. In Figure 4.14, the energy $\int_0^L r_h^2 dx$ as a function of time for the numerical solutions is shown. The energy for the zero-viscosity limit case is conservative. The energy decays faster when $\varepsilon$ increases.

4.3 The zero-dispersion limit of the HS equation

In this section, we consider the regularization with dispersion of the HS equation

$$u_{xxt} + 2u_xu_{xx} + uu_{xxx} - \varepsilon u_{xxxxx} = 0. \quad (4.10)$$

where $\varepsilon \geq 0$ is a constant.

Example 4.6.
In this example, we consider the zero-dispersion limit of the equation (4.10) with the initial data

\[
u(x, 0) = \begin{cases} 
2, & \text{if } x \leq 0.25, \\
3 - 4x, & \text{if } 0.25 < x < 0.75, \\
0, & \text{if } x \geq 0.75.
\end{cases}
\]
The exact solution of the HS equation (4.1) is

\[
  u(x, t) = \begin{cases} 
  -4(t - 0.5), & \text{if } x \leq 0.75 - 2(t - 0.5)^2, \\
  \frac{2(x - 0.75)}{t-0.5}, & \text{if } 0.75 - 2(t - 0.5)^2 < x < 0.75, \\
  0, & \text{if } x \geq 0.75.
  \end{cases}
\]  

(4.12)

The computational domain is \([-10, 10]\). We use \(P^2\) elements with \(N = 160\) cells.
in our computation of the LDG method. No limiter is used in this test. In Figures 4.15–4.17, we show the zoomed-in results for the solution \( u \) with different values of \( \varepsilon \) at time \( t = 0.1, 0.2, 0.4 \). Here, we take \( \varepsilon = 0.0625, 0.00125, 0.00025 \) and 0.0. From the results, we can see that the oscillations tend to zero with the decrease of \( \varepsilon \). These numerical results are consistent with the numerical results in [15]. In Figure 4.18, the energy \( \int_0^L r^2 \, dx \) as a function of time for the numerical solutions is shown. The energy for the zero-dispersion limit case is conservative. The energy decays when \( \varepsilon > 0 \).

\[ \begin{align*}
\text{Figure 4.15: Solution } u \text{ for the equation (4.10) the initial condition (4.11) for different } \varepsilon \\
\text{at } t = 0.1.}
\end{align*} \]

5 Conclusion

We have developed a local discontinuous Galerkin method to solve the HS equation. Energy stability is proven for general solutions. Numerical examples are given to illustrate the accuracy and capability of the methods. Although not addressed in this paper, the LDG methods are flexible for general geometry, unstructured meshes and \( h-p \) adaptivity. That will be our future work on solving nonlinear equations in mathematical physics.
Figure 4.16: Solution $u$ for the equation (4.10) the initial condition (4.11) for different $\varepsilon$ at $t = 0.2$.

References


Figure 4.17: Solution $u$ for the equation (4.10) the initial condition (4.11) for different $\varepsilon$ at $t = 0.4$.

Figure 4.18: Energy for zero-dispersion limit with the initial condition (4.11) for different $\varepsilon$.


