Question

Consider an exchange economy with two goods, \(x_1\) and \(x_2\). Consumer A has preferences represented by the Cobb-Douglas utility function

\[ U_A = \left( x_A^1 \right)^\frac{1}{2} \left( x_A^2 \right)^\frac{1}{3} \]  
(1)

and an endowment \(\omega_A = (1,0)\). Consumer B has preferences represented by the Cobb-Douglas utility function

\[ U_B = \left( x_B^1 \right)^\frac{1}{3} \left( x_B^2 \right)^\frac{1}{2} \]  
(2)

and an endowment \(\omega_B = (0,1)\). Find a competitive equilibrium.

Solution

We know that in competitive equilibrium (i) both consumers are making a utility-maximizing choice and (ii) markets must clear. First we will find the demand functions for each consumer to deal with (i). To find each consumer’s income, we calculate the value of the endowment that the consumer owns as a function of prices \(p_1\) and \(p_2\):

\[ m_A = p_1 \omega_A^1 + p_2 \omega_A^2 = (p_1 \ast 1) + (p_2 \ast 0) = p_1, \]
(3)

\[ m_B = p_1 \omega_B^1 + p_2 \omega_B^2 = (p_1 \ast 0) + (p_2 \ast 1) = p_2. \]
(4)

Next, we will solve for each consumer’s optimal choice of good 1. In this example we can use the Cobb-Douglas demand functions that we learned earlier in the course. In general though you might need to use tangency or graphical methods here.

\[ x_A^1 = \frac{c}{c + d} \frac{m_A}{p_1} = \frac{1}{2} + \frac{1}{3} \frac{p_1}{p_1} = \frac{1}{2} \]  
(5)

\[ x_B^1 = \frac{c}{c + d} \frac{m_B}{p_1} = \frac{1}{3} + \frac{2}{3} \frac{p_2}{p_1} = \frac{p_2}{3p_1} \]  
(6)

Now we can exploit our fact (ii) that markets must clear. We know that both consumers like good 1, and so we know that the market clearing condition will hold with equality:

\[ x_A^1 + x_B^1 = \omega_A^1 + \omega_B^1. \]
(7)

We can now combine equations 5, 6 and 7 to find a price ratio that will clear the market for good 1 at each consumer’s optimal choice:

\[ \frac{1}{2} + \frac{p_2}{3p_1} = 1 \]
(9)

\[ \frac{p_1}{p_2} = \frac{2}{3} \]  
(10)
At this price ratio, consumer 1’s optimal choice of good 1 is $\frac{1}{2}$ and consumer 2’s optimal choice of good 1 is $\frac{1}{2}$, which adds up to precisely the total endowment of 1. By Walras’ Law, we know that the market for the remaining good will also clear at this price ratio; let’s make sure this is the case. By our Cobb-Douglas demand functions, we have:

$$x^*_A = \frac{d}{c+d} \frac{m_A}{p_2} = \frac{1}{2} \frac{p_1}{\frac{1}{2} + \frac{1}{2} p_2} = \frac{1}{2} * \frac{2}{3} = \frac{1}{3},$$  \hspace{1cm} (11)$$

$$x^*_B = \frac{d}{c+d} \frac{m_B}{p_2} = \frac{2}{3} \frac{p_2}{\frac{1}{3} + \frac{2}{3} p_2} = \frac{2}{3}. $$  \hspace{1cm} (12)$$

And we’re done! The competitive equilibrium is at $x_A = \left(\frac{1}{2}, \frac{1}{3}\right)$, $x_B = \left(\frac{1}{2}, \frac{2}{3}\right)$, $p_1/p_2 = \frac{2}{3}$. Remember that a competitive equilibrium is an allocation and a price ratio that implements it, so you must write down both things. To check your understanding, what if the price ratio that the auctioneer had called out had instead been $p_1/p_2 = 1$? Why would this not have resulted in a competitive equilibrium? Can you relate the answer to that question to the consumers’ preferences?