1 Explain the welfare theorems (20 points)

There are (at least) two ways you could go with this question. You could focus on the ‘inside the model’ message: the first theorem tells us that a complete set of competitive markets is Pareto efficient, and the second theorem tells us that Pareto efficient outcomes can be achieved as the equilibrium of a complete set of competitive markets as long as we can reallocate the endowment. You could expand on what Pareto efficiency means (all gains from trade are achieved) or on what exactly we mean by complete and competitive markets.

Another way to go would be to talk about potential ‘real world’ messages from the theorems. One interpretation: let prices do the work of allocating goods and signaling the relative scarcity of stuff, rather than dictate prices to achieve more equal distributions of stuff; try where possible to redistribute opportunity and starting conditions rather than redistributing the product of people’s choices.

2 Working with the second welfare theorem (20 points)

a) Since both consumers have convex, monotonic preferences, we know that Pareto efficient points are characterized by tangency between an indifference curve for consumer A and an indifference curve for consumer B. The slope of indifference curves is $\text{MRS}$, and so we are looking for points where $MRS_A = MRS_B$. Calculating: \[ \frac{x_A^2}{x_A^3} = \frac{x_B^2}{x_B^3}. \]

b) By the second welfare theorem, any Pareto efficient point can be supported as a competitive equilibrium with a suitable reallocation of the endowment. (The conditions of the second welfare theorem on the convexity of preferences are satisfied here.) Checking this allocation in the equation from a), we see that $MRS$ for each consumer is $\frac{1}{2}$, and so this allocation is Pareto efficient.

To support it as a competitive equilibrium, the auctioneer needs to do things. First, call the price ratio $\frac{p_1}{p_2} = \frac{1}{2}$ (price ratio equal to the value of $MRS$ for each consumer). Second, move the endowment to any point on the budget line with slope $\frac{1}{2}$ that passes through the required allocation. One such point is the allocation $x_A = (2, 3), x_B = (2, 1)$ itself.

c) Same method as b). $MRS = \frac{2}{3}$ for each consumer at this allocation, so it is Pareto efficient and can be supported as a competitive equilibrium. The auctioneer should set a price ratio $\frac{p_1}{p_2} = \frac{2}{3}$ and move the endowment to a point on the required budget line.

3 Production (20 points)

a) Average product is $\frac{y}{x} = \frac{4}{\sqrt{x}}$. Marginal product is $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$.

b) $\pi = py - wx = 4\sqrt{xp} - wx$. The first order condition for a maximum is $\frac{dx}{dx} = \frac{2p}{\sqrt{x}} - w = 0$, which we can solve for $x = \frac{4p^2}{w^2}$.

c) Back to the production function to find $y^* = 4\sqrt{x^*} = \frac{8p}{w}$. 


d) With \( w = 1, y^* = 8p. \) Sketch this on the usual supply curve axes, output on the x axis and price on the y axis. Notice that if we take different \( w \) values the supply curve moves in the intuitive direction.

4 Jim no function well coffee without (20 points)

The production function has decreasing returns to scale. Scale up both inputs by \( k > 1: 10 \min\{\sqrt{kx_1}, \sqrt{kx_2}\} = \sqrt{k}10 \min\{\sqrt{x_1}, \sqrt{x_2}\}. \) This is less than \( k \) times the original amount.

We know that at the profit-maximizing choice, Jim Corp. will choose \( x_1 = x_2 \) (since otherwise it could reduce whichever is bigger and save on costs without losing any output). Call \( x_1 = x_2 = x. \) Therefore we can be sneaky and write the profit function as a function of just one variable:

\[
\pi = py - c(y) = p10\sqrt{x} - (w_1 + w_2)x = 120\sqrt{x} - 12x
\]

We can maximize this in the usual way. The first order condition is \( 60\sqrt{x} = 12, \) so we solve for \( x^* = 25, \) and the production function tells us that output is then \( y^* = 50. \)

5 Costs and perfect competition (20 points)

a) Fixed costs are 25, variable \( 5y + y^2. \)

b) \( MC = \frac{dc}{dy} = 5 + 2y \) and \( AC = \frac{c}{y} = \frac{25}{y} + 25 + y. \)

c) \( \pi = py - c(y) = 35y - (25 + 5y + y^2). \) At the optimal choice

\[
\frac{d\pi}{dy} = 0 \quad (4)
\]
\[
35 - 5 - 2y = 0 \quad (5)
\]
\[
y^* = 15. \quad (6)
\]

At this output, \( \pi = (35 \times 15) - (25 + (5 \times 15) + 15^2) = 200. \)

d) In the long run, since this industry is perfectly competitive, the supply curve is where price equals minimum average cost:

\[
\frac{dAC}{dy} = 0 \quad (7)
\]
\[
-\frac{25}{y^2} + 1 = 0 \quad (8)
\]
\[
y = 5. \quad (9)
\]

Putting this back into average cost, we get \( p = 15 \) at the minimum.

e) We already got \( y = 5 \) in the previous part as the output that minimizes average cost. So \( \pi = (15 \times 5) - (25 + (5 \times 5) + 5^2) = 0, \) which is as we expected in a long run perfectly
competitive industry.