1. Let \( W^{(1)} \) and \( W^{(2)} \) be skew-symmetric tensors with axial vectors \( w^{(1)} \) and \( w^{(2)} \) respectively. Show that

(a) \( W^{(1)} W^{(2)} = w^{(1)} \otimes w^{(2)} - (w^{(1)} \cdot w^{(2)}) I \)

(b) \( \text{tr} (W^{(1)} W^{(2)}) = -2 w^{(1)} \cdot w^{(2)} \)

By making appropriate choices of \( W^{(1)} \) and \( W^{(2)} \), deduce from (b) the vector identity

\[(a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \quad \forall \ a, b, c, d, \in \mathbb{E}.\]

Soln.

(a) Using the relation between the Skew-Symmetric matrix and its axial vector from Ogden Pg. 24 (One can easily show this by writing \( W a = w \wedge a \) in indicial notation)

\[ W^{(1)}_{ij} = -\epsilon_{ijk} w^{(1)}_k \]  
\[ W^{(2)}_{jl} = -\epsilon_{jlr} w^{(2)}_r \]  

\[ W^{(1)}_{ij} W^{(2)}_{jl} = (-\epsilon_{ijk} w^{(1)}_k)(-\epsilon_{jlr} w^{(2)}_r) \]
\[ = (\epsilon_{ijk} \epsilon_{jlr}) W^{(1)}_k W^{(2)}_r \]
\[ = - (\epsilon_{ijk} \epsilon_{jlr}) W^{(1)}_k W^{(2)}_r \]
\[ = - (\delta_{il} \delta_{kr} - \delta_{ir} \delta_{kl}) W^{(1)}_k W^{(2)}_r \]
\[ = (\delta_{il} \delta_{kr} - \delta_{ir} \delta_{kl}) W^{(1)}_k W^{(2)}_r \]
\[ = w^{(1)}_i w^{(2)}_l - (w^{(1)}_i w^{(2)}_k) \delta_{il} \]
\[ = (w^{(1)}_i w^{(2)}_l - (w^{(1)}_i w^{(2)}_k) \delta_{il} \]

\[ (W^{(1)} W^{(2)})_{il} = (w^{(2)} \otimes w^{(1)})_{il} - (w^{(1)} \cdot w^{(2)}) \delta_{il} \]  
\[ W^{(1)} W^{(2)} = w^{(2)} \otimes w^{(1)} - (w^{(1)} \cdot w^{(2)}) I \]
\[ \text{tr} \left( W^{(1)} W^{(2)} \right) = \text{tr} \left( w^{(2)} \otimes w^{(1)} \right) - (w^{(1)} \cdot w^{(2)}) \text{tr} I \]  

Using the fact \( \text{tr} a \otimes b = a \cdot b \) and \( \text{tr} I = 3 \)

\[ \text{tr} \left( W^{(1)} W^{(2)} \right) = (w^{(2)} \cdot w^{(1)}) - 3(w^{(1)} \cdot w^{(2)}) \]

\[ = -2(w^{(2)} \cdot w^{(1)}) \quad (7) \]

(c)

Now choose
\[ W^{(1)} = (a \otimes b) - (b \otimes a) \] and
\[ W^{(2)} = (c \otimes d) - (d \otimes c) \]

Then the axial vector are \( b \wedge a \) and \( d \wedge c \) respectively(Chadwick pg. 30)

\[ W^{(1)}_{ij} W^{(2)}_{jl} = (a \otimes b - b \otimes a) \cdot (c \otimes d - d \otimes c) \]

\[ = (a \otimes b), (c \otimes d) - (a \otimes b), (d \otimes c) - (b \otimes a), (c \otimes d) + (b \otimes a)(d \otimes c) \]

\[ = (a \otimes d)(b \cdot c) - (a \otimes c)(b \cdot d) - (b \otimes d)(a \cdot c) + (b \otimes c)(a \cdot d) \]

\( (8) \)

Now taking the trace of the above equation

\[ \text{tr} \left( W^{(1)} W^{(2)} \right) = \text{tr} \left( (a \otimes d)(b \cdot c) \right) - \text{tr} \left( (a \otimes c)(b \cdot d) \right) - \text{tr} \left( (b \otimes d)(a \cdot c) \right) + \text{tr} \left( (b \otimes c)(a \cdot d) \right) \]

Using the fact \( \text{tr} (\alpha A) = \alpha \text{tr} A \) and \( \text{tr} (u \otimes v) = u \cdot v \)

\[ \text{tr} \left( W^{(1)} W^{(2)} \right) = (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d) - (b \cdot d)(a \cdot c) + (b \cdot c)(a \cdot d) \]

\[ = 2(a \cdot d)(b \cdot c) - 2(a \cdot c)(b \cdot d) \quad (10) \]

Now in the Eqn(7) Substitute the axial vectors \( w^{(1)} = b \wedge a \) and \( w^{(2)} = d \wedge c \)

\[ \text{tr} \left( W^{(1)} W^{(2)} \right) = -2(b \wedge a) \cdot (d \wedge c) \]

\[ = -2(a \wedge b) \cdot (c \wedge d) \quad (11) \]

LHS of Eqns(10 and 11) are one and the same, hence on equating the RHS of both the equation we get

\[ (a \wedge b) \cdot (c \wedge d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \]  

(12)
2). Let \( \phi u \) and \( T \) be differentiable scalar, vector and tensor fields. Show that

(i) \( \text{grad}(\phi u) = u \otimes \text{grad} \phi + \phi \text{grad} u \)

(ii) \( \text{div}(Tu) = u \cdot \text{div} T + \text{tr}(T \text{grad} u) \)

(iii) \( \text{grad}(\phi T) = T^T \text{grad} \phi + \phi \text{div} T \)

Sln.

(i) Let \( v = \phi u \)

\[
\text{grad } v = \frac{\partial v_p}{\partial x_q} e_p \otimes e_q
\]

\[
= \left[ \frac{\partial (\phi u)_p}{\partial x_q} \right] e_p \otimes e_q
\]

\[
= \left[ \phi \frac{\partial u_p}{\partial x_q} + \frac{\partial \phi}{\partial x_q} u_p \right] e_p \otimes e_q
\]

\[
= \left[ \phi \frac{\partial u_p}{\partial x_q} e_p \otimes e_q + \frac{\partial \phi}{\partial x_q} u_p e_p \otimes e_q \right]
\]

\[
= \phi \text{grad } u + u \otimes \text{grad } \phi
\] (13)

(ii) Let \( v = Tu \)

\[
\text{div } v = \frac{\partial v_p}{\partial x_p}
\]

\[
= \frac{\partial (Tu)_i}{\partial x_i} u_i
\]

\[
= \frac{\partial T_{pi}}{\partial x_p} u_i + T_{pi} \frac{\partial u_i}{\partial x_p}
\]

\[
= u \cdot \text{div } T + \text{tr}(T \text{grad } u)
\] (14)

(iii) Let \( P = \phi T \)

\[
\text{div } P = \frac{\partial P_{ij}}{\partial x_i} e_j
\]

\[
= \frac{\partial (\phi T)_{ij}}{\partial x_i} e_j
\]

\[
= \phi \frac{\partial T_{ij}}{\partial x_i} + T_{ij} \frac{\partial \phi}{\partial x_i}
\]

\[
= \phi \text{div } T + T^T \text{grad } \phi
\] (15)
3). A Pure strain is defined by the deformation gradient

\[ A = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + e_3 \otimes e_3, \]

where \( e_1, e_2, e_3 \) are mutually orthogonal unit vectors. If two line elements are aligned with \( M = \cos \phi e_1 + \sin \phi e_2 \) and \( M' = -\sin \phi e_1 + \cos \phi e_2 \) in the reference configuration, calculate the change in the angle between them due to the deformation. Deduce that the maximum change among all such pairs of line elements is

\[ \sin^{-1}\left( \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2} \right) \]

Soln.

Strain is defined as

\[ A = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + e_3 \otimes e_3 \]  \hspace{1cm} \hspace{1cm} (16)

The matrix representation of \( A \) in the basis of \( e_1, e_2, e_3 \) is

\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]  \hspace{1cm} \hspace{1cm} (17)

\[ M = \cos \phi e_1 + \sin \phi e_2; \quad M' = \begin{bmatrix}
\cos \phi \\
\sin \phi \\
0 \\
\end{bmatrix} \]  \hspace{1cm} \hspace{1cm} (18)

\[ M' = -\sin \phi e_1 + \cos \phi e_2; \quad M' = \begin{bmatrix}
-\sin \phi \\
\cos \phi \\
0 \\
\end{bmatrix} \]  \hspace{1cm} \hspace{1cm} (19)

angle between \( M \) and \( M' \) is

\[ \cos \Theta = M \cdot M' = 0 \]  \hspace{1cm} \hspace{1cm} (21)

so, \( \Theta = (n + \frac{1}{2})\pi; \quad \Theta = \frac{\pi}{2}, \frac{3\pi}{2} \) but since \( M' = -\sin \phi e_1 + \cos \phi e_2 \), as opposed to, \( \sin \phi e_1 - \cos \phi e_2 \), hence \( \Theta = \frac{\pi}{2} \)

After deformation

\[ m = A M = \begin{bmatrix}
\lambda_1 \cos \phi \\
\lambda_2 \sin \phi \\
0 \\
\end{bmatrix} \]  \hspace{1cm} \hspace{1cm} (22)

\[ m' = A M' = \begin{bmatrix}
-\lambda_1 \sin \phi \\
\lambda_2 \cos \phi \\
0 \\
\end{bmatrix} \]  \hspace{1cm} \hspace{1cm} (23)
so, unit vectors along \( m' \) and \( m \) are

\[
\begin{align*}
m_1 &= \frac{1}{\sqrt{\lambda_1^2 \sin^2 \phi + \lambda_2^2 \cos^2 \phi}} \langle \lambda_1 \cos \phi, \lambda_2 \sin \phi, 0 \rangle^T \quad (24) \\
m_1' &= \frac{1}{\sqrt{\lambda_1^2 \cos^2 \phi + \lambda_2^2 \sin^2 \phi}} \langle -\lambda_1 \sin \phi, \lambda_2 \cos \phi, 0 \rangle^T \quad (25)
\end{align*}
\]

so,

\[
\cos \theta = m_1 \cdot m_1' = \frac{(\lambda_2^2 - \lambda_1^2) \cos \phi \sin \phi}{\sqrt{\lambda_2^2 \sin^2 \phi + \lambda_2^2 \cos^2 \phi}} \quad (26)
\]

Let \( \alpha = \Theta - \theta = \frac{\pi}{2} - \theta \)

we are looking for maximum \( \alpha \)

take \( \sin \alpha = \cos \theta \)

hence for maximum \( \alpha \) we have to find maximum \( \cos \theta \)

If one looks at the Eqn(26) and interchanges \( \cos \phi \) by \( \sin \phi \) the Eqn. remains unchanged hence the maximum of Eqn(26) is obtained by solving the maximum of either the numerator or denominator in the LHS.

The maximum of numerator is easily obtained to be when \( \cos \phi = \sin \phi \),

Hence \( \phi = \frac{\pi}{4}, \frac{3\pi}{4} \)

so on substituting either \( \phi = \frac{\pi}{4}, \frac{3\pi}{4} \) in Eqn(26) we obtain

\[
\sin \alpha_{max} = \frac{|\lambda_2^2 - \lambda_1^2|}{\lambda_1^2 + \lambda_2^2} \quad (27)
\]

or,

\[
\alpha_{max} = \sin^{-1} \left( \frac{|\lambda_2^2 - \lambda_1^2|}{\lambda_1^2 + \lambda_2^2} \right) \quad (28)
\]