EN221 - Fall2008 - HW # 9 Solutions

Prof. Vivek Shenoy

1.) Consider the observer transformation discussed in class is defined by the relation

\[ x^* = c(t) + Q(t)x \]  

where \( Q \) is an orthogonal tensor. (a) Show that

\[ \text{div}_x^* \mathbf{T}^*(x^*, t) = Q(t) \text{div}_x \mathbf{T}(x, t) \]  

(b) Show that the acceleration transforms according to

\[ \dot{v}^*(x^*, t) = Q(t) \dot{v}(x, t) + \ddot{c}(t) + 2Q(t)v(x, t) + Q(t)x \]  

(c) Show further that if the body force transforms according to

\[ b^*(x^*, t) = Q(t)b(x, t) \]

then the equation of motion is given by

\[ \text{div}_x^* \mathbf{T}^* + \rho^* b^* + k = \rho^* \dot{v}^* \]

where \( \rho^*(x^*) = \rho(x, t) \) and determine a relation for \( k \)

(d) because of the additional term \( k \), the equation of motion is not not invariant under all changes in observer. Show that \( k \) vanishes for observer for whom \( Q \) and \( \dot{c} \) are constant. Such observer are called Galilean and have the property of being accelerationless with respect to the underlying inertial observer.

Soln.
The observer transformation is

\[ x^* = c(t) + Q(t)x \]

(a) \[ \text{div}_x^* \mathbf{T}^* = \frac{\partial T^*_{ij}}{\partial x^*_i} e^*_j \]

\[ T^*_{ij}(x^*, t) = T_{ij}(x, t) \]

\[ \frac{\partial T^*_{ij}(x^*, t)}{\partial x} = \frac{\partial T_{ij}(x, t)}{\partial x} \]
\[
\begin{align*}
\text{div}_x \cdot T^* &= \frac{\partial T_{ij}}{\partial x_i} e_j \\
&= \frac{\partial T_{ij}}{\partial x_i} Q(t) e_j \\
&= Q(t) \frac{\partial T_{ij}}{\partial x_i} e_j \\
&= Q(t) \text{div}_x T
\end{align*}
\]

(b)
\[
\begin{align*}
\text{div}_x \cdot T^* &= Q(t) \text{div}_x T \\
&= Q(t)(-\rho b + \rho \dot{v}) \\
&= -\rho Q(t)b + \rho Q(t)\dot{v}
\end{align*}
\]

But,
\[
\begin{align*}
x^* &= c(t) + Q(t)x \\
\dot{x}^* &= \dot{v}^* = \ddot{c}(t) + Q(t)\ddot{x} + \dot{Q}(t)x \\
\ddot{x}^* &= \dddot{v}^* = \dddot{c}(t) + Q(t)\dddot{x} + \ddot{Q}(t)x + 2\dot{Q}(t)\dot{x} \\
&= \dddot{c}(t) + Q(t)\dddot{v} + \dot{Q}(t)x + 2\dot{Q}(t)v
\end{align*}
\]

Thus,
\[
\begin{align*}
\text{div}_x \cdot T^* &= -\rho Q(t)b + \rho Q(t)\dot{v} \\
&= -\rho Q(t)b + \rho \left[ \dddot{v}^* - \dddot{c}(t) - \dot{Q}(t)x - 2\dot{Q}(t)v \right] \\
\end{align*}
\]

using \( \rho^* = \rho \)
\[
\begin{align*}
\text{div}_x \cdot T^* + \rho^* b^* + \rho^* \left[ \dddot{c}(t) + \dddot{Q}(t)x + 2\dot{Q}(t)v \right] &= \rho^* \dddot{v}^* \\
\end{align*}
\]

where Internal body force in the accelerating frame is
\[
k = \dddot{c}(t) + \dddot{Q}(t)x + 2\dot{Q}(t)v \text{ if } Q \text{ and } c \text{ are constant then } k = 0
2.) The Transformation rule for the Cauchy stress tensor $\mathbf{T}$ for relative motion of observers is

$$ T^* = QTQ^T \quad (10) $$

Tensors that transform in this manner are called objective tensors. Is the material rate of change (or material derivative) of Cauchy stress $\dot{T}$ objective?, i.e., is $\dot{T}^* = Q\dot{T}Q^T$?

(b) Show that if the Cauchy stress tensor is objective, then the Jaumann stress rate defined by

$$ \dot{T} = \dot{T} - WT + TW \quad (11) $$

is objective. Note that $W$ in the above equation represents the spin tensor, $\frac{1}{2}(L - L^T)$. This stress rate is also called the co-rotational rate of Cauchy stress. Why is this a reasonable name for this stress rate?

Soln.

(a)

$$ T^* = QTQ^T $$
$$ \dot{T}^* = \dot{QTQ}^T + QT \dot{Q}^T + Q\dot{T}Q^T \quad (12) $$

Additional terms $\dot{QTQ}^T + QT \dot{Q}^T$, so $\dot{T}$ is not objective

(b)

$$ \dot{T} = \dot{T} - WT + TW $$
$$ \dot{T}^* = \dot{T}^* - W^*T^* + T^*W^* \quad (13) $$

$$ \Omega^* = \dot{Q}Q^T = -Q\dot{Q}^T \quad (14) $$
$$ W^* = QWQ^T + \Omega \quad (15) $$

Substituting Eqn(12 and 15) in Eqn(13)

$$ \ddot{T}^* = \dot{QTQ}^T + QT \dot{Q}^T + Q\dot{T}Q^T - QWQ^T T^* - \dot{Q}Q^T T^* + T^*QWQ^T - T^*Q\dot{Q}^T $$

$$ = \dot{QTQ}^T + QT \dot{Q}^T + Q\dot{T}Q^T - QWQ^T QTQ^T - \dot{Q}Q^T QTQ^T + QTQ^T QWQ^T - QTQ^T Q\dot{Q}^T $$

$$ = \dot{QTQ}^T + QT \dot{Q}^T + Q\dot{T}Q^T - QWTQ^T - \dot{QTQ}^T + QTWQ^T - QTQ^T $$

$$ = Q(\dot{T} - WT + TW)Q^T \quad (16) $$

Thus Jaumann stress rate is frame independent.

This is the rate of change of $\mathbf{T}$ relative to a basis rotating with the local body spin $\Omega$. 

3
3. For each of the following constitutive equations decide whether or not the principle of objectivity is satisfied. (α and β are scalar constants, p a scalar valued function and f a symmetric tensor-valued function.)

(i)  \[ \sigma = -p(t)I \]
(ii)  \[ \sigma = \alpha(F + F^T) \]
(iii)  \[ \sigma = f(v) \]
(iv)  \[ \sigma = \alpha \{ \text{grad} \ a + (\text{grad} \ a)^T + 2L^T L \} \]
(v)  \[ \sigma = f(b) \]
(vi)  \[ \dot{\sigma} = W\sigma - \sigma W + (\alpha \text{ tr } D)I + \beta D \]

Soln.

(i)
\[
\sigma = -p(t)I \\
\sigma^*(x^*, t^*) = -p(t^*)I \\
= -p(t - t_0^*)I
\]

(17)

\[ t^* = t - t_0^*; \ t = t_0^* \] is origin for \( t^* \) (There is some origin in time \( t_0(t_0^*) = 0 \) but, \( t - t_0^* = t \)
\[ \Rightarrow \sigma(x^*, t^*) = -p(t)I = \sigma(x, t) \]

(ii)
\[
\sigma = \alpha(F + F^T) \\
\sigma^* = \alpha(F^* + F^{*T}) \\
F^* = *QF \\
\Rightarrow \sigma^* = \alpha(QF + F^{T}Q^{T}) \\
= \alpha(Q(F + F^{T})Q^{T} - \alpha QFQ^T - Q
\]

so, \( \sigma^* \) is not objective

(iii)
\[
\sigma = f(v) \\
\sigma^* = f(v^*) \\
= f(\dot{c} + \dot{Q}x + Qv) \\
= Qf(v)Q^T \text{ for the relation to be objectivet} \quad (18) \\
\Leftrightarrow f(\dot{c} + \dot{Q}x + Qv) = Qf(v)Q^T \quad (19)
\]

putting \( v = 0 \) and \( Q = I \)
\[
f(\dot{c}) = f(0) \quad \forall \dot{c} \quad (20)
\]
⇒ \( f(v) \) is a constant = \( C \)

\[
f(\ddot{c} + \dot{Q}x + Qv) = Qf(v)Q^T
\]  

(21)

putting \( c = 0 \) and \( \dot{Q} = 0 \)

\[
⇒ f(Qv) = C = QCQ^T \quad \forall Q
\]  

(22)

putting a few values of \( Q \) we will obtain

\( C = pI \); where \( p \) is a constant is a necessary and sufficient for objectivity

(iv)

\[
σ = α\{\text{grad } a + (\text{grad } a)^{T} + 2L^{T}L\}
\]  

(23)

Let \( A_2 = \text{grad } a + (\text{grad } a)^{T} + 2L^{T}L \) from problem 4.1 Chadwick

From the hint given on pg. 169 and problem 4.1

\[
A_2 = A_1 \quad (A_1 = 2D) \\
A_1 = \dot{A}_1 + L^T A_1 + A_1 L
\]

\[
= 2 \left( \frac{\dot{L} + L^T}{2} \right) + L^T 2 \left( \frac{L + L^T}{2} \right) + 2 \left( \frac{L + L^T}{2} \right) L
\]

\[
= \dot{L} + \dot{L}^T + L^2 + L^T L + 2L^T L
\]  

(24)

\[
⇒ σ = αA_2
\]  

(25)

\( A_1 \) is \( 2D \) and clearly objective (pg. 134 Chadwick)

\( A_2 = A_1 \)

and from problem 4 (pg. 134)

Since \( A_1 \) is objective \( A_2 \) is objective

Thus \( σ = αA_2 \) is objective

(v)

\[
σ = f(b)
\]  

(26)

Using Problem 1 of the HW

\[
b^* = b + (\ddot{c} + 2\dot{Q}v + \ddot{Q}x)
\]

\[
⇒ σ^* = f(b^*)
\]

\[
= f(b + \ddot{c} + 2\dot{Q}v + \ddot{Q}x)
\]  

(27)

\[
σ^* = QσQ^T
\]  

(28)

\[
⇔ f(b + \ddot{c} + 2\dot{Q}v + \ddot{Q}x) = Qf(b)Q^T
\]  

(29)

Making \( \dot{Q} = 0, \ddot{Q} = 0, b = 0, Q = I \)

\[
f(\ddot{c}) = Qf(0)Q^T = f(0) \quad \forall e
\]  

(30)
Thus, \( f(\ddot{c}) = f(0) = C(\text{constant}) \) Putting \( \ddot{c} = \ddot{Q} = \dddot{Q} = 0 \)

\[
f(b) = C = QCQ^T
\]  

(31)

Thus, by the same argument as in (iii) \( c = bI \), where \( b \) is some constant, is necessary and sufficient for objectivity.

(vi)

\[
\dot{\sigma} = W\sigma - \sigma W + (\alpha \text{ tr } D)I + \beta D
\]

\[
\dot{\sigma}^* = W^*\sigma^* - \sigma^* W^* + (\alpha \text{ tr } D^*)I + \beta D^*
\]

\[
W^* = QWQ^T + \Omega
\]

\[
\Omega = \dot{Q}Q^T
\]

\[
\sigma^* = Q\sigma Q^T
\]

\[
D^* = QDQ^T
\]

\[
\Rightarrow \text{ tr } D^* = \text{ tr } (QDQ^T) = \text{ tr } (Q^TQD) = \text{ tr } D
\]  

(32)

\[
\dot{\sigma}^* = (QWQ^T + \Omega)Q\sigma Q^T - Q\sigma Q^T(QWQ^T + \Omega) + \alpha \text{ tr } D + \beta Q\sigma Q^T
\]

\[
= QW\sigma Q^T + \dot{Q}Q^TQ\sigma Q^T - Q\sigma WQ^T + Q\sigma Q^TQ^T\sigma Q^T + Q^T\alpha \text{ tr } (D)IQ^T + \beta(QDQ^T)
\]

\[
= Q(W\sigma + \sigma W + \alpha \text{ tr } (D)I + \beta D)Q^T + \dot{Q}\sigma Q^T + Q\dot{\sigma} Q^T
\]

\[
= Q\dot{\sigma} Q^T + \dot{Q}\sigma Q^T + Q\sigma Q^T
\]

\[
= (Q\sigma Q^T)
\]  

(33)

\[
\Rightarrow \dot{\sigma} \text{ is objective.}
\]
4). The stress response of a certain type of material which exhibits both elastic and viscous properties is described by the consecutive equation

\[ \sigma = f(F, \dot{F}) \] (34)

\( f \) being a symmetric tensor-valued function. Investigate the restriction imposed on \( f \) by the principle of objectivity and hence show that the most general form of Eqn (34) is

\[ F^* = QF \] (35)

\[ \Rightarrow \sigma^* = f(F^*, \dot{F}^*) = Qf(F, \dot{F})Q^T \] (for objectivity)

\[ \Rightarrow f(QF, (QF)) = Qf(F, \dot{F})Q^T \]

writing \( F = RU \), we get

\[ f(QRU, (QRU\dot{)}) = Qf(F, \dot{F})Q^T \]

\[ (QRU\dot{)} = \dot{QRU} + Q\dot{RU} + QR\dot{U} \] (36)

Since \( Q \) is arbitrary, on putting \( Q = R^T \)

\[ (QRU\dot{)} = \dot{R}^T RU + R^T \dot{R}U + R^T R\dot{U} \]

\[ = (\dot{R}^T R + R^T \dot{R})U + \dot{U} \]

but \( R^T R = I \) and

\( \dot{R}^T R + R^T \dot{R} = 0 \)

hence

\[ (QRU\dot{)} = \dot{U} \]

\[ \Rightarrow f(U, \dot{U}) = Qf(F, \dot{F})Q^T = R^T f(F, \dot{F})R \]

\[ \Rightarrow f(U, \dot{U}) = \sigma = R^T f(U, \dot{U})R \] (37)