\[ \mathbf{\dot{y}} = \dot{x} \hat{i} + \dot{\theta} \hat{j} \]

For mass \( m_0 \), velocity
\[ v = \dot{x} \hat{i} + \dot{\theta} \hat{j} \]

For mass \( m \), velocity
\[ v = \dot{x} \hat{i} + \dot{\theta} \hat{j} + \dot{\phi} \hat{k} \]

\[ (\dot{x} + \dot{\phi} \cos \theta)^2 + \dot{\phi} \sin \theta \]

\[ \Rightarrow \text{Kinetic Energy} \quad T = \frac{1}{2} m_0 \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + \dot{\theta}^2 + 2 \dot{\phi} \dot{\theta} \cos \theta) \]

Potential Energy
\[ V = \frac{1}{2} k (x - l_0)^2 + m g l (1 - \cos \theta) \]

Lagrangian
\[ L = T - V \]

Non-conservative force
\[ \vec{F}_{nc} = - \frac{\partial}{\partial \theta} (c \theta) \rightarrow \vec{F}_{nc} = - c \theta \hat{j} \]

\[ Q_x = 0, \quad Q_\theta = 0 \]

Then,
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = Q_{x \dot{x}} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = Q_{\theta \dot{\theta}} \]

Equations of motion:

17 \( (m + m_0) \ddot{x} + m \dot{x} (\dot{\theta} \cos \theta - \dot{\phi} \sin \theta) + k (x - l_0) + c \ddot{\theta} = 0 \)

27 \( x \ddot{x} + \dot{x} \cos \theta + g \sin \theta = 0 \)

Kinetic Energy:

$$T = \frac{1}{2} 3m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m \ell^2 \dot{\theta}^2$$

(1.1)

Non-holonomic Constraint:

$$\ell = \ell \theta - x \sin \theta + y \cos \theta = 0$$

(1.2)

Lagrange’s Equations:

$$3m \ddot{x} = -\lambda \sin \theta$$

(1.3)

$$3m \ddot{y} = \lambda \cos \theta$$

(1.4)

$$m \ell^2 \ddot{\theta} = -\frac{1}{2\sqrt{3}} \ell \lambda$$

(1.5)

Eliminate $\lambda$:

$$\dot{x} \cos \theta + \dot{y} \sin \theta = 0$$

(1.6)

$$\ell \ddot{\theta} + \frac{\sqrt{3}}{2} \dot{x} \sin \theta - \frac{\sqrt{3}}{2} \dot{y} \cos \theta = 0$$

(1.7)

Introduce basis $e_x, e_y$:

$$e_x = i \cos \theta + j \sin \theta$$

(1.8)

$$e_y = -i \sin \theta + j \cos \theta$$

Note that for $\vec{v} = \vec{v}_x + \vec{v}_y$, Eqn. (1.6) corresponds to:

$$\dot{e}_x = 0 \Rightarrow \dot{\vec{v}}_x = 0$$

(1.9)

and Eqn. (1.5) corresponds to:

$$\ell = \ell \theta + \vec{v}_y = 0.$$ 

(1.10)

Eliminating $\vec{v}_y$ from Eqns. one obtains

$$\ddot{\vec{v}} + \frac{\ell}{2\sqrt{3}} \vec{v} = 0$$

(1.11)

whereas differentiation of (1.5) gives

$$\frac{5\ell}{2\sqrt{3}} \dot{\theta} - \theta \ddot{\theta} = 0.$$ 

(1.12)

Eliminating $\vec{v}$ between (1.11) and (1.12) gives

$$5\dot{\theta} - 5(\dot{\theta})^2 + (\ddot{\theta})^2 = 0.$$ 

(1.13)

Eqn. (1.13) is a nonlinear ordinary differential equation for $\dot{\theta}(t)$. Because this equation is homogeneous (of degree 4) in the time derivatives, this equation does not determine the time dependence of $\dot{\theta}(t)$. Instead it suggests looking for a solution $\dot{\theta} = \dot{\theta}(\theta)$. Because exponentials can cancel out as a common factor we look for a solution of the form
\[ \dot{\theta} = \exp\left[f(\theta)\right] \]  

(1.14)

for which

\[ \dot{\theta} = \left[\exp\left[f(\theta)\right]\right]^2 f'(\theta) \]

\[ \dot{\theta} = \left[\exp\left[f(\theta)\right]\right]^2 \left[f'(\theta) + 2\left(f'(\theta)^2\right)\right]. \]

(1.15)

Substitution of (1.14) and (1.15) into (1.13) and cancellation of the common exponential term leads to the following differential equation for \( f(\theta) \).

\[ 5f''(\theta) + 5\left[f'(\theta)^2\right] + 1 = 0 \]

(1.16)

Equation (1.16) has the solutions

\[ f'(\theta) = \pm \frac{i}{\sqrt{5}}, \quad i = \sqrt{-1}. \]

(1.17)

Integration of (1.17) gives

\[ f(\theta) = \pm i \theta + c \]

(1.18)

where \( c \) is a constant. Substitution of (1.18) into (1.14) gives a solution of the form

\[ \dot{\theta} = C \left[ \cos\left(\theta / \sqrt{5}\right) \pm i \sin\left(\theta / \sqrt{5}\right) \right] \]

(1.19)

where \( C = \exp[c] \) is a constant to be determined from the initial condition for \( \dot{\theta}(0) \), which, from the initial conditions \( \theta(0) = 0, \quad \dot{\theta}(0) = \nu_0 \), and the non-holonomic constraint (1.5) is given by

\[ C = -\frac{2\sqrt{3} \nu_0}{\ell}. \]

(1.20)

Substitution of (1.20) into (1.19) and selecting only the real part of (1.19), which is also a solution of (1.13), the solution of (1.13) becomes

\[ \dot{\theta} = \frac{2\sqrt{3} \nu_0}{\ell} \cos\left(\theta / \sqrt{5}\right) \]

(1.21)

for \( \left[\theta / \sqrt{5}\right] \leq -\pi / 2 \); once \( \theta \) becomes equal to \( -\sqrt{5}\pi / 2 \) it becomes constant and stays at this value indefinitely. Substitution of (1.21) into (1.4) gives

\[ \lambda = \frac{12\sqrt{3} \nu_0^2}{\ell \sqrt{5}} \sin\left(2\theta / \sqrt{5}\right) \]

(1.22)

for \( \left[\theta / \sqrt{5}\right] \geq -\pi / 2 \); afterwards, \( \lambda = 0, \quad \nu_0 = 0, \quad \nu = -\sqrt{5} \nu_0 \).
\[ \epsilon'_1 = \frac{1}{\sqrt{3}} (\epsilon_1 + \epsilon_2 + \epsilon_3) \]
\[ \epsilon'_2 = \frac{1}{16} (\epsilon_1 + \epsilon_2 - 2\epsilon_3) \]
\[ \epsilon'_3 = \frac{1}{\sqrt{2}} (-\epsilon_1 + \epsilon_2) \]

\[ I \Rightarrow I_{xx} = I_{yy} = I_{zz} = \frac{ma^2}{6} \] (for cube of side a)

\[ [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{ma^2}{6} \]

\[ [K] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \]

\[ [I'] = [K][I][K]^T = \frac{ma^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(Since \( I \) is a diagonal matrix)