Linear elasticity solution in polar coordinates

Typical problems: Stress around a circular hole in an elastic solid.

Boundary conditions:

Traction free $r = a$: \( \sigma_r \vec{e}_r + \sigma_{\theta \phi} \vec{e}_\theta = 0 \), i.e. \( \sigma_r = 0 \), \( \sigma_{\theta \phi} = 0 \)

Boundary conditions:

\( \sigma_r = -p \), \( \sigma_{\theta \phi} = 0 \) \( r = a \)

\( \sigma_r = 0 \), \( \sigma_{\theta \phi} = 0 \) \( r = b \)

Governing equation: \( \nabla^2 \nabla^2 \phi = 0 \)

In Cartesian coordinates: \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0 \)
Proposition: use \((r, \theta)\) instead of \((x, y)\), \(\phi = \phi(r, \theta)\)

\[
\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}
\]

\[
\nabla^2 = \nabla \cdot \nabla \left( \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \right) = \left( \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \right) \left( \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \right)
\]

\[
\tilde{e}_r = \cos \theta \tilde{e}_x + \sin \theta \tilde{e}_y
\]

\[
\tilde{e}_\theta = -\sin \theta \tilde{e}_x + \cos \theta \tilde{e}_y
\]

\[
\frac{\partial \tilde{e}_r}{\partial \theta} = -\sin \theta \tilde{e}_x + \cos \theta \tilde{e}_y = \tilde{e}_\theta
\]

\[
\frac{\partial \tilde{e}_\theta}{\partial \theta} = -\cos \theta \tilde{e}_x - \sin \theta \tilde{e}_y = -\tilde{e}_r
\]

It follows from above that \(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\)

Governing equation in polar coordinates: \(\phi = \phi(r, \theta)\)

\[
\nabla^2 \nabla^2 \phi = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0
\]

Stress components in Cartesian coordinates:

\[
\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2}
\]

\[
\sigma_{xx} + \sigma_{yy} = \nabla^2 \phi
\]

Stress components in polar coordinates:

\[
\sigma_{rr} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right), \quad \sigma_{\theta\theta} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}
\]

Equilibrium equations in polar coordinates:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0
\]

\[
\frac{\partial \sigma_{\theta\theta}}{\partial r} - \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2 \sigma_{r\theta}}{r} + f_\theta = 0
\]
Hooke’s law in polar coordinates:
\[
\varepsilon_{rr} = \frac{1}{E} \left( \sigma_{rr} - \nu \sigma_{\theta \theta} \right) \]
\[
\varepsilon_{\theta \theta} = \frac{1}{E} \left( \sigma_{\theta \theta} - \nu \sigma_{rr} \right) \]
\[
\varepsilon_{r \theta} = \frac{1 + \nu}{E} \sigma_{r \theta} \]

Strain-displacement relations in polar coordinates:
\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta \theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad 2 \varepsilon_{r \theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \]

Example 1: Thick-walled pressure vessel

Since the problem is axisymmetric,
\[
\phi = \phi(r) \]
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) \]

Boundary conditions:
\[
\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} = -p_i, \quad \sigma_{r \theta} = 0 \quad @ \quad r = a \]
\[
\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} = -p_0, \quad \sigma_{r \theta} = 0 \quad @ \quad r = b \]

In mathematical description, the problem becomes an ordinary differential equation
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) \phi = 0 \]

with boundary conditions
\[
\phi'(a) = -ap_i
\]
\[
\phi'(b) = -bp_0
\]

The above differential equation can be directly integrated and has the solution
\[
\phi = A \ln r + Br^2 \ln r + Cr^2 + D
\]

The constant term \( D \) is nothing but a rigid body motion and can be neglected in stress analysis, i.e. \( D = 0 \).

The tangential displacement associated with the term \( Br^2 \ln r \) comes out to be
\[
u_\theta = \frac{4B}{E} r \theta
\]
plus a rigid body motion, which is not a single-valued function. Actually, the term \( Br^2 \ln r \) represents a so-called disclination (think of gluing a cut-opened ring back into a circle). For the present problem, take \( B = 0 \). Therefore, the solution to thickwalled cylinder is
\[
\phi = A \ln r + Cr^2
\]

\[
\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{A}{r^2} + 2C
\]

\[
\sigma_{\theta \theta} = \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + 2C
\]

The constants \( A, \ C \) are determined from the boundary conditions:
\[
\sigma_{rr} \bigg|_{r=a} = \frac{A}{a^2} + 2C = -p_i
\]
\[
\sigma_{rr} \bigg|_{r=b} = \frac{A}{b^2} + 2C = -p_0
\]

The results are
\[
A = \frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2}
\]
\[
2C = \frac{p_i a^2 - p_0 b^2}{b^2 - a^2}
\]

In the special case of \( p_0 = 0 \):
\[
\sigma_{rr} = \frac{a^2 p_i}{b^2 - a^2} \left( 1 - \frac{b^2}{r^2} \right) \quad (< 0, \text{ compressive})
\]
Consider the tensile hoop stress,

\[ \sigma_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2} \left( 1 + \frac{b^2}{r^2} \right) \quad (> 0 , \text{tensile}) \]

Consider the tensile hoop stress,

\[ @ \ r = b , \ \sigma_{\theta\theta} = \frac{2a^2 p_i}{b^2 - a^2} \]

\[ @ \ r = a , \ \sigma_{\theta\theta} = \frac{a^2 + b^2}{b^2 - a^2} p_i = SCF \cdot p_i \]

Maximum stress occurs @ \( r = a \).

\[ SCF = \frac{a^2 + b^2}{b^2 - a^2} \] is called stress concentration factor.

In the case of a pressurized circular hole in an infinite medium, i.e. \( p_0 = 0 \) and \( b = \infty \):

\[ \sigma_{rr} = -p_i \frac{a^2}{r^2} , \quad \sigma_{\theta\theta} = p_i \frac{a^2}{r^2} \]

Example 2: Pressurized underground tunnel
The solution is discussed in Timoshenko’s book (Timoshenko and Goodier, 1987). The interesting features are that the maximum stress occurs at two potential sites

@ point $P$: $\sigma_{xx} = \frac{4a^2}{d^2 - a^2} p_i$

@ point $Q$: $\sigma_{\theta\theta} = \frac{d^2 + a^2}{d^2 - a^2} p_i$

For $d = \sqrt{3}a$, $\sigma^P_{xx} = \sigma^Q_{\theta\theta}$

If $d < \sqrt{3}a$, maximum stress occurs at ground point $P$.

If $d > \sqrt{3}a$, maximum stress occurs at the hole boundary point $Q$.

Example 3:

This is a special case of example 1. Take $p_i = 0$, $p_0 = -\sigma_\infty$, $b \to \infty$. We find $A = -a^2 \sigma_\infty$. 

$A = -a^2 \sigma_\infty$
The stress fields are:

\[
\sigma_{rr} = \sigma_\infty \left(1 - \frac{a^2}{r^2}\right)
\]

\[
\sigma_{\theta\theta} = \sigma_\infty \left(1 + \frac{a^2}{r^2}\right)
\]

The maximum stress occurs at \( r = a \), \( \sigma_{\theta\theta}\big|_{r=a} = 2\sigma_\infty \) with a stress concentration factor of 2.

The general solution of \( \nabla^2 \nabla^2 \phi = 0 \) in polar coordinates (i.e. for any 2D elasticity problem) can be expressed as:

\[
\phi = \left(a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r\right) + \left(d_0 r^2 \theta + a_0' \theta\right) + \left(\frac{a_1}{2} r \theta \sin \theta - \frac{c_1}{2} r \theta \cos \theta\right) + \left(b_1 r^3 + a_1 r^{-1} + b_1' r \ln r\right) \cos \theta + \left(d_1 r^3 + c_1' r^{-1} + d_1' r \ln r\right) \sin \theta + \sum_{n=2}^{\infty} \left(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}\right) \cos n\theta + \sum_{n=2}^{\infty} \left(c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}\right) \sin n\theta
\]

The general solution can be conveniently used to solve boundary value problems.

Example 4: Circular hole under uniaxial tension (remote)

Governing equation: \( \nabla^2 \nabla^2 \phi = 0 \)

Boundary conditions: \( \sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} = -p_\infty \), \( \sigma_{\theta\theta} = 0 \) \( @ \ r = a \)

\[
\sigma_{xx} = \sigma_\infty, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0 @ r = \infty
\]

First, let us transform the remote stresses into polar coordinates

\[
\sigma = \sigma_\infty \tilde{e}_x \otimes \tilde{e}_x
\]
\[
\begin{align*}
\sigma_{rr} &= \bar{e}_r \cdot \bar{\sigma}_r = \sigma_\infty \cos^2 \theta = \frac{\sigma_\infty}{2} (1 + \cos 2\theta) \\
\sigma_{\theta\theta} &= \bar{e}_\theta \cdot \bar{\sigma}_\theta = \sigma_\infty \sin^2 \theta = \frac{\sigma_\infty}{2} (1 - \cos 2\theta) \\
\sigma_{r\theta} &= \bar{e}_r \cdot \bar{\sigma}_\theta = -\frac{\sigma_\infty}{2} \sin 2\theta
\end{align*}
\]

The above expressions suggest that the Airy stress function should have the form of \( \phi = C_1 \ln r + C_2 r^2 + f(r) \cos 2\theta \). Pick the corresponding expression in the general solution associated with \( \cos 2\theta \), we see \( f(r) = a_2 r^2 + a_4 r^{-2} + b_2 \) (the \( r^4 \) term is discarded since it generates infinite stress at large \( r \)). Applying the boundary conditions allow all the parameters to be determined. The solution is

\[
\phi = -\frac{\sigma_\infty a^2}{2} \ln r + \frac{\sigma_\infty}{4} r^2 + \left( -\frac{r^2}{4} - \frac{a_2}{2} - \frac{a_4}{4} \right) \sigma_\infty \cos 2\theta
\]

The associated stress fields are

\[
\begin{align*}
\sigma_{rr} &= \frac{\sigma_\infty}{2} \left( 1 - \frac{a^2}{r^2} \right) + \frac{\sigma_\infty}{2} \left( 1 - \frac{a_2}{r^2} \right) \left( 1 - \frac{3a_4}{r^2} \right) \cos 2\theta \\
\sigma_{\theta\theta} &= \frac{\sigma_\infty}{2} \left( 1 + \frac{a^2}{r^2} \right) - \frac{\sigma_\infty}{2} \left( 1 + \frac{a_2}{r^2} \right) \left( 1 + \frac{3a_4}{r^2} \right) \cos 2\theta \\
\sigma_{r\theta} &= -\frac{\sigma_\infty}{2} \left( 1 - \frac{a^2}{r^2} \right) \left( 1 + \frac{3a_2}{r^2} \right) \sin 2\theta
\end{align*}
\]

The maximum tensile stress occurs at \( r = a, \ \theta = \frac{\pi}{2} \)

\[
\sigma_{\theta\theta}^{\text{max}} = 3\sigma_\infty
\]

Therefore, the stress concentration factor is 3.

The above problem can also be directly treated without knowing the general solution (next lecture).
For an elliptic hole under uniaxial tension (remote),

\[
\sigma_{\theta\theta} = \sigma_\infty \left(1 + \frac{b}{a}\right)
\]

The stress concentration factor is \(1 + \frac{2b}{a}\), which depends on the aspect ratio of the elliptic hole.