EN4 HW 6 Solutions

1. From last week's MATLAB, we had $M = 2\, \text{kg}$, $k = 25\, \text{N/m}$, $C = 3\, \text{N-s/m}$, $x_0 = 0.1\, \text{m}$, and $v_0 = -1.0\, \text{m/s}$.

For a forced vibration, $F(t) = F_0 \sin \omega t$, we need to specify $F_0$ and examine the steady-state amplitude of vibration $X$. We normalize $X$ by the static deflection caused by a force $F_0$, i.e., $X_0 = F_0/k$.

In my MATLAB, I used $F_0 = 1\, \text{N}$ so $X_0 = 0.04\, \text{m}$.

For various $\omega$, the response is:

<table>
<thead>
<tr>
<th>$\omega$ (rad/s)</th>
<th>$X/X_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>~1.0</td>
</tr>
<tr>
<td>1.0</td>
<td>~1.1</td>
</tr>
<tr>
<td>1.5</td>
<td>~1.2</td>
</tr>
<tr>
<td>2.0</td>
<td>~1.4</td>
</tr>
<tr>
<td>2.5</td>
<td>~1.75</td>
</tr>
<tr>
<td>3.0</td>
<td>~2.2</td>
</tr>
<tr>
<td>3.5</td>
<td>~2.3</td>
</tr>
<tr>
<td>4.0</td>
<td>~1.75</td>
</tr>
<tr>
<td>4.5</td>
<td>~1.2</td>
</tr>
<tr>
<td>5.0</td>
<td>~0.8</td>
</tr>
<tr>
<td>6.0</td>
<td>~0.5</td>
</tr>
<tr>
<td>7.0</td>
<td>~0.3</td>
</tr>
</tbody>
</table>

* Natural frequency $\omega_n = 3.5355$  
  $f = 0.2121$  
  Amplitude at resonance should be $X = \frac{1}{25} = 0.04\, \text{m}$.
The buoyancy force is
\[ F_b = \rho_w A(L-(x-y))g \]

The force of gravity is
\[ F_g = mg = \rho_b ALg \quad [m = \rho_b AL] \]

b. Summing forces in the vertical direction (positive up),

\[ \sum F = F_b - mg - (C_w + C_c) \frac{dx}{dt} = m \frac{d^2x}{dt^2} \]

Using part a,
\[ \rho_w A(L-(x-y))g - \rho_b ALg - (C_w + C_c) \frac{dx}{dt} = \rho_b AL \frac{d^2x}{dt^2} \]

Rearranging,
\[ \frac{d^2x}{dt^2} + \frac{C_w + C_c}{\rho_b AL} \frac{dx}{dt} - \frac{\rho_w A g (L-(x-y))}{\rho_b AL} + g = 0 \]

Moving \( y \), which is same as \( y(t) \), to the right hand side,
\[ \frac{d^2x}{dt^2} + \frac{C_w + C_c}{\rho_b AL} \frac{dx}{dt} - \frac{\rho_w A g (L-x)}{\rho_b L} - g = \frac{\rho_w A g y}{\rho_b L} \]
c. The static position is obtained by setting \( \frac{d^2x}{dt^2} = \frac{dx}{dt} = 0 \) and \( y = 0 \) (no motion). Then,
\[
-\frac{P_w g (L - x_0)}{P_{bl}} - g = 0 \Rightarrow x_0 = L \left( 1 - \frac{P_{bl}}{P_w} \right)
\]

d. Let \( z = x - x_0 \). Then
\[
\frac{d^2z}{dt^2} = \frac{d^2x}{dt^2}, \quad \frac{dz}{dt} = \frac{dx}{dt}
\]
as the constant terms in the E.O.M. drop out:
\[
\frac{d^2z}{dt^2} + \frac{C_w + C_c}{P_{bl}A_L} \frac{dz}{dt} + \frac{P_w g z}{P_{bl}} = \frac{P_w g y}{P_{bl}L}
\]

e. \( y = y_{max} \sin \omega t \)
f. \( \frac{d^2z}{dt^2} + \frac{C_w + C_c}{P_{bl}A_L} \frac{dz}{dt} + \frac{P_w g z}{P_{bl}} = \frac{P_w g y_{max} \sin \omega t}{P_{bl}L} \)

So,
\[
\omega_n = \sqrt{\frac{P_w g}{P_{bl}L}} \quad \text{(somewhat like a pendulum!)}
\]

Then
\[
2 \pi \omega_n = \frac{C_w + C_c}{P_{bl}A_L} \Rightarrow \omega = \frac{C_w + C_c}{2P_{bl}A_L \frac{P_{bl}}{P_{w}g}} = \frac{C_w + C_c}{2A \sqrt{P_{w}gL}}
\]

and \( \frac{F_0}{m} = \frac{P_w g y_{max}}{P_{bl}} \) so \( F_0 = P_{bl}A_L \cdot \frac{P_w g y_{max}}{P_{bl}} = \frac{P_w g A y_{max}}{P_{bl}} \)

\[
Z = \frac{F_0}{m \omega_n^2} M \quad \text{in general, where } M \text{ is the "Magnification Factor"}
\]

But \( \frac{F_0}{m \omega_n^2} = \frac{P_w g y_{max}}{P_{bl}L} \left( \frac{P_{bl}}{P_{w}g} \right) = y_{max} \)

so \( Z = y_{max} M \) Buoy motion amplified by \( M \) over wave motion.
h. For small $f$, we want $w_n = w$ (Resonance) for a given wave frequency $w$. So 
\[ \sqrt{\frac{P}{\rho L}} = w \text{ or } \rho L = \frac{P}{w^2} \]
Any combo of $\rho$, $L$ (within limits not considered here) that satisfies the above equation will match the buoy response to the wave motion.

i. At resonance, $M \sim \frac{1}{2} \rho$ so 
\[ Z_{\text{max}} = \frac{y_{\text{max}}}{2} \]

j. Power = (Force) \cdot (Velocity).
The force in the electrical dashpot is $-c_{e}x$. So, the power is 
\[ P = -c_{e}V \cdot x = -c_{e}V^2 \text{ (minus indicates energy lost from the buoy motion).} \]
But 
\[ V = \frac{dz}{dt} = Z_{\text{max}} w \cos(wt + \phi) \]
So 
\[ P = -c_{e} Z_{\text{max}}^2 w^2 \cos^2(wt + \phi) \]
Averaging over one period of vibration ($T = \frac{2\pi}{\omega}$) we have 
\[ \bar{P} = \frac{1}{T} \int_{0}^{T} P \, dt = -c_{e} Z_{\text{max}}^2 W^2 \frac{1}{2} \int_{0}^{2\pi} \cos^2(wt + \phi) \, dt \]
(Integral: \( \frac{w}{2\pi} \int_{0}^{2\pi} \cos^2(wt + \phi) \, dt = \frac{1}{2} \); do it yourself for practice)
So 
\[ \bar{P} = \frac{1}{2} c_{e} Z_{\text{max}} w^2 = \frac{c_{e} y_{\text{max}}^2 w^2}{8 \pi^2} \text{ [drop "-" sign now]} \]
\[ P = \frac{c_c y_{\text{max}}^2}{\sqrt{4A^2 (\rho / \rho _d g) / \Phi}} \frac{\rho _d g}{\Phi} \frac{1}{(c_c + c_w)^2} \]

or

\[ P = \frac{\frac{y_{\text{max}}^2 A^2 \rho_d^2 g^2}{2}}{(c_c + c_w)^2} \frac{c_c}{(c_c + c_w)^2} \]

\[ \frac{\partial P}{\partial c_c} = 0 \Rightarrow \frac{1}{(c_c + c_w)^2} - \frac{2c_c}{(c_c + c_w)^3} = 0 \]

\[ \Rightarrow c_c + c_w = 2c_c \Rightarrow c_c = c_w \]

For \( c_c = c_w \), \( \frac{c_c}{(c_c + c_w)^2} = \frac{1}{4c_w} \)

Thus,

\[ P = \frac{y_{\text{max}}^2 A^2 \rho_d^2 g^2}{8c_w} \]

Check units: Power = \( F \cdot v = \text{kg} \cdot \text{m} \cdot \frac{\text{m}}{\text{s}} = \text{kg} \cdot \frac{\text{m}^2}{\text{s}^3} \)

\[ \frac{y_{\text{max}}^2 A^2 \rho_d^2 g^2}{c_w} = \frac{(m^2)(m^4)(\text{kg} / \text{m}^3)(m^2)}{(\text{kg} \cdot \text{m}^2 / \text{s})^3} = \text{kg} \cdot \frac{\text{m}^2}{\text{s}^3} \checkmark \]
3. This is a "Rotor Excitation" problem.
To solve, we need to know \( \omega_n \), \( \omega \), the mass imbalance \( \Delta m \), the eccentricity \( e \), and the operating frequency \( \omega \).

Here, we are given
\[
\Delta m = 0.5 \text{ kg}, \quad e = 0.3 \text{ m}, \quad \omega = 0 \text{ (no damping)}, \quad m = 35 \text{ kg} \quad \text{and} \quad \omega \equiv 275 \text{ RPM}
\]

From the static info (35 kg compresses four springs a distance of 0.01 m) we have
\[
m g = 4 k \delta \quad \Rightarrow \quad \delta = \frac{m g}{4 k} \quad \Rightarrow \quad \omega_n = \sqrt{\frac{4 k}{m}} = \sqrt{\frac{g}{\delta}} = 31.32 \text{ rad/s}.
\]

With 275 RPM corresponding to
\[
\omega = \frac{2 \pi \times 275 \text{ rad}}{60 \text{ s}} = 28.8 \text{ rad/s},
\]
the steady-state amplitude of vibration \( X \) satisfies
\[
\frac{X}{(\frac{\Delta m e}{m})} = (\frac{\omega}{\omega_n})^2 M \quad \Rightarrow \quad M = \frac{1}{(1 - (\frac{\omega}{\omega_n})^2)^2 + (2 \pi \frac{\omega}{\omega_n})^2}
\]

Plugging in,
\[
X = \frac{\Delta m e}{m} \left(\frac{\omega}{\omega_n}\right)^2 \frac{1}{1 - (\omega/\omega_n)^2} = \frac{0.5 \text{ kg}}{35 \text{ kg}} \times 0.3 \text{ m} \times \left(\frac{28.8}{31.32}\right)^2 \frac{1}{1 - \left(\frac{28.8}{31.32}\right)^2}
\]

\[
X = 0.023 \text{ m} = 2.3 \text{ mm}
\]

2.3 x the static deflection but not too big, perhaps.
4. a. Governing equation is
\[ L \frac{dI}{dt} + RI + \frac{1}{C} I \cdot g = V_o \sin \omega t \]
Subbing in \( I = \frac{dq}{dt} \) yields
\[ L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} I \cdot g = V_o \sin \omega t \]
or
\[ \frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} I \cdot g = \frac{V_o}{L} \sin \omega t \]

b. \( \omega_n = \sqrt{\frac{1}{LC}} \) (square root of coefficient multiplying \( g \))

c. At steady-state, we have
\( q(t) = Q \sin(\omega t + \phi) \)
where
\( Q = \frac{(V_o/L)}{\omega_n^2} M \) since "\( E_n = \frac{V_o}{m} \) here"
So,
\( I(t) = Q \omega_n \cos(\omega t + \phi) = I \cos(\omega t + \phi) \)
and at \( \omega = \omega_n \), \( M = \frac{1}{2} \) so
\[ I = Q \omega_n M = \frac{V_o/L}{\omega_n^2} \frac{1}{2} \]

From a., \( 2\delta \omega_n = R/L \) so \( \frac{1}{2} \delta = \omega_n L/R \). Hence
\[ I = \frac{V_o/L}{\omega_n^2} \cdot \omega_n L/R = \frac{V_o}{R} = I \]
5. a.  

\[ r_0 \quad \rightarrow \quad r_0 \rightarrow 1 \]

5. b. [Assuming equal masses]

1. \[ m \frac{dx_1}{dt^2} = k(x_2 - x_1) \]
2. \[ m \frac{dx_2}{dt^2} = -k(x_2 - x_1) + k(x_3 - x_2) \]
3. \[ m \frac{dx_3}{dt^2} = -k(x_3 - x_2) \]

5. c. Let \( y = x_2 - x_1 \) and \( z = x_3 - x_2 \).

Now, subtract Eq. 1 from Eq. 2 to get

\[ m \frac{d^2(x_2 - x_1)}{dt^2} = -2k(x_2 - x_1) + k(x_3 - x_2) \]

or

\[ m \frac{d^2y}{dt^2} = -2ky + kz \]

Subtracting Eq. 2 from Eq. 3 gives

\[ m \frac{d^2(x_3 - x_2)}{dt^2} = -2k(x_3 - x_2) + k(x_2 - x_1) \]

or

\[ m \frac{d^2z}{dt^2} = -2kz + ky \]

5. d. 3 atoms, 3 directions of motion \( \Rightarrow \) 9 degrees of freedom

6. 3 atoms, 1 direction of motion \( \Rightarrow \) 3 degrees of freedom

\[ 3 - 1 = 2 \] degrees of freedom for vibrations, matching w/c.
e. In matrix form,
\[ m \frac{d^2 [y]}{dt^2} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} [y] \]

We want the eigenvalues \( \lambda^2 \) of the matrix, which are equal to the natural frequencies squared, \( \omega_n^2 \).

We take the determinant:
\[
\det \begin{bmatrix} -k_m - \lambda^2 & \frac{k}{m} \\ \frac{k}{m} & -2k_m - \lambda^2 \end{bmatrix} = \left( -2k_m - \lambda^2 \right) \left( \frac{k}{m} \right) = 0
\]

and solve for \( \lambda^2 \):
\[
\lambda^2 = \pm \frac{k}{m} - \frac{2k}{m} = -\frac{k}{m} > -\frac{3k}{m}
\]
so the natural frequencies are
\[
\omega_n = |\lambda| = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}}
\]

f. From last week, we already found \( \sqrt{\frac{k}{m}} = 6.73 \times 10^{13} \) rad/s

so
\[
\omega_n = 6.73 \times 10^{13} \text{ rad/s} \quad \text{and} \quad 11.66 \times 10^{13} \text{ rad/s}
\]

The wave numbers are \( k = \frac{\omega_n}{c} \) [See Problem definitions]

and so
\[
k = \frac{\omega_n}{\sqrt{3 \times 10^8 \text{ m/s}}} = \frac{1295 \text{ cm}^{-1}, 2243 \text{ cm}^{-1} = k}{\text{Close to actual numbers quoted in HW]}}
\]

This is in the INFRARED (as discussed in class)
If O atoms move oppositely while C is fixed, i.e.

\[
\begin{array}{ccc}
\text{←} & \text{O} & \text{→} \\
\end{array}
\]

Then the O atoms are like a mass on a spring \( \epsilon_0 \)

\[
\omega_n = \sqrt{\frac{k}{m}} = 6.73 \times 10^3 \text{ s}^{-1} \quad (\text{wavenumber 1295 cm}^{-1})
\]

If the O atoms move in the same direction while C moves in the other direction, by twice the amount so the center of mass stays fixed:

\[
\begin{array}{ccc}
\text{O} & \text{←} & \text{O} \\
\text{O} & \text{→} & \text{O} \\
\end{array}
\]

\[
\begin{array}{c}
\Delta x \\
-2 \Delta x \\
\Delta x \\
\end{array}
\]

\[
\text{then the forces on the O atoms are} \ 3k \Delta x \text{ and one could surmise that the vibration frequency would be} \ 
\omega_n = \sqrt{\frac{3k}{m}} \quad (\text{wavenumber 2243 cm}^{-1})
\]

These motions can be determined by finding the "eigenvectors" of the eigenvalues \( \lambda^2 \), as done in class for a slightly different system.

\[
\begin{array}{ccc}
\text{O} & \text{O} & \text{O} \\
\text{e} & +2e & -e \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow \text{E}_0 \\
\end{array}
\]

F.B.D.:

\[
\begin{array}{ccc}
\text{←} & \text{O} & \text{→} \\
\text{O} & \text{→} & \text{O} \\
\end{array}
\]

\[
\begin{array}{c}
-eE \\
+2eE \\
-eE \\
\end{array}
\]

Consistent with the 2243 cm\(^{-1}\) mode of vibration

\[
\omega = \omega_n \text{ for 2243 cm}^{-1}, \text{ i.e. } \omega = 11.66 \times 10^3 \text{ s}^{-1}
\]

\[i \quad O_2, \text{N}_2 \text{ are symmetric. An E field cannot force them to vibrate.} \]
\[j \quad \text{CH}_4 \text{ has 2 different atom types and could vibrate (it is worse than O}_2).\)