1) Let the displacement field in a cube of material aligned with the Cartesian coordinate axes \( e_1, \ e_2, \ e_3 \), be given as
\[
  u_1 = k x_2, \quad u_2 = 0, \quad u_3 = 0
\]
where \( k \) is a constant. Such a deformation is known as \textit{simple shear}.

a) In the \( x_1-x_2 \) plane, schematically sketch the final shape of unit square (take the square in the first quadrant, with the origin as one of its corners) after undergoing the above deformation.

b) Determine the displacement gradient and deformation gradient matrices.

c) Determine the Lagrangian and small strain matrices. Recall that small strain matrix is given by \( \varepsilon_{ij} = (u_{ij}+u_{ji})/2 \).

d) What is the volumetric strain (using small strain tensor)?

e) For what range of \( k \) do the norms of the Lagrangian strain matrix and the small strain matrix not differ by more than 5%? Norm of a matrix \( [A_{ij}] \) can be defined as \( \sqrt{A_{ij}A_{ij}} \) (don’t forget that repeated index denotes summation). In other words, it is the square root of the sum of squares of all individual components.

f) Consider a material element along the vector \( e_1+2e_2 \). What is the stretch ratio of this element for \( k = 0.1 \)? Use both the Lagrangian strain tensor and the small strain tensor to calculate the stretch ratio.

g) Consider the above vector and a second vector \( e_1-e_2 \). What is the angle between the two vectors before deformation? What is the angle after deformation, for \( k = 0.1 \)? Use both the Lagrangian strain and small strain matrices to calculate the angle after deformation.

h) For \( k = 0.1 \), find the eigen values and the eigen vectors of the Lagrangian strain tensor (Normalize the eigen vectors to make their magnitude equal to 1). Now, what are the components of the Lagrangian strain tensor with respect to a basis formed by the eigen vectors? Now, take a unit cube, whose edges are parallel to the eigen basis vectors and sketch how it deforms. This sketch does not look like the sketch in part (a) above. How do you interpret this difference?
2) A cube of material with its edges aligned with the coordinate directions undergoes a
displacement given by

\[ u_1 = b \, x_1 + b \, x_2; \quad u_2 = -(1/3) \, b \, x_2; \quad u_3 = 0 \]

where \( b \) is a positive constant.

(a) Determine the small strain matrix \( \varepsilon_{ij} \) in terms of \( b \).
(b) Using the small strain matrix, determine the extensional strain along each of the four cube
diagonals whose components are:

\[ m_i^a = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\} \quad m_i^b = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\} \]

\[ m_i^c = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\} \quad m_i^d = \left\{ -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\} \]

(c) What is the change in angle between the diagonals \( m^a \) and \( m^b \). Is the angle between these
material lines in the deformed configuration larger or smaller than \( \pi/2 \)?

(d) What is the volumetric strain?

3. You are given the following strain field.

\[ \varepsilon_{11} = (1/2) \, x_1 \, x_2, \varepsilon_{22} = 2 \, x_1^2 \, x_2^2, \varepsilon_{12} = x_1^2/4 + x_1 \, x_2^2 \]

All other strain components are zero. Does this represent a physical deformation situation (use
compatibility conditions to check)? If yes, what displacement field gives rise to this strain state?