Statistics of Atmospheric Circulations from Cumulant Expansions

Figure: NASA
Outline

• Prototypical problem: Barotropic Jet
• Direct Statistical Simulation (DSS) by Cumulant Expansions
• Two-layer primitive equations
• Stochastic Jet
• Conclusions
Barotropic Point Jet

+ J. E. Nielson, JAS 43, 1045 (1986)
Barotropic Point Jet

\[
\frac{\partial \omega}{\partial t} + (\vec{v} \cdot \vec{\nabla})(\omega + f) = \frac{\omega_{jet} - \omega}{\tau}
\]

\[
\omega \equiv \hat{r} \cdot (\vec{\nabla} \times \vec{v})
\]

\[
f(\phi) = 2\Omega \sin \phi
\]
Observations

Statistics are much smoother in space than instantaneous flows & hence require fewer degrees of freedom to describe.

Correlations are *non-local* in space.

Statistics are also much stiffer in time than instantaneous flows & hence may be described by a fixed point or by a nearly fixed point.
Equations of Motion and Statistics

\[ \dot{q}_i = A_i + B_{ij} q_j + C_{ijk} q_j q_k + f_i(t) \]

\[ \langle f_i(t) f_j(t') \rangle = \Gamma_{ij} \delta(t - t') \]

\[ q_i = \langle q_i \rangle + q'_i \quad \text{with} \quad \langle q'_i \rangle = 0 \]

\[ c_i \equiv \langle q_i \rangle = m_i \]

\[ c_{ij} \equiv \langle q'_i q'_j \rangle = \langle q_i q_j \rangle - \langle q_i \rangle \langle q_j \rangle = m_{ij} - m_i m_j \]
Eddies sheared apart by mean flows

Mean Flows and Eddies

\[ q'(m) \]

\[ \langle q \rangle \]

\[ q'(m) \]

\[ q'(-m) \]

\[ q'(m) \]

\[ q'(m_1 + m_2) \]

\[ q'(m_1) \]

\[ q'(m_2) \]

Eddies sheared apart by mean flows
Flow only weakly non-linear. Mixing occurs even in absence of eddy-eddy interactions.
Two-Layer Primitive Equations

(Held & Suarez 1978 & 1994)
P. O’Gorman & T. Schneider, GRL 34, 524 (2007)

45 K equilibrium pole-to-equator gradient + 10 K stable stratification

Mean Zonal Velocity vs. Latitude

Mean Potential Temperature vs. Latitude
60 K equilibrium pole-to-equator gradient + 10 K stable stratification

DNS:

CE2:
DNS:

CE2:
30 K equilibrium pole-to-equator gradient + 20 K stable stratification

DNS:

CE2:

In electrical circuits, a simple chaotic attractor known as the Chua attractor has been observed with an extremely simple autonomous circuit. It is third-order reciprocal and has only one non-linear element: a piecewise linear resistor. A modified Chua-type attractor that replaces the piecewise linear function with a Heaviside function can be obtained by varying the parameters in the equation of motion for the three variables.

The resulting attractor shows a basin of attraction and well-defined statistical moments of dynamical variables given that the initial conditions are within the basins. The numerical values for the two parameters used in the simulation are:

\[ A_1 = -0.6 \text{ and } A_3 = -0.6 \]

The equations of motion for the Chua-type attractor with a Heaviside non-linearity is given by:

\[
\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = x_3 \\
\dot{x}_3 & = A_3 x_3 - x_2 + A_1 x_1 + H(x_1)
\end{align*}
\]

The shape of the Chua attractor with a Heaviside term is shown in the figure. From the equations of motion for the dynamical variables, we can derive the equations of motions for the first and second cumulants. The EOM for the cumulants contains terms such as:

\[
\begin{align*}
\dot{c}_{11} & = c_{12} \\
\dot{c}_{12} & = c_{13} \\
\dot{c}_{13} & = A_1 c_{11} + A_2 c_{12} + A_3 c_{13} + H(x_1)
\end{align*}
\]

These terms can be expressed in terms of the first and second cumulants by assuming that the probability density function of the dynamical variables is a Gaussian in three-dimensional space equivalent to truncating the cumulants at second order. The results for the cumulants are compared with the results stemming from direct numerical simulation (DNS) in the table.

<table>
<thead>
<tr>
<th></th>
<th>1st cumulant</th>
<th>2nd cumulant</th>
</tr>
</thead>
<tbody>
<tr>
<td>DNS</td>
<td>1.43</td>
<td>1.57 0.00 -1.50</td>
</tr>
<tr>
<td>CE2</td>
<td>1.52</td>
<td>1.23 0.00 -1.23</td>
</tr>
</tbody>
</table>

The EOM terms containing the Heaviside function are evaluated symbolically in Mathematica, yielding error functions and subsequently the EOM themselves are solved numerically within the same scientific software package. The results for the cumulants are compared with the results stemming from direct numerical simulation (DNS) in the table.
Stochastically Driven Jet

DNS:

CE2:
Jet formation *not* due to inverse-cascade processes

Conclusions

• Direct Statistical Simulation, by integrating out fast modes, focuses on the slow modes of most interest.
• Works for problems with non-trivial mean flows.
• Accuracy can be systematically improved (eg. CE3)
• DSS can lead to improved understanding.
• DSS can be significantly faster than DNS.
• Offers a natural way to incorporate statistical models of subgrid physics.
Statistics of Atmospheric Circulations from Cumulant Expansions

J. B. Marston and F. Sabou

Large-scale atmospheric flows are not so nonlinear as to preclude their direct statistical simulation (DSS) by systematic expansions in equal-time cumulants. Such DSS offers a number of advantages: (i) Low-order statistics are smoother in space and stiffer in time than the underlying instantaneous flows, hence statistically stationary or slowly varying fixed points can be described with fewer degrees of freedom and can also be accessed rapidly. (ii) Convergence with increasing resolution can be demonstrated. (iii) Finally and most importantly, DSS leads more directly to understanding, by integrating out fast modes, leaving only the slow modes that contain the most interesting information. This makes the approach ideal for simulating and understanding modes of the climate system, including changes in these modes that are driven by climate change. The equations of motion for the cumulants form an infinite hierarchy. The simplest closure is to set the third and higher order cumulants to zero. We extend previous work (Marston, Conover, and Schneider 2008) along these lines to two-layer models of the general circulation which has previously been argued to be only weakly nonlinear (O'Gorman and Schneider, 2006). Equal-time statistics so obtained agree reasonably well with those accumulated by direct numerical simulation (DNS) reproducing efficiently the midlatitude westerlies and storm tracks, tropical easterlies, and non-local teleconnection patterns (Marston 2010). Low-frequency modes of variability can also be captured. The primitive equation model of Held & Suarez, with and without latent heat release, is investigated, providing a test of whether DSS accurately reproduces the responses to simple climate forcings as found by DNS.