Bilateral Trading Processes, Pairwise Optimality, and Pareto Optimality [1]

ALLAN M. FELDMAN
Brown University

INTRODUCTION

In a world where transferals are costless and multilateral trades can be arranged at no expense, and where utility functions are continuous and depend only on allocations, voluntary trade will lead inevitably to Pareto optimality. However, when transferals are costly and multilateral trades difficult or impossible, optimality is problematical. In this paper we will consider a pure exchange economy in which bilateral trades are costless, but multilateral trades are infinitely costly. We will show that under certain conditions a sequence of bilateral trades will carry the economy to a pairwise optimal allocation, that is, an allocation which cannot be improved upon by bilateral trade. Then we shall prove a theorem which says that, under fairly general conditions, a pairwise optimal allocation is Pareto optimal if it assigns each trader a positive quantity of one good (which we will call " money "), for which each trader has a positive marginal utility. As a corollary, we shall show that certain types of sequences of bilateral trades will always carry an economy to its set of Pareto optimal allocations.

The question of Pareto optimality for a pairwise optimal allocation has been treated before by Rader [8], who showed, essentially, that if there is a trader who can deal in all goods under a given pairwise optimal allocation, that allocation is Pareto optimal. Rader's theorems suggest how important a broker might be in an exchange economy where multilateral trades are impossible. Our theorems, on the other hand, suggest how important a money commodity might be. The role of the money commodity in the model of this paper is different from the role of money in the models of those authors, who, like Niehans [5, 6], view money as something which minimizes transferal (or transport) costs. We will ignore such costs; our analysis is about costly trade only in the sense that all bilateral trades are costless and all multilateral trades prohibitively costly. These are costs associated with the formation of trading groups, irrespective of the nature of the commodities traded.

We are especially interested in bilateral trading processes for two reasons. First, very casual empiricism suggests that bilateral commodity trades are more common than multilateral ones. Second, and much more important, there is good reason to believe that the nontransport costs of effecting trades rise much faster than the size of the groups making those trades. If, for example, information costs are proportional to the number of pairs in a group, they will go up as the square of the size of the group; if they are proportional to the number of subgroups in a group, they will increase exponentially with the size of the group. The assumption that bilateral trades are costless and multilateral trades prohibitively costly is a first approximation in investigating the effects of trading-group-formation costs which rise faster than group size.

Now the connection of bilateral trade and money is not new. Ostroy [7] and Starr [9] have both developed models of bilateral trade in which the introduction of money allows an economy to reach a competitive equilibrium via a finite series of bilateral trades

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constrained by a *quid pro quo*, or bilateral balance requirement. However, these analyses, unlike the one presented here, assume that equilibrium prices are known by all traders, and they allow trades which may make some parties to them (temporarily) worse off, but which are nonetheless reasonable because of the price information assumed. Our model, on the other hand, is a model of exchange in which no prices are known, and in which traders, having no information or subjective probabilities about trading opportunities, only make trades when they can make themselves better (or no worse) off by so doing. They are truly groping in the dark. Yet, we will show that under general conditions their myopic trading will lead to a pairwise optimal allocation, and that, if there is a universally held money commodity, that pairwise optimal allocation will also be Pareto optimal.

THE MODEL

Consider an n-trader, m-good economy. The goods space is the non-negative orthant of m-dimensional Euclidean space. An *allocation* $x = (x_1, \ldots, x_n)$ is an n-vector of non-negative m-vectors, or a point in R_+^{nm}, whose ith component, x_i, is the *bundle* of goods assigned to trader i. We will let ω_i represent i's initial bundle; $\omega = (\omega_1, \ldots, \omega_n)$ will be a fixed *initial allocation*, and we will let $A(\omega)$ denote society's set of feasible allocations: $A(\omega) \equiv \{x = (x_1, \ldots, x_n) | \, x_i \in R^m, \, x_i \geqq 0$ for all i and $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i\}$.[1] We shall assume that all m goods are in positive supply; so $\sum_{i=1}^n \omega_i > 0$.

We will suppose that every trader has preferences which can be represented by a continuous utility function $u_i(x_i)$ which maps R_+^m into R. We will say that the utility function u_i *represents convex preferences* if $u_i(y_i) > u_i(x_i)$ implies $u_i(ty_i + (1-t)x_i) > u_i(x_i)$ whenever $0 < t < 1$. We will say that u_i *represents strictly convex preferences* if

$$u_i(y_i) \geqq u_i(x_i)$$

implies $u_i(ty_i + (1-t)x_i) > u_i(x_i)$ whenever $0 < t < 1$ and $y_i \neq x_i$. The ith individual is said to be *insatiable* if for any bundle x_i there is a bundle y_i such that $u_i(y_i) > u_i(x_i)$.

For any allocation x, $u(x)$ will be the corresponding utility vector; so

$$u(x) \equiv (u_1(x_1), u_2(x_2), \ldots, u_n(x_n)).$$

The set of feasible utility vectors will be called $U(\omega)$; $U(\omega) \equiv \{u \in R^n \mid$ there exists an $x \in A(\omega)$ such that $u = u(x)\}$.

The letter N will represent the set of all n traders in the economy, and we will define as a *coalition* any non-empty subset of N. We will say that a coalition S can *block* x *from* y is there exist bundles $s_i \geqq 0$, for all $i \in S$, such that

$$\sum_{i \in S} s_i = \sum_{i \in S} y_i,$$

$$u_i(s_i) \geqq u_i(x_i) \quad \text{for all} \quad i \in S, \text{ and}$$

$$u_j(s_j) > u_j(x_j) \quad \text{for some} \quad j \in S.$$

The *core from* y, which we will write $C(y)$, is the set of allocations which cannot be blocked from y by any coalitions. The *core* is $C(\omega)$.

[1] We will use the following (standard) vector inequality notation:
If x and y are two k-dimensional vectors, we define

$$x = y \text{ if } x_i = y_i \text{ for } i = 1, \ldots, k$$
$$x \geqq y \text{ if } x_i \geqq y_i \text{ for } i = 1, \ldots, k$$
$$x \geq y \text{ if } x \geqq y \text{ and } x \neq y$$
$$x > y \text{ if } x_i > y_i \text{ for } i = 1, \ldots, k.$$

We will also use 0 to denote a zero vector of the appropriate dimension.

An allocation x is *Pareto optimal* or *efficient* if it cannot be blocked from ω by N. $P(\omega)$ will denote the set of Pareto optimal allocations. An allocation x will be called *pairwise optimal* if no two-member coalition can block it from x.

An m-vector of prices p, and an allocation y, together form a *competitive equilibrium from* x, if $p \cdot z_i \leqq p \cdot x_i$ implies $u_i(z_i) \leqq u_i(y_i)$ for all i in N and any bundle z_i. If (p, y) is a competitive equilibrium, y is called a *competitive allocation from* x and p is an *equilibrium price vector.*

We can finally define two types of bilateral exchange: We will say that the trade from an allocation x to an allocation y is a *bilateral trade move*, or BTM, if (i) it makes no trader worse off; that is, if $u(y) \geqq u(x)$, and (ii) $x_i = y_i$ for all traders but (at most) two. As a trivial consequence of this definition, if i and j are the two individuals who *are* trading, then $x_i + x_j = y_i + y_j$. Our definition of a BTM does not exclude the possibility that $x_i = y_i$ and $x_j = y_j$, but if $x_i \neq y_i$, then necessarily $x_j \neq y_j$.

If i and j trade to a (sub)allocation which is optimal for them, we will say they are making an optimizing bilateral trade move. More rigorously, let us define, for any allocation x and any given pair of traders $\{i, j\}$, an " optimal with respect to x " set

$O(i, j, x) \equiv \{y \in A(\omega) | \, u(y) \geqq u(x),\ x_l = y_l \text{ for } l \neq i, j, \text{ and } \{i, j\} \text{ cannot block } y \text{ from } x\}.$

($O(i, j, x)$ is non-empty by the continuity of u_i and u_j.)

An *optimizing bilateral trade move between i and j* is a move from an allocation x to an allocation $y \in O(i, j, x)$.

WHERE DOESN'T BILATERAL TRADE LEAD?

Let us imagine that the traders in our economy make a sequence of bilateral trade moves.

Where does that sequence of moves take them? Does it lead to the core, to the set of competitive equilibria, or to the set of Pareto optimal allocations? Does it, at least, bring the economy to the set of pairwise optimal allocations? Or does it lead nowhere?

It is easy to construct an example to show that bilateral trade moves need not lead to the core, even under the severest assumptions on the utility functions and the initial allocation ω. Suppose there are three individuals and two goods in the economy, and all three individuals have utility functions of the form $u_i(x_i) = u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$, where x_{ij} is the amount of good j in bundle x_i. Assume that the initial allocation is

$$\omega = (\omega_1, \omega_2, \omega_3) = ((1, 9), (5, 5), (9, 1)).$$

Now consider the move from ω to $y = ((3, 3), (5, 5), (7, 7))$. It is clear that

$$y \in O(1, 3, \omega);$$

that is, the move from ω to y is an optimizing bilateral trade move between 1 and 3. Moreover, y is Pareto optimal, and therefore, no further non-degenerate bilateral trades are possible from y. But y is not in the core, because it can be blocked from ω by the coalition of traders 1 and 2: Take $s_1 = (3, 3)$, $s_2 = (3, 11)$; then

$$s_1 + s_2 = \omega_1 + \omega_2;\ u_1(s_1) = u_1(y_1) \text{ and } u_2(s_2) > u_2(y_2).$$

In the above example, bilateral trade leads to an allocation which is, if not in the core, at least Pareto optimal. However this is not always possible, and the fact that it's not always possible motivates the central theorems of this paper. The following example illustrates the dilemma of an allocation which cannot be improved upon except by multilateral trade, that is, an allocation which is pairwise optimal but not Pareto optimal.

Suppose there are three individuals and three goods, with utility functions and initial bundles given by the table below:

	$u_i(x_i)$	ω_i
Trader 1	$u_1(x_1) = 3x_{11}+2x_{12}+x_{13}$	(0, 1, 0)
Trader 2	$u_2(x_2) = 2x_{21}+x_{22}+3x_{23}$	(1, 0, 0)
Trader 3	$u_3(x_3) = x_{31}+3x_{32}+2x_{33}$	(0, 0, 1)

The initial allocation ω is not efficient because it is dominated by

$$y = ((1, 0, 0), (0, 0, 1), (0, 1, 0)),$$

since $u(y) = (3, 3, 3) > u(\omega) = (2, 2, 2)$. However, it is impossible to move from ω via bilateral trade: individual 1 would be glad to offer some good 2 in exchange for some of 2's good 1, but individual 2 values good 1 twice as highly as he values good 2, and individual 1 is not prepared to make an exchange at such a ratio. Similar considerations prohibit trade between 1 and 3, and 2 and 3. Consequently, no non-degenerate bilateral trades from ω are possible, and the economy cannot reach an optimum, in the sense of Pareto, via bilateral exchange. Moreover, a slight shift of ω into the interior of $A(\omega)$ would not alter the nature of this example. If we started, say, at

$$\omega^* = ((0{\cdot}01, 0{\cdot}98, 0{\cdot}01), (0{\cdot}98, 0{\cdot}01, 0{\cdot}01), (0{\cdot}01, 0{\cdot}01, 0{\cdot}98)),$$

which is *not* pairwise optimal, a sequence of bilateral trades might lead to an allocation like $y^* = ((0{\cdot}03, 0{\cdot}97, 0), (0{\cdot}97, 0, 0{\cdot}03), (0, 0{\cdot}03, 0{\cdot}97))$. Now y^* is pairwise optimal, but it is not Pareto optimal.

Is it possible to construct a sequence of bilateral trade moves which doesn't even lead to the set of pairwise optimal allocations? The answer is obviously yes since we allow for degenerate moves: the sequence of trades from ω to ω to ω... is an example. However, even if we require *optimizing* bilateral trades we aren't through. An example of an infinite sequence of rounds of optimizing bilateral trades which does not lead to a pairwise optimal allocation is given in the concluding section of this paper.

WHERE DOES BILATERAL TRADE LEAD?

In this section, we are going to show that the assumption of a rotating trading pattern, which forces every pair to trade periodically, coupled with an assumption of strictly convex preferences, will ensure that a sequence of optimizing BTM's converges to the set of pairwise optimal allocations. First, however, we will make a definition, motivated by our quite intuitive expectation that bilateral trade *ought* to lead to pairwise optimal allocations.

Consider an infinite sequence $\{b^k\}$ of bilateral trade moves which gives rise to a sequence of allocations $\{x^k\}$. Since the space of allocations $A(\omega)$ is compact, $\{x^k\}$ will have cluster points in $A(\omega)$. We will say that $\{b^k\}$ is *effective* if, whenever x^* is a cluster point of $\{x^k\}$, x^* is pairwise optimal.

In what follows we are going to assume that if b^k is a move from x^k to x^{k+1}, then x^{k+1} is contained in $O(i, j, x^k)$, for some pair $\{i, j\}$. Now for a given pair $\{i, j\}$ and a fixed x, $O(i, j, x)$ is simply a set of allocations, but if we let x vary over $A(\omega)$, $O(i, j, x)$ can be interpreted as a point-to-set correspondence. In the appendix we prove the following proposition, whose usefulness will become obvious as we proceed:

Proposition. *If u_i and u_j are continuous and represent strictly convex preferences, then $O(i, j, x)$ is a continuous correspondence.*

With this result in hand, we can now impose more structure on our as yet loosely defined sequence of bilateral trade moves. It is rather obvious that if one pair of traders is forever prevented from trading, a sequence of bilateral trades will not in general bring an economy to a pairwise optimal allocation. The easiest way to rule out such an occurrence is to assume that the trading pattern is a rotating one; that is, one in which trader 1 trades with trader 2, then with 3, then with 4, ..., then with n, and then 2 trades with 3, then with 4, ..., then with n, and so on, until $n-1$ trades with n, after which the round repeats, *ad infinitum*. If each step b^k of the process involves an optimizing trade, or, for an $i-j$ trade, a move from x to $O(i, j, x)$, we will say that $\{b^k\}$ is a *rotating sequence of optimizing bilateral trade moves*. In Theorem 1 below we will give sufficient conditions for a rotating sequence of optimizing bilateral trade moves to be effective, or to have pairwise optimal cluster points.

Before proceeding with the theorem, we need two preliminary lemmas.

Lemma 1. *Define* $h(i, j, x) \equiv \inf_{y \in O(i, j, x)} [\Delta u(i, j, x, y)]$,

where $\Delta u(i, j, x, y) \equiv \sum_{l=1}^{n} u_l(y_l) - \sum_{l=1}^{n} u_l(x_l) = u_i(y_i) - u_i(x_i) + u_j(y_j) - u_j(x_j)$.

If u_i and u_j are continuous and represent strictly convex preferences, $h(i, j, x)$ is a continuous function.

Proof. By the Proposition, $O(i, j, x)$ is closed for any $x \in A(\omega)$. Since $A(\omega)$ is bounded, $O(i, j, x)$ is compact. $\Delta u(i, j, x, y)$ is continuous by the continuity of u_i and u_j.

Therefore,

$$h(i, j, x) = \min_{y \in O(i, j, x)} [\Delta u(i, j, x, y)].$$

Therefore,

$$-h(i, j, x) = \max_{y \in O(i, j, x)} [-\Delta u(i, j, x, y)].$$

Since $O(i, j, x)$ is a continuous correspondence by the Proposition, the continuity of h follows from 1.8 (4) in [3], or [2, p. 116]. Q.E.D.

Lemma 2. *Suppose all utility functions are continuous and represent strictly convex preferences. Suppose the move from x to y is an optimizing BTM between traders i and j; that is, suppose $y \in O(i, j, x)$. Then for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $d(x, y) \geqq \varepsilon$ implies $\Delta u(i, j, x, y) \geqq \delta(\varepsilon)$, where δ is independent of $\{i, j\}$.*[1]

Proof. Consider one pair of traders $\{i, j\}$. By the upper semi-continuity of O, the set $\{(x, y) | \, y \in O(i, j, x)\}$ is closed in $A(\omega) \times A(\omega)$. Given $\varepsilon > 0$ the set $\{(x, y) | \, d(x, y) \geqq \varepsilon\}$ is also closed. Therefore,

$$X \equiv \{(x, y) | \, y \in O(i, j, x)\} \cap \{(x, y) | \, d(x, y) \geqq \varepsilon\}$$

is closed. By the boundedness of $A(\omega)$, X is compact. The function $\Delta u(i, j, x, y)$ is strictly positive on X by the strict convexity assùmption. Therefore,

$$\delta(i, j, \varepsilon) \equiv \inf_{(x, y) \in X} \Delta u(i, j, x, y) > 0,$$

[1] If x and y are two allocations, we will let $d(x, y)$ denote the Euclidean distance between x and y. Further, if S is a set of allocations, we will define

$$d(x, S) = \inf_{y \in S} d(x, y).$$

if X is non-empty. If X is empty, define $\delta(i, j, \varepsilon) \equiv 1$. Let $\delta(\varepsilon) \equiv \min_{\{i, j\} \subset N} \delta(i, j, \varepsilon) > 0$, and the lemma follows. Q.E.D.

Now we finally have

Theorem 1. *Suppose all utility functions are continuous and represent strictly convex preferences. Suppose $\{b^k\}$ is a rotating sequence of optimizing bilateral trade moves. Then $\{b^k\}$ is effective.*

Proof. Let $\{x^k\}$ be the sequence of allocations associated with $\{b^k\}$; so b^k is the move from x^k to x^{k+1}. Since $A(\omega)$ is compact, $\{x^k\}$ has cluster points in $A(\omega)$; let x^* be one such point.

First, we claim that $d(x^k, x^{k+1}) \to 0$. If not, there is an $\varepsilon > 0$ such that for an infinite set of indices $\{r\}$, $d(x^r, x^{r+1}) \geqq \varepsilon > 0$. This implies, by Lemma 2, that

$$\sum_{i=1}^{n} [u_i(x^{r+1}) - u_i(x^r)] \geqq \delta(\varepsilon) > 0$$

for all r. But this means $\sum_{i=1}^{n} u_i(x^k) \to \infty$ as $k \to \infty$, which is impossible since $U(\omega)$ is compact, by the compactness of $A(\omega)$ and the continuity of u.

Suppose $\{b^k\}$ is *not* effective. Assume, without loss of generality, that traders 1 and 2 could make a profitable trade from x^*. Therefore, there is a $y \in O(1, 2, x^*)$ such that $\Delta u(1, 2, x^*, y) > 0$. By the upper semi-continuity of O, $O(1, 2, x^*)$ is closed; by the boundedness of $A(\omega)$, $O(1, 2, x^*)$ is compact, and since

$$y \in O(1, 2, x^*) \Rightarrow \Delta u(1, 2, x^*, y) > 0, \; h(1, 2, x^*) > 0.$$

Now consider the subsequence $\{x^s\}$ of allocations which immediately precedes trades between 1 and 2:

$$\{x^s\} = \{x^1, x^{1+\frac{n}{2}(n-1)}, x^{1+2\left[\frac{n}{2}(n-1)\right]}, \ldots\}.$$

We claim that x^* is a cluster point of $\{x^s\}$. This is the case because (i) x^* is a cluster point of $\{x^k\}$; (ii) any element of $\{x^k\}$ is no further than $n(n-1)/2$ places removed from some allocation which is also in $\{x^s\}$; and (iii) $d(x^k, x^{k+l}) \to 0$ for $l \leqq n(n-1)/2$.

Finally, by Lemma 1, $h(1, 2, x)$ is continuous at x^*; therefore, since

$$h(1, 2, x^*) > 0, \; h(1, 2, x^s) \nrightarrow 0.$$

But this implies that $u_1(x_1^k) + u_2(x_2^k)$ grows without bound, contradicting the fact that $U(\omega)$ is compact. Q.E.D.

We might remark at this point that Theorem 1 would still be true under a much broader definition of a rotating sequence of bilateral trade moves. All that is necessary for our proof is that $\{b^k\}$ be a sequence of rounds of a fixed maximum length, and that, in each round, each pair trade at least once. The particular trading pattern $1-2$, $1-3$, ..., $1-n$, $2-3$, ..., $2-n$, ..., $(n-1)-n$, is not essential.

PAIRWISE OPTIMALITY AND PARETO OPTIMALITY

Now we must consider the following question. When is a pairwise optimal allocation also Pareto optimal? This is not a vacuous issue, as we have shown above it is easy to construct a pairwise optimal allocation that is not efficient. Such examples show that bilateral trade moves are not entirely satisfactory for some economies; because of the difficulty, in a barter economy, of multilateral trades, it is important to know how likely it is that bilateral trade will prove unsatisfactory. The theorem below gives relatively simple conditions that ensure that a pairwise optimal allocation is in fact a Pareto optimal allocation.

The proof will turn on the assumption that there is a good for which everyone has a positive marginal utility and of which everyone has a positive quantity. Intuitively, the existence of such a " money " commodity opens up trading possibilities that would otherwise remain closed, as they are in our second example above. We will write $u_{ij}(y_i)$ for the partial derivative of u_i with respect to y_{ij}, that is, the marginal utility of good j for trader i.

Theorem 2. *Suppose u is continuous on $A(\omega)$, u_i represents convex preferences for $i = 1, \ldots, n$, and every u_i has continuous first partial derivatives.*[1]

[1] I am obliged to Harl Ryder for observing that this differentiability assumption is crucial; in fact, the theorem fails without it. (An assumption analogous to differentiability is also essential for Rader's [8] pairwise optimality—Pareto optimality equivalency result.) Consider the following economy:

$$u_1(x_1) = \min\,[100x_{11}, 3x_{11}+97]+2x_{12}+ x_{13};\quad \omega_1 = (1, 1, 0);$$
$$u_2(x_2) = \min\,[100x_{21}, 2x_{21}+98]+ x_{22}+3x_{23};\quad \omega_2 = (2, 0, 0);$$
$$u_3(x_3) = \min\,[100x_{31}, x_{31}+99]+3x_{32}+2x_{33};\quad \omega_3 = (1, 0, 1).$$

Now ω is pairwise optimal, all the u_i's are continuous and represent convex preferences, the first good is held in positive quantities by all the traders, and, incidentally, all the utility functions are monotonically increasing in it. But ω is *not* Pareto optimal since it is dominated by ((2, 0, 0), (1, 0, 1), (1, 1, 0)).

If, however, we smooth the kink at $x_{i1} = 1$, our difficulties vanish. Suppose we change the utility functions to

$$u_i^*(x_i) = \begin{cases} \sum_{j=1}^{3} \alpha_{ij}x_{ij}+(2x_{i1}-x_{i1}^2)(100-\alpha_{i1}) & \text{for } x_{i1} \leqq 1 \\ \sum_{j=1}^{3} \alpha_{ij}x_{ij}+(100-\alpha_{i1}) & \text{for } x_{i1}>1, \end{cases}$$

where $\| \alpha_{ij} \| = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ is the same matrix of constant marginal utilities (for $x_{i1} > 1$) as in the example above. A graph of u_i against x_{i1}, for fixed x_{i2}, x_{i3}, shows what has happened to the utility functions.

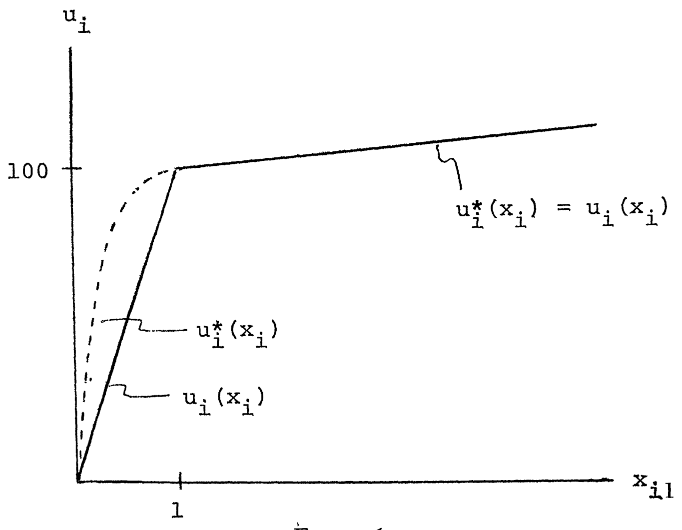

FIGURE 1

Suppose again that we start at $\omega = ((1, 1, 0), (2, 0, 0), (1, 0, 1))$. Now ω is no longer pairwise optimal. For example, traders 1 and 3 can make an exchange to bring the economy to $x^1 = ((1.01, 0.99, 0), (2, 0, 0), (0.99, 0.01, 1))$, and $u(x^1) \geqslant u(\omega)$. The reader can verify that a bilateral trading process in this economy can reach a Pareto optimal allocation.

It is important to observe that any effective bilateral trading sequence in this example will pivot around trader 3's one unit of good one. This is where the differentiability assumption is useful. Trader 3 must take small steps back and forth off the constant slope part of his utility hill: he can do this because the hill is smooth. When his utility function is $u_3(x_3) = \min\,[100x_{31}, x_{31}+99]+3x_{32}+2x_{33}$, he can't make those small pivoting steps, because there's a cliff behind him!

Suppose y is a pairwise optimal allocation, and for some good, say the first, $u_{i1}(y_i)>0$ and $y_{i1}>0$, for all i.

Then y is a competitive equilibrium allocation, it is in $C(y)$ and it is Pareto optimal.

Proof. To lighten the notation, we will drop the y_i in $u_{ij}(y_i)$; it is understood that all the partial derivatives are to be evaluated at y. Our strategy is to define a price vector p with the property that (p, y) is a competitive equilibrium from y. Then we will use Theorem 1 of Debreu and Scarf [4] to establish that y is in the core for y, and is, therefore, Pareto optimal.

Let $p_1 = 1$.

Now choose an individual, say i_2, to whom y assigns a positive amount of good 2, and let

$$p_2 = p_1 u_{i_2 2}/u_{i_2 1} = u_{i_2 2}/u_{i_2 1}.$$

By assumption, $u_{i_2 1}$ is positive.

We claim that p_2 is well defined, in the sense that if j and k are two individuals to whom y assigns positive amounts of good 2, then $u_{j2}/u_{j1} = u_{k2}/u_{k1}$. For if

$$u_{j2}/u_{j1} \neq u_{k2}/u_{k1},\ y_{j1}, y_{k1}, y_{j2}, y_{k2}>0 \text{ and } u_{j1}, u_{j2}, u_{k1}, u_{k2}$$

are continuous, y is not pairwise optimal, contradicting our assumption.

In general, let $p_k = p_1 u_{i_k k}/u_{i_k 1}$, for $k = 2, 3, \ldots, m$, where i_k is an individual to whom y assigns a positive amount of good k.

Let $p = (p_1, p_2, \ldots, p_m)$. We claim that (p, y) is a competitive equilibrium from y. We must show that, for $i = 1, 2, \ldots, n$, y_i maximizes $u_i(z_i)$, subject to $0 \leqq z_i$ and

$$p \cdot z_i \leqq p \cdot y_i.$$

By the convex preferences assumption, y_i solves this maximization problem if and only if (i) $u_{ij}-\lambda_i p_j \leqq 0$ for all $j = 1, 2, \ldots, m$ and some non-negative λ_i, and (ii) $u_{ij}-\lambda_i p_j = 0$ whenever y_{ij} is positive (see, e.g., [1]). Let $\lambda_i = u_{i1}$, which is, by assumption, positive. Then if y_{ij} is positive, $u_{ij}-\lambda_i p_j = u_{ij}-u_{i1}p_j = u_{ij}-u_{i1}(u_{ij}/u_{i1}) = 0$, so (ii) is satisfied. To establish (i), we must show that if $y_{ij} = 0$, and if k is an individual for whom y_{kj} is positive, then $u_{ij}-u_{i1}(u_{kj}/u_{k1}) \leqq 0$. But if this is not the case, then

$$y_{i1}, y_{k1}>0,\ y_{kj}>0,\ u_{ij}/u_{i1}>u_{kj}/u_{k1},$$

and by the continuity of u_{ij}, u_{i1}, u_{kj}, u_{k1}, y is not pairwise optimal, which is again a contradiction. Therefore, (y, p) is a competitive equilibrium from y.

Because $u_{i1}>0$ for all i, no i is satiated at y_i. Moreover, all the u_i are continuous and represent convex preferences; therefore, we can apply Theorem 1 of [4]; therefore y is in $C(y)$. Consequently, y is Pareto optimal. Q.E.D.

Theorems 1 and 2 can be combined to provide a set of sufficient conditions under which sequences of bilateral trade moves bring the economy to Pareto optimality.

Theorem 3. *Suppose all utility functions are continuous, have continuous first partial derivatives, represent strictly convex preferences, and are monotonic increasing in at least one common argument.*

Suppose $\{x^k\}$ is a sequence of allocations which results from a sequence of bilateral trade moves $\{b^k\}$. Assume that for some $\hat{k}$, the set $\{x \mid u(x) \geqq u(x^{\hat{k}})\} \subset \mathring{A}(\omega) \equiv$ the interior of $A(\omega)$.

Let x^* be a cluster point of $\{x^k\}$.

Then if $\{b^k\}$ is a rotating sequence of optimizing BTM's, x^* is Pareto optimal.

Proof. Since $x^* \in \{x \mid u(x) \geqq u(x^{\hat{k}})\}$, $x^* \in \mathring{A}(\omega)$; i.e., $x^*>0$. By Theorem 1, x^* is pairwise optimal, and by a trivial application of Theorem 2, x^* is Pareto optimal.

Q.E.D.

CONCLUSIONS

In this paper we have treated sequences of bilateral trade moves which are (i) unconstrained by any *quid pro quo* requirements, and (ii) always utility non-decreasing. We think our sequences of moves reflect what is meant by " bilateral barter " in one of its most primitive forms, and we have proved that under the assumptions of continuity and strict convexity of preferences, rotating sequences of optimizing bilateral trade moves lead to pairwise optimal allocations. We further showed that pairwise optimal allocations are Pareto optimal, under rather general conditions, if there is one good which everyone values and possesses in positive amounts.

These theorems tempt us to interpret the real world. Let us consider those primitive economies in which a commodity like a staple food product is used as a medium of exchange. It is probably the case that no household will want to trade away all its yams, or all its cattle, or all of whatever. If the preferences of households are strictly convex, Theorems 1 and 2 suggest that (i) an endless series of bilateral trades will establish pairwise optimality, and (ii), the existence of a commonly held commodity, like yams, or cattle, will guarantee that pairwise optimality is Pareto optimality. Moreover, this is the case even though there are no markets, no well-defined prices, traders are completely ignorant of possible future exchange ratios, and they are dreadfully conservative in the sense that they make no trades which do not directly and immediately gratify them.

We think this a rather striking result. But how general is it? In particular, how important is the stringent assumption of strict convexity? The answer to this question is not yet entirely clear. However, we do know that the correspondence O is not upper semi-continuous if, say, preferences are merely assumed to be convex, rather than strictly convex, and the function h is not necessarily continuous under these conditions.

We also can construct rather perverse examples of optimizing bilateral trading rounds, in which preferences are assumed to be convex, but not strictly convex, which do not establish pairwise optimality. Consider a three-person two-good economy and a trading round described by the following sequence $\{b^k\}$.

$$\omega = x^1 \xrightarrow{b^1} x^2 \xrightarrow{b^2} x^3 \xrightarrow{b^3} x^4 \xrightarrow{b^4} x^5$$

	$\omega = x^1$	x^2	x^3	x^4	x^5
$u_1(x_1) = x_{11}$	(1, 0)	(1, 2)	(1, 2)	(1, 0)	(1, 0)
$u_2(x_2) = x_{21}$	(0, 2)	(0, 0)	(0, 0)	(0, 2)	(0, 2)
$u_3(x_3) = x_{31}+x_{32}$	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)

Now b^1 is an optimizing trade between 1 and 2, b^2 is a degenerate optimizing trade between 2 and 3, b^3 is again an optimizing trade between 1 and 2, and b^4 is a degenerate optimizing trade between 1 and 3. Suppose after one round the process repeats, *ad infinitum*. Then $x^1 = ((1, 0), (0, 2), (1, 0))$ is a cluster point of $\{x^k\}$. However, it is not pairwise optimal, because traders 2 and 3 can block it. In other words, the process is not effective.

The difficulty here is apparently the perpetually disappearing opportunity of a profitable 2-3 trade; such a difficulty would be ruled out in a strictly convex preferences model, where no pair of players can make an endless sequence of optimizing trades of lengths $\geqq \varepsilon > 0$. Obviously, reasonable assumptions other than strict convexity of preferences might also eliminate such pathological cases. When we say that we are uncertain about the importance of the strict convexity assumption, we simply mean that we have not weighed the significance of perverse cases like this one, and have not weighed the reasonableness of assumptions other than strict convexity which would rule them out.

It would be ideal to have a theorem which gives interesting necessary and sufficient conditions for the effectiveness of bilateral trade move sequences, and it would be ideal to have a much stronger theorem than Theorem 3 for the convergence of such sequences to Pareto optimal allocations. However the ideal is elusive. We hope what little we have done is interesting in itself.

APPENDIX

Proposition. *Suppose u_i and u_j are continuous and represent strictly convex preferences. Then the correspondence $O(i, j, x)$ is continuous.*

Proof. We will show that $O(i, j, x)$ is continuous at any allocation x^0.

Step 1. O is upper semi-continuous. We must show that if $x^q \to x^0$, (where of course $\{x^q\}$ is a sequence of allocations), if $y^q \in O(i, j, x^q)$ for all q and $y^q \to y^0$, then $y^0 \in O(i, j, x^0)$.

Suppose not. It is clear that we must have $y_i^0 + y_j^0 = x_i^0 + x_j^0$, so there must exist a $\hat{y} \in O(i, j, x^0)$ with $(u_i(\hat{y}_i), u_j(\hat{y}_j)) \geqslant (u_i(y_i^0), u_j(y_j^0))$. Because $\hat{y} \in O(i, j, x^0)$,

$$\hat{y}_i + \hat{y}_j = x_i^0 + x_j^0 = y_i^0 + y_j^0,$$

and therefore, since either $\hat{y}_i \neq y_i^0$ or $\hat{y}_j \neq y_j^0$, we must have $\hat{y}_i \neq y_i^0$ and $\hat{y}_j \neq y_j^0$. By the strict convexity assumption, we can choose a $\hat{\hat{y}} \in O(i, j, x^0)$ such that

$$(u_i(\hat{\hat{y}}_i), u_j(\hat{\hat{y}}_j)) > (u_i(y_i^0), u_j(y_j^0)).$$

By the continuity of u_i and u_j, there exists an $\varepsilon > 0$ such that for any $y \in N_{\hat{\hat{y}}}^{\varepsilon}$, where $N_{\hat{\hat{y}}}^{\varepsilon}$ is a closed ε-ball centred on $\hat{\hat{y}}$, and any $y^q \in N_{y^0}^{\varepsilon}$, we have $(u_i(y_i), u_j(y_j)) > (u_i(y_i^q), u_j(y_j^q))$. But since $\hat{\hat{y}}_i + \hat{\hat{y}}_j = x_i^0 + x_j^0$, and since $x_i^q + x_j^q \to x_i^0 + x_j^0$, there exists a Q such that $q \geqq Q$ implies $x_i^q + x_j^q$ can be distributed to i and j in a way which gives an allocation in $N_{\hat{\hat{y}}}^{\varepsilon}$. However, this contradicts the assumption that $y^q \in O(i, j, x^q)$. Therefore, O is upper semi-continuous.

Step 2. O is a lower semi-continuous. Now we must show that if $x^q \to x^0$, and $y^0 \in O(i, j, x^0)$, then there exists a sequence $\{y^q\}$ of allocations, with $y^q \in O(i, j, x^q)$ for all q, such that $y^q \to y^0$.

Step 2*a*. First, we will show that the bilateral trade move correspondence is lower semi-continuous. Let us define

$$B(i, j, x) \equiv \{y \in A(\omega) |\ x \to y \text{ is a BTM between traders } i \text{ and } j\}$$
$$= \{y \in A(\omega) |\ x_l = y_l \text{ for } l \neq i, j, \text{ and } (u_i(y_i), u_j(y_j)) \geqq (u_i(x_i), u_j(x_j))\}.$$

We wish to establish that, under our continuity and strict convexity assumptions, if $\{x^q\}$ is a sequence of allocations satisfying $x^q \to x^0$, and if $y^0 \in B(i, j, x^0)$, then there exists a sequence of allocations $\{y^q\}$, with $y^q \in B(i, j, x^q)$ for all q, such that $y^q \to y^0$.

Case 1. $y^0 = x^0$. Now we can take $y^q = x^q$ for all q, and we are done.

Case 2. $y^0 \neq x^0$. Since $y^0 \in B(u, j, x^0)$, $y_k^0 = x_k^0$ for all $k \neq i, j$, and therefore, $y_i^0 \neq x_i^0$ and $y_j^0 \neq x_j^0$. Now suppose B is *not* lower semi-continuous. Then there is an $\varepsilon > 0$ such that for an infinite number of q's, $B(i, j, x^q) \cap N_{y^0}^{\varepsilon} = \varnothing$, where $N_{y^0}^{\varepsilon}$ is a closed ε-ball centred on y^0. Let $\{x^r\}$ be the subsequence of $\{x^q\}$ for which $B(i, j, x^r) \cap N_{y^0}^{\varepsilon} = \varnothing$.

Consider the allocation $\hat{y} = ty^0 + (1-t)x^0$, where t is chosen so that the distance between $\hat{y}$ and y^0 is exactly $\varepsilon/2$. By the strong convexity of preferences assumption,

$$(u_i(\hat{y}_i), u_j(\hat{y}_j)) > (u_i(x_i^0), u_j(x_j^0)).$$

Since u is continuous, there exists an $\varepsilon' > 0$ which we can choose smaller than or equal to $\varepsilon/2$, such that for any $y \in N_{\hat{y}}^{\varepsilon'}$, and any $x^r \in N_{x^0}^{\varepsilon'}$, $(u_i(y_i), u_j(y_j)) > (u_i(x_i^r), u_j(x_j^r))$. Because $\varepsilon' \leqq \varepsilon/2$, $N_{\hat{y}}^{\varepsilon'} \subset N_{y^0}^{\varepsilon}$.

Now since $x^r \to x^0$, there exists an R_1 such that $r \geqq R_1$ implies $x^r \in N^{\varepsilon'}_{x^0}$, and therefore, $(u_i(y_i), u_j(y_j)) > (u_i(x^r_i), u_j(x^r_j))$ for any $y \in N^{\varepsilon'}_{\hat{y}}$. However, because $x^r_i + x^r_j \to \hat{y}_i + \hat{y}_j$, there must be an R_2 such that $r \geqq R_2$ implies that for some $y \in N^{\varepsilon'}_{\hat{y}}$, $y_i + y_j = x^r_i + x^r_j$. Let $R = \max(R_1, R_2)$. Then $r \geqq R$ implies there exists a $y \in N^{\varepsilon'}_{\hat{y}} \cap B(i, j, x^r) \subset N^{\varepsilon}_{y^0} \cap B(i, j, x^r)$, which contradicts our construction of $\{x_r\}$. Therefore, B must be lower semi-continuous.

Step 2b. To establish the lower semi-continuity of O, we must show that if x^q is a sequence of allocations such that $x^q \to x^0$, and $y^0 \in O(i, j, x^0)$, then there exists a sequence $\{y^q\}$ of allocations, with $y^q \in O(i, j, x^q)$ for all q, such that $y^q \to y^0$.

By the lower semi-continuity of B, there exists a sequence of allocations $\{z^q\}$, with $z^q \in B(i, j, x^q)$ for all q, such that $z^q \to y^0$. Choose $y^q \in O(i, j, z^q)$. Therefore, $y^q \in O(i, j, x^q)$. We claim that $y^q \to y^0$. Let y^* be a cluster point of $\{y^q\}$. Clearly $y^* \in B(i, j, x^0)$. Moreover,

$$(u_i(y^*_i), u_j(y^*_j)) \geqq (u_i(y^0_i), u_j(y^0_j)),$$

since for all q

$$(u_i(y^q_i), u_j(y^q_j)) \geqq (u_i(z^q_i), u_i(z^q_j)), \; z^q \to y^0,$$

and u is continuous.

It must be the case that $(u_i(y^*_i), u_j(y^*_j)) = (u_i(y^0_i), u_j(y^0_j))$, because an inequality would contradict the assumption that $y^0 \in O(i, j, x^0)$. But by the strict convexity assumption, if $(u_i(y^*_i), u_j(y^*_j)) = (u_i(y^0_i), u_j(y^0_j))$, $y^0 \in O(i, j, x^0)$, and $y^* \in B(i, j, x^0)$, then $y^* = y^0$, as claimed. Therefore, O is lower semi-continuous at x^0. Q.E.D.

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