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ENVY, WEALTH, AND CLASS HIERARCHIES

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A persion i is said to *not envy* another person j if he likes his own bundle of goods as well as he would like j's bundle. This paper explores the social structure defined by the non-envy relation, and relates it to the social structure defined by market values of bundles, or wealth.

1. Introduction

Recent literature on the concept of fairness has developed an intuitively appealing and analytically productive idea of fairness or equity: an allocation of goods is *fair* if no person prefers another person's bundle of goods to his own. [For example, Kolm (1971), Schmeidler and Vind (1972), Feldman and Kirman (1974), Varian (1974), and Varian (1976).] In this paper we take the crucial ingredient of the fairness discussion – the idea of people not envying each other – and relate it to the stratification of the agents of an economy according to wealth.

The connection between fairness in the non-envy sense and equality of incomes has been analyzed by Varian (1976), whose main result, loosely speaking, is this: When there are many agents, if an allocation of goods is fair, efficient, and differentiable, then that allocation must give all equal incomes. In this paper, we stick to the finite (nondifferentiable) case, and go beyond the consideration of fair allocations by looking at the binary relation of non-envy for arbitrary efficient allocations. Person i is *non-envious* of person j if he would not prefer j's bundle of goods to his own, so an allocation is fair if for every pair i and j, i is non-envious of j. But any allocation, fair or not, defines a non-envy relation, and the only distinguishing feature of a fair allocation in this sense is the fact that for a fair allocation, *every* pair of individuals 'belongs to' the relation. Similarly, we go beyond equality of incomes by looking at the binary relation of wealth: Person i is *as-wealthy-as* person j if his bundle of goods, valued at competitive equilibrium prices, is worth as much as j's. While Varian (1976) is mainly concerned with the connection between a fair allocation and an equal-income allocation, we are mainly concerned with the connection between the non-envy relation and the as-wealthy-as relation.

Our interest in these two binary relations leads to an interest in the question of whether or not they partition the set of economic agents into equivalence classes. The as-wealthy-as relation obviously does this: There is a hierarchy of equal wealth classes; members of one equal wealth class are all as-wealthy-as each other, and any member of a higher equal wealth class is wealthier than any member of a lower one. This belabors the obvious only because as-wealthy-as is a complete and transitive relation. When we wonder about class structures for the non-envy relation the answers are not so obvious. It is not obvious that there is a hierarchy of non-envy classes, such that members of the same class are non-envious of each other, and such that any member of a higher non-envy class is non-envious of any member of a lower class, while the lower class member *is* envious of the higher. Why wonder? We often express an interest in social stratification by wealth (or income). But wealth does not enter utility functions in the usual microeconomics models; bundles of goods do. The most natural analog for the (wealth) statement 'i's wealth is as much as j's' is the (non-envy) statement 'i likes his bundle of goods as much as he would like j's.' In our view, wealth comparisons are the proxies, envy is the real thing! Well then, if there is a proxy stratification of the economy, is there necessarily a real one? How might the proxy and the real stratifications be related?

In this paper we ask these questions: What are the connections between the as-wealthy-as relation and the non-envy relation? Does the non-envy relation partition society into a hierarchy of non-envy classes? And if it does, how is that hierarchy related to the wealth hierarchy?

2. The model

We will discuss an exchange economy, with a set of individuals N, indexed by $i=1,2,\dots$. An allocation is a list of bundles of goods, with one bundle for each person. Let x_i represent i's bundle. An allocation x is *feasible* iff

$$\sum_{i \text{ in } N} x_i = \Omega,$$

where Ω is a vector of total endowments of goods in the economy. An arbitrary bundle of goods will occasionally be represented with the letter a.

Person i's preferences are represented by a continuous, selfish, utility function $u_i(x_i)$. We assume that $u_i(x_i)$ is monotonic, strictly quasiconcave, and differentiable. Also, we restrict our attention to the interior of the set of all possible allocations by requiring that every bundle x_i contains positive amounts of every good.

An allocation x is *efficient* iff there is no other feasible allocation y for which

$$u_i(y_i) > u_i(x_i) \quad \text{for all} \quad i.$$

Under the above assumptions, if x is efficient, then there exists a unique *competitive equilibrium price vector* p, with all positive prices, such that:

$$x_i \text{ maximizes } u_i(a), \text{ subject to } p \cdot a \leqq p \cdot x_i, \text{ for all } i.$$

Our main concern is with the feelings people have about the relative desirability of their own and others' bundles of goods. To formalize these feelings, we define two binary relations on the set N. First, let

$$iRj \quad \text{iff} \quad u_i(x_i) \geqq u_i(x_j).$$

Thus iRj means i likes his own bundle as well as he likes j's, or *i doesn't envy j*. Naturally R is a function of the particular allocation x which obtains. Naturally, also, a person's view of himself within society is rarely dependent only on the distribution of possessions which x represents; the other intangibles are overlooked by R. Second, let

$$i\hat{R}j \quad \text{iff} \quad p \cdot x_i \geqq p \cdot x_j.$$

Thus $i\hat{R}j$ means i's bundle is worth as much as j's, or i is *as wealthy as* j. Note that $\hat{R}$ is uniquely defined since the equilibrium price vector p is unique. However, if p were to change, $\hat{R}$ would also change, since $\hat{R}$ depends on p.

We think that the non-envy relation R is more fundamental than the as-wealthy-as relation $\hat{R}$, because a person's utility depends directly on his bundle of goods, rather than his net worth. However, $\hat{R}$ is empirically significant: wealth (or income) is observable and measurable. Wealth (or income) distributions are widely studied, and any observed wealth distribution defines an $\hat{R}$. Yet what matters to people is goods, not money. Therefore, observed wealth (or income) differences are interesting mainly because they say something about the likelihood that people might envy each other – that is, they say something about R, or, possibly, about a 'potential' R.

From the relations R and $\hat{R}$ we can derive associated asymmetric and symmetric relations:

$$iPj \quad \text{iff} \quad iRj \quad \text{and not} \quad jRi, \quad \text{that is, iff}$$

$$u_i(x_i) \geqq u_i(x_j) \quad \text{and} \quad u_j(x_i) > u_j(x_j),$$

and

$$iIj \quad \text{iff} \quad iRj \quad \text{and} \quad jRi, \quad \text{that is, iff}$$
$$u_i(x_i) \geqq u_i(x_j) \quad \text{and} \quad u_j(x_j) \geqq u_j(x_i).$$

Note that iPj means i doesn't envy j but j does envy i, which suggests i is better off than j: while iIj means neither one envies the other. Similarly,

$$i\hat{P}j \quad \text{iff} \quad i\hat{R}j \quad \text{and not} \quad j\hat{R}i, \quad \text{that is, iff} \quad p \cdot x_i > p \cdot x_j,$$

and

$$i\hat{I}j \quad \text{iff} \quad i\hat{R}j \quad \text{and} \quad j\hat{R}i, \quad \text{that is, iff} \quad p \cdot x_i = p \cdot x_j.$$

An arbitrary relation Q defined on N is *complete* iff, for any pair i, j in N, either iQj or jQi. It is *transitive* iff, for any i, j, k in N, if iQj and jQk, then iQk. If Q is complete and transitive it is an order. An arbitrary relation Q on N is *acyclic* iff its associated asymmetric part has no cycles. For example, R is acyclic iff there is no subset $\{i_1, i_2, \ldots, i_k\}$ of N for which $i_1 P i_2$, $i_2 P i_3, \ldots, i_{k-1} P i_k$, and $i_k P i_1$. If Q is complete and acyclic it is a *suborder*.

Any order Q on N partitions N into non-overlapping exhaustive equivalence classes, such that (i) i and j are in the same equivalence class if iQj and jQi, and (ii) i is in a higher class than j if iQj and not jQi [see, e.g., Birkhoff and MacLane (1953)]. The significance of this elementary result for binary relations is obvious when $Q = \hat{R}$: then an equivalence class is simply an equal-wealth class, and the hierarchy of equivalence classes is simply the wealth hierarchy. Obviously, this particular hierarchical structure is interesting to economists. But does the hierarchy of values-of-bundles, or wealth, correspond to a hierarchy for the more fundamental relation R? It is clear that $\hat{R}$ is always an order and that there is always a wealth hierarchy. Now if R is an order, then there exists a non-envy hierarchy, that is, a partition of N into non-overlapping exhaustive equivalence classes such that (i) i and j are in the same equivalence class if neither one envies the other, and (ii) i is in a higher class than j if i does not envy j but j does envy i. However, it is not clear that $\hat{R}$ need be an order, and, even if it is one, the correspondence between a wealth hierarchy and a non-envy hierarchy is problematical.

These observations motivate this paper. Since the distribution of wealth is a central empirical question, and since we believe $\hat{R}$ is only interesting because it somehow represents a more fundamental R, we think it is important to attempt, in a precise way, to examine the properties of R and $\hat{R}$ and the connections between R and $\hat{R}$.

3. The properties of R and $\hat{R}$

Suppose x is efficient. With arguments like those used by Kolm (1971), Feldman and Kirman (1974), and Varian (1974), it is possible to show R must be complete and acyclic. For if R were not complete, there would be a pair $\{i, j\}$ for whom $u_i(x_i) < u_i(x_j)$ and $u_j(x_j) < u_j(x_i)$. Switching the bundles x_i and x_j between i and j would make both better off. By monotonicity and continuity of the utility functions a further change to another feasible allocation could be made to make all people better off than at x. This would contradict efficiency for x. Similarly, if R were not acyclic, there would be a set of people for whom $i_1 P i_2$, $i_2 P i_3, \ldots, i_{k-1} P i_k$, and $i_k P i_1$. Giving i_1's bundle to i_2, i_2's bundle to i_3, and so on, would make them all better off. This would again lead to a contradiction of efficiency for x. Therefore, we have

Proposition 1. If x is efficient, the non-envy relation is complete and acyclic; that is, R is a suborder.

However, R need not be transitive, as the following shows:

Example 1. Under general assumptions, R need not be transitive. Consider the three-person, two-good economy illustrated in fig. 1:

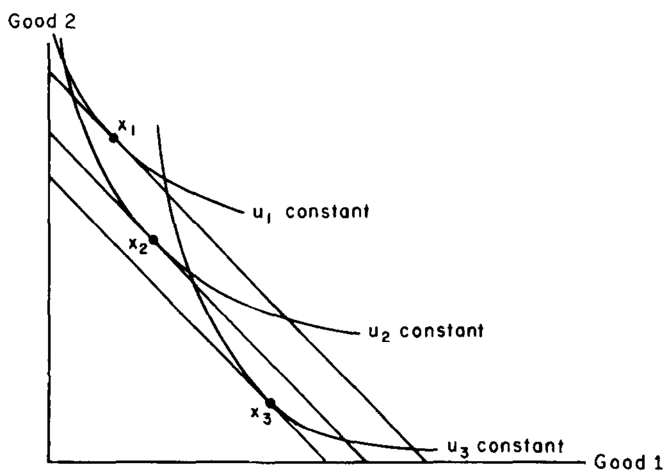

Fig. 1

Here are three (parallel) budget lines, three indifference curves, and three bundles x_1, x_2, x_3. Since x_1, x_2, and x_3 lie on budget lines defined by a unique competitive equilibrium price vector, the allocation $x = (x_1, x_2, x_3)$ is a competitive equilibrium allocation in the economy in which $x_1 + x_2 + x_3 = \Omega$, and it is therefore efficient. Clearly, $2R3$, since person 2 does not envy

3's bundle, and $3R1$. However, 2 does envy 1's bundle; that is, *not* $2R1$, and therefore, R is *not* transitive in this case.

It is occasionally useful to illustrate the relation R with a diagram in which an arrow from i to j, $i \to j$, stands for iPj, and a two-headed arrow between i and j, $i \leftrightarrow j$, stands for iIj. For instance, the non-envy relation R of example 1 is diagrammed in fig. 2(a).

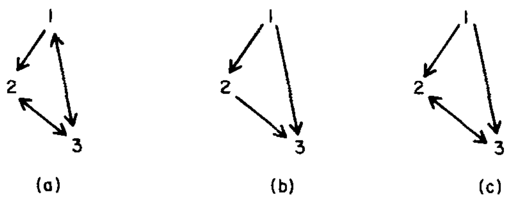

Fig. 2

The 2(a) diagram makes it clear that it is impossible to partition the three individuals into a non-envy-based hierarchy of classes. For example, if $\{1\}$ is proposed as the top class, and $\{2, 3\}$ as the bottom, the $1 \leftrightarrow 3$ link stands in the way, since it says 3 is not envious of 1. A similar diagram can be constructed to represent the as-wealthy-as relation $\hat{R}$. The $\hat{R}$ of example 1 is illustrated in fig. 2(b), which reaffirms what we already know: it *is* possible to partition the three individuals into a wealth-based hierarchy of classes, with $\{1\}$ at the top, $\{2\}$ in the middle, and $\{3\}$ at the bottom.

These two diagrams for the example 1 economy show that, although a wealth hierarchy is readily observable, a more fundamental R-based hierarchy need not exist, and so a wealth hierarchy need not accurately reflect anything fundamental.

It is sometimes argued, however, that $\hat{R}$ is what 'really' matters because if $i\hat{P}j$, j envies the *purchasing power implicit* in i's (more valuable) bundle. Person j cares not about i's bundle *per se*, but about what that bundle will trade for. Although this sounds plausible, it is incorrect when the number of individuals is small, in which case the competitive equilibrium price vector p depends heavily on who owns what bundle. Two person general equilibrium examples can easily be constructed in which person 1 is a half or a third as wealthy (given p) as person 2, but would under no circumstances be interested in swapping bundles with him. When there aren't many people, the relation R makes more sense than $\hat{R}$!

4. The $R - \hat{R}$ connections, and non-envy hierarchies

To this point we have argued that the unobservable non-envy relation is fundamentally more important – since it depends on utilities attached to bundles of goods – than the observable as-wealthy-as relation. Also, the hierarchical structure which is necessarily present when wealth is considered might be absent when non-envy is considered. Nonetheless, we will show that a wealth-based hierarchy is in a sense an *approximation* to a non-envy hierarchy, because the as-wealthy-as relation is in a sense an *approximation* to the non-envy relation. Moreover, we will show $\hat{R}$ in fact *coincides* with R under certain circumstances. Insofar as $\hat{R}$ does approximate R, and inasmuch as $\hat{R}$ might coincide with R, partitions of society according to wealth are, in certain senses, indicative of a more fundamental partition according to comparisons of bundles, or envy. We turn them to these questions: (1) What do we mean by an 'approximation' to a binary relation? (2) Is $\hat{R}$ an approximation to R? (3) When does $\hat{R}$ coincide with R?

The intuitive rationale for our definition of an approximation can be seen by reconsidering the diagrammatic illustration of R for example 1, that is, fig. 2(a). If this diagram were changed by the erasure of the head of the arrow from 3 to 1, we would be left with fig. 2(c). (The erasure means $1I3$ was changed to $1P3$, or $3R1$ was nullified, or 3 became envious of 1 – because of a change in bundles, or utility functions, presumably.) Now the modified diagram is transitive, and does define a hierarchical structure: the top equivalence class is $\{1\}$, and the bottom equivalence class is $\{2,3\}$. Such a diagram represents what we would call an approximation to R: that is, a binary relation derived from R by erasing some of the arrows in R.

More formally, any binary relation Q defined on N can be viewed as a subset of $N \times N$; that is, writing iQj is equivalent to saying the pair (i,j) is an element of the subset Q of $N \times N$. Since binary relations are sets, one can contain another; that is, we can have $Q' \subset Q$. In this case we will say Q' is an *approximation to Q from below*, or for short, Q' is an *approximation* to Q. In this paper, *approximation simply means containment.*

By the definition, if Q is an approximation to the non-envy relation R, then iQj (or (i,j) is in Q) implies iRj (or (i,j) is in R), that is, i does not envy j's bundle of goods. Now we formally observe that $\hat{R}$, the as-wealthy-as relation, is in fact an approximation to R. (This proposition is a consequence of a proof of Kolm (1971).)

Proposition 2. Suppose x is efficient. Then $\hat{R}$ is contained in R, that is, $i\hat{R}j$ implies iRj, for all i and j.

Proof. $i\hat{R}j$ means $p \cdot x_i \geqq p \cdot x_j$. Since x_i maximizes $u_i(a)$ subject to $p \cdot a \leqq p \cdot x_i$, $u_i(x_i) < u_i(x_j)$ is impossible. Q.E.D.

The reader might note that $\hat{R}$ of example 1, that is, fig. 2(b) is an approximation of R of example 1, that is, fig. 2(a). As the proposition requires, $\hat{R}$ is contained in R.

What does proposition 2 mean in terms of partitions of society according to wealth? Simply this: In a competitive equilibrium situation, if i and j are equally wealthy, neither can be envious of the other's bundle of goods. If i is wealthier than j, then i cannot envy j's bundle of goods. However, if i is poorer than j, it does not follow that i envies j's bundle of goods. In this sense social stratification by wealth is necessary, but not sufficient, for social stratification by envy.

According to proposition 2, $\hat{R} \subset R$ always holds when x is efficient. The reverse containment, however, is problematical. Obviously, if all utility functions are identical, then $R \subset \hat{R}$ must be true. This suggests that if preferences are somehow *similar*, $\hat{R}$ ought to be close to R. This is the idea we explore below.

Let $B_i(x_i) = \{a | u_i(a) > u_i(x_i)\}$. $B_i(x_i)$ is the set of bundles person i prefers to his own bundle. Now suppose that $B_i(x_i)$ is contained in $B_j(x_j)$. We claim this implies $u_i(x_i) \geqq u_i(x_j)$; that is, iRj. If not, then $u_i(x_i) < u_i(x_j)$. Therefore, $x_j \in B_i(x_i) \subset B_j(x_j)$, so $u_j(x_j) > u_j(x_j)$, a contradiction.

What if the converse holds, that is, if

$$iRj \quad \text{implies} \quad B_i(x_i) \subset B_j(x_j)? \tag{1}$$

Condition (1) is illustrated for the two-dimensional case in fig. 3.

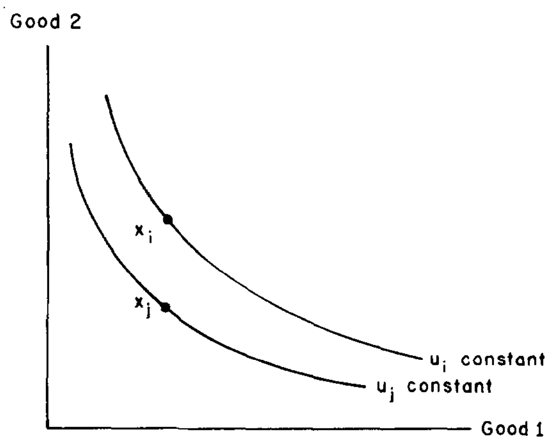

Fig. 3

In this figure, individual i's indifference curve through x_i lies entirely within the area on or above individual j's indifference curve through x_j; the two curves must not cross. The condition imposes a conceptually simple sort of uniformity of tastes and bundles, and it requires, among other things, that if iRj and jRi then i's and j's indifference curves through x_i and x_j must in fact coincide.

We will say the economy satisfies the *strong uniformity* assumption iff for all i, j in N, iRj implies $B_i(x_i) \subset B_j(x_j)$.

The significance of strong uniformity is that it forces the non-envy relation and the as-wealthy-as relation to be identical.

Proposition 3. Suppose x is efficient. Suppose the strong uniformity assumption is satisfied. Then $R = \hat{R}$.

Proof. In light of proposition 2, it will suffice to show that R is contained in $\hat{R}$, or iRj implies $i\hat{R}j$. Let iRj. Assume it is *not* the case that $i\hat{R}j$. Therefore, $p \cdot x_i < p \cdot x_j$. By monotonicity and strict quasi-concavity we can clearly find a sequence of bundles $\{a^k\}$ such that $u_i(a^k) > u_i(x_i)$ and $a^k \to x_i$. Because $a^k \to x_i$, $p \cdot a^k \to p \cdot x_i < p \cdot x_j$. Therefore, we can choose a k^* for which $p \cdot a^{k^*} \leqq p \cdot x_j$. Now we have $a^{k^*} \in B_i(x_i)$, and $B_i(x_i) \subset B_j(x_j)$ because of iRj and strong uniformity. Therefore, $u_j(a^{k^*}) > u_j(x_j)$. Therefore, x_j does not maximize j's utility subject to the budget constraint $p \cdot a \leqq p \cdot x_j$, contradicting the efficiency of x. Q.E.D.

By proposition 3, under the strong uniformity assumption R and $\hat{R}$ exactly coincide. It follows that R is an order (since $\hat{R}$ is) and that R partitions N into non-overlapping, exhaustive, non-envy-based equivalence classes. There is a non-envy class hierarchy. This is a fundamental hierarchical structure, since it is based on utility evaluations of bundles. Moreover, it precisely coincides with the wealth-based hierarchy. Therefore, observable equal wealth classes are *precisely* non-envy classes, and every poor man would *truly* like to trade places with every rich man.

Strong uniformity, however, is a substantial assumption. Let us see how it might be weakened.

Suppose iRj but $B_i(x_i) \not\subset B_j(x_j)$; that is, suppose the strong uniformity assumption breaks down for some pair i, j in N. This breakdown by itself will not destroy transitivity for R, since transitivity always involves three individuals, and, so far, we have only two. However, as Example 1 shows, transitivity *is* destroyed when a bundle owned by a third party is placed between crossed indifference curves. In that example, $2R3$, $B_2(x_2) \not\subset B_3(x_3)$, and x_1 lies above 2's indifference curve and below 3's. This possibility is ruled out by the following assumption:

We will say that the economy satisfies the *uniformity* assumption iff for all i, j, k in N, iRj implies x_k is not a member of $B_i(x_i) - B_j(x_j)$. Here $B_i(x_i) - B_j(x_j)$ is the set theoretic difference, so x is in $B_i(x_i) - B_j(x_j)$ if and only if x is in $B_i(x_i)$ but not in $B_j(x_j)$.

The difference between strong uniformity, on the one hand, and uniformity, on the other, is that strong uniformity requires that iRj implies $B_i(x_i)-B_j(x_j)$ is *empty*, while uniformity requires that iRj implies $B_i(x_i)-B_j(x_j)$ contains no bundle which *belongs* to someone. Obviously strong uniformity implies uniformity. The uniformity assumption is illustrated in fig. 4, where we require that no x_k be included in the shaded area.

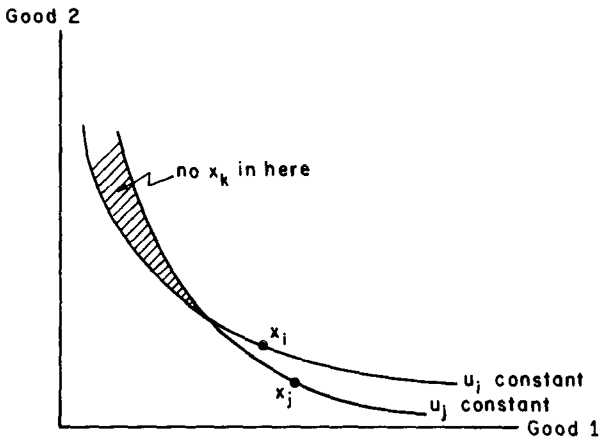

Fig. 4

The significance of uniformity is that it forces the non-envy relation to be transitive.

Proposition 4. R is transitive if and only if the uniformity assumption is satisfied.

Proof. First, suppose R is transitive, and iRj. We must show that, for all x_k, x_k is not in $B_i(x_i)-B_j(x_j)$; that is, either x_k is not in $B_i(x_i)$ or x_k is in $B_j(x_j)$; that is, either $u_i(x_k)\leqq u_i(x_i)$ or $u_j(x_k)>u_j(x_j)$. Now either jRk holds, or it doesn't. If jRk, iRk must hold by transitivity; that is, $u_i(x_i)\geqq u_i(x_k)$, and we are done. If not jRk, then $u_j(x_k)>u_j(x_j)$, and we are done again. *Second,* suppose the uniformity assumption is satisfied, and iRj and jRk. We must show that iRk. By iRj and uniformity, x_k is not in $B_i(x_i)-B_j(x_j)$; that is, either $u_i(x_k)\leqq u_i(x_i)$ or $u_j(x_k)>u_j(x_j)$. Since jRk by assumption, $u_j(x_j)\geqq u_j(x_k)$. Therefore, $u_i(x_k)\leqq u_i(x_i)$ must hold, i.e. iRk, and R is transitive. Q.E.D.

An immediate consequence is:

Proposition 5. Suppose x is efficient and the uniformity assumption is satisfied. Then R is an order. Moreover, R partitions N into a hierarchy of *non-overlapping, exhaustive, non-envy-based equivalence classes, with the property that within an equivalence class no one envies anyone else, but members of lower classes always envy members of the higher classes. Conversely, if R is an order, the uniformity assumption is satisfied.*

Thus uniformity is a necessary and sufficient condition, when x is efficient, for there to be a class hierarchy based on utilities on bundles of commodities. How will this envy-based class hierarchy correspond to the wealth-based partition of N? It will be a *coarser* partition. That is, every equal-wealth class will be contained in a non-envy class, but not necessarily vice versa. Therefore, under uniformity we do not have the strong result of proposition 3, where the observable net worth hierarchy has to correspond to a non-envy hierarchy. But we have more than proposition 2, since now a non-envy class hierarchy *must* exist, although it is still the case that i's being poorer than j does not imply that i envies j's bundle of goods.

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