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Fairness and Envy

By Allan Feldman and Alan Kirman*

"Envy's a stronger spur than pay."
John Gay

"This only grant me that my means may lie Too low for envy, for contempt too high." Abraham Cowley

Standard neoclassical economic analysis is typically concerned with individual utility maximization. In this paper we shall consider a problem of constrained social welfare maximization. Our criterion of social welfare is "fairness," and we shall discuss how this may be maximized by a move from an initial allocation to a final fairer allocation, subject to the constraint that no one be made worse off by the move. We think the goal of fairness maximization characterizes, albeit in a simplistic way, the goals pursued by "enlightened" governments in their redistributional policies. We shall also discuss a concept of complete fairness and illustrate some of its weaknesses.

The fairness problem is ancient and dates back at least to classical Greece. It has been treated recently by mathematicians who typically are concerned with the existence of a "fair division" of a nonuniform object among n persons; that is, a division with the property that each party thinks he is getting at least $1/n$th of the value of the object. (See, for example, Lester Dubins and Edwin Spanier, Harold Kuhn, and Hugo Steinhaus.) This is not the approach we will take, since we will assume a world of homogeneous infinitely divisible goods in which the mathematical fair division problem becomes trivial.

The concept of fairness has also been treated extensively by philosophers. The most recent philosophical approach is that of John Rawls, who argues at length for a social contract theory of justice: a society which maximizes the welfare of its worst off members is most just and that is the sort of society people will, from an initial position of ignorance about their endowments and interests, contract to enter. Rawls' approach has been extended to a theory of taxation by Edmund Phelps. Again, Rawlsian fairness, or "justice," is not the fairness we are interested in; we do not assume a precontractual state of ignorance, we do assume that knowledge of wealth and tastes is given. In fact, knowledge about one's own and others' bundles of goods is crucial in our discussion.

What then is our notion of fairness? It is fairness in the sense of *non-envy*. A completely fair social state is one in which no citizen would prefer what another has to what he himself has; a relatively fair social state is one in which few citizens would prefer what others have to what they themselves have; a totally unfair state is one in which every citizen finds his position to be inferior to that of everyone else. This concept of fairness is appealing because it only depends, like other economic concepts, on individual tastes and endowments.

Fairness in the non-envy sense has been discussed in several recent papers by economists. Serge Christophe Kolm considers allocative fairness, and shows that there exist allocations which are both completely fair and efficient.[1] David Schmeidler and Karl Vind define fair *trades* as

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[1] In a recent paper Hal Varian extends this sort of analysis. Also Richard Zeckhauser discusses one of Kolm's concerns at some length: what does a fairness-minded planner do when there is an unfair distribution of nontransferrable goods, like I.Q.?

trades which exclude envy (a transaction between a price discriminating monopolist and two customers—each getting a different price—is an "unfair" trade in this sense). In both the Kolm and the Schmeidler-Vind papers, things are either completely fair (or totally devoid of envy) or not. Neither paper considers the problem of establishing an index of fairness and maximizing it, as we will below.

In this paper we will first discuss the question of the preservation of complete fairness. We will show that such fairness is *not* preserved by competitive equilibrium trades, by trades to the core, or *even* by fair trades. In short, perfect fairness is a delicate condition. Whether this says something about our definitions, or whether it says something about the real world, we leave to the reader to decide. Our own feeling is that it says something about both.

Then we will define three social measures of envy. One is an ordinal-utility counting measure which sums up the instances of envy inherent in an allocation. The second and third are weighted sums of cardinal-utility individual envy measures. Finally we will characterize the solutions of some fairness maximization (or envy minimization), problems in an economy in which all traders have identical utility functions.

I. The Model

Consider an n-trader, m-good economy. We will assume that each of the m commodities is homogeneous and infinitely divisible. An *allocation* $x=(x_1, x_2, \ldots, x_n)$ is an n vector of nonnegative m vectors, i.e., a point in R_+^{nm}, whose ith component, x_i, is the bundle of goods assigned to trader i under x. We will let ω be a fixed initial allocation, we assume that $\sum_{i=1}^n \omega_i > 0$.[2] We also define $A(\omega)$ to be the set of allocations which are feasible in our economy. That is, $A(\omega)\equiv\{x\colon x\geqq 0 \text{ and } \sum_{i=1}^n x_i=\sum_{i=1}^n \omega_i\}$.

We will assume that every trader has preferences which can be represented by a continuous utility function $u_i(x_i)$ which maps R^m into R.

We will assume that the utility functions of all traders are strictly quasi concave: If x_i and y_i are distinct nonnegative m vectors, $0<\lambda<1$, and $u_i(x_i)\geqq u_i(y_i)$, then $u_i(\lambda x_i+(1-\lambda)y_i)>u_i(y_i)$. We will also assume that all u_i are strictly monotonic.

Pareto optimal (or *efficient*), *core*, and *competitive equilibrium* allocations are defined in the usual ways. Consider an allocation x. If a subset S of traders can redistribute its own resources in a way which makes all of its members at least as well off as x makes them, and makes some of them better off, we say that S can block x. Formally, if there are bundles s_i, for all i in S, such that $\sum_{i \text{ in } S} s_i=\sum_{i \text{ in } S}\omega_i$ $u_i(s_i)\geqq u_i(x_i)$ for all i in S, and $u_j(s_j)>u_j(x_j)$ for some j in S, then S *blocks* x. An allocation is in the core if no group of traders can block it. An allocation is Pareto optimal if it cannot be blocked by the whole set of traders. If p is a vector of prices and $\hat{x}$ is an allocation, and if $\hat{x}_i$ maximizes $u_i(x_i)$ subject to $p\cdot x_i\leqq p\cdot\omega_i$ for all i, then we say that $(p, \hat{x})$ is a competitive equilibrium, and that $\hat{x}$ is a competitive equilibrium allocation.

Following Kolm and Schmeidler and Vind, we will say that an allocation x is *fair* if for every pair of traders $\{i, j\}$, $u_i(x_i)\geqq u_i(x_j)$. If $t=(t_1, t_2, \ldots, t_n)$ is a vector of m vectors satisfying $x_i+t_i\geqq 0$ for all i and $\sum_{i=1}^n t_i=0$, we will say that t is a *feasible trade from* x. We will say that t is a *fair trade from* x if it is feasible and if

[2] We use the following vector inequality notation: If x and y are k dimensional vectors, $x\geqq y$ means $x_i\geqq y_i$ for $i=1, \ldots, k$; $x\geq y$ means $x_i\geqq y_i$ for $i=1, \ldots, k$, and $x_j>y_j$ for some j; $x>y$ means $x_i>y_i$ for $i=1, \ldots, k$.

for every pair $\{i, j\}$ of traders, $u_i(x_i+t_i) \geqq u_i(x_i+t_j)$ whenever $x_i+t_j \geqq 0$.

According to our definitions, a fair allocation is an allocation with the property that no trader would prefer another's bundle of goods to his own, and a fair trade is a trade with the property that no trader would prefer another's *exchange* to his own, providing that he could have made it.

Now we will define $C(x) \equiv$ the number of pairs $\{i, j\}$ for whom $u_i(x_i) < u_i(x_j)$. That is, $C(x)$ counts the number of instances of envy associated with the allocation x. Clearly $C(x)=0$ if and only if x is fair and $C(x)$ attains its maximum of n^2-n when every trader is envious of every other trader. When we maximize fairness in the sense of $C(x)$, we will be maximizing $-C(x)$.

Our second and third, nondiscrete, measures of envy presume that individuals have cardinal utility functions, so that utility sums and differences are meaningful for each trader.

Let us define

$$e_i(x) \equiv \sum_{j=1}^{n} [u_i(x_j) - u_i(x_i)]$$

The function $e_i(x)$ measures i's total envy by adding up his envy (positive or negative) of every other trader. We can define a vector of envies as follows:

$$e(x) \equiv (e_1(x), e_2(x), \ldots, e_n(x))$$

The vector $e(x)$ can be treated in a manner analogous to the usual economic treatment of utility vectors; for example, it is possible to find allocations undominated in envy just as it is possible to find allocations undominated in utility. We might remark that if x is a fair allocation, then $e(x) \leqq 0$, but not vice versa.

The measure $e_i(x)$ has the property that a man who isn't on the bottom rung of the economic ladder is compensated (in envy) by the misfortune of those below him. Such compensation may not seem entirely natural, and for this reason we will define a third measure of envy without this feature. Let

$$e_i^*(x) \equiv \sum_{j: u_i(x_j) \geqq u_i(x_i)} [u_i(x_j) - u_i(x_i)]$$

Also, let

$$e^*(x) \equiv (e_1^*(x), e_2^*(x), \ldots, e_n^*(x))$$

Now we can remark that x is a fair allocation if and only if $e^*(x) = (0, 0, \ldots, 0)$.

The two vectors $e(x)$ and $e^*(x)$ are not comparable to the scalar $C(x)$, so we will define two more social envy (or unfairness) measures as follows:

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a set of positive weights, we will let the *total social envy* (given α) for an allocation x be

$$E(\alpha, x) \equiv \sum_{i=1}^{n} \alpha_i e_i(x)$$

Similarly, we will let the *total social envy** (given α) for an allocation x be

$$E^*(\alpha, x) \equiv \sum_{i=1}^{n} \alpha_i e_i^*(x)$$

Now for any $\alpha > 0$, $E(\alpha, x) \leqq 0$ if x is fair, and $E^*(\alpha, x) = 0$ if and only if x is fair. When we maximize fairness in the sense of $E(\alpha, x)$ or $E^*(\alpha, x)$, we will be maximizing $-E(\alpha, x)$ or $-E^*(\alpha, x)$.

II. The Delicate Nature of Allocative Fairness

Partly for our immediate gratification, and partly to motivate the discussion of fairness maximization which will follow, we will now show that a number of reasonable looking, useful, sensible, and comforting conjectures about fairness are false.

In this section we are particularly concerned with the preservation of allocative fairness. A goal of social policy ought to have some "stability" properties; if the goal is "unstable" in some sense, the policy maker's job is much bigger than if it is "stable": he must not only institute a change, he must also remain around forever to prevent backsliding. Pareto optimality has an obvious stability property since deviations from it create self-corrective incentives. This is not the case for fairness, however.

Our first two examples are motivated by a theorem of Kolm which says that (under general conditions) there exist allocations which are simultaneously fair and efficient. The proof of the theorem uses the fact that a trade from the equal allocation (which assigns every trader an identical bundle of goods) to a competitive equilibrium preserves fairness. The reason for this is transparent, for if the economy starts at the equal allocation and x is a competitive allocation based on it, then x_j must be in the ith trader's budget set for every pair $\{i, j\}$, and so envy (that is, an inequality of the form $u_i(x_j) > u_i(x_i)$) contradicts utility maximization. But what if we start at an arbitrary fair allocation and make a competitive equilibrium trade? Do we end at a competitive equilibrium allocation which is fair?

The following Edgeworth box diagram shows that we need not. In Figure 1, i_1 and i_2 are two of trader I's indifference curves; j_1 and j_2 are two of trader J's; $\omega = (\omega_i, \omega_j)$ is the initial allocation; $\omega^{-1} \equiv (\omega_j, \omega_i)$ is the allocation which switches the bundles between i and j. Now the allocation $x = (x_i, x_j)$ is a competitive allocation (from ω), but it is *not* fair, because $x^{-1} \equiv (x_j, x_i)$ lies above the indifference curve labelled i_2 which means that trader I envies trader J at x.

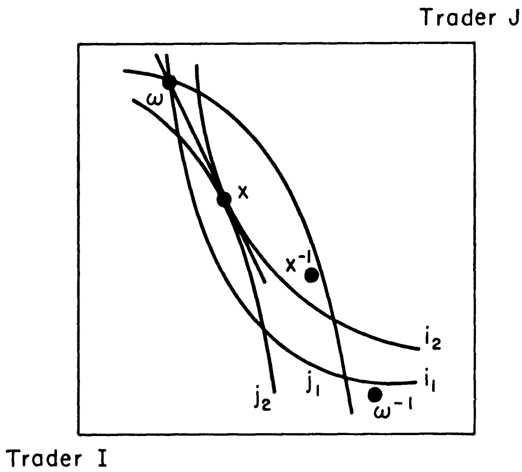

FIGURE 1

We've shown that competitive equilibrium trades may destroy fairness. Moreover, a trade from the equal allocation (surely the fairest of the fair) to the core may also destroy fairness. An example of this perverse result is illustrated in Table 1. In this three-person three-good economy, the initial allocation is the equal allocation. Some examination will convince the reader that x is in the core: no subset of the three traders could, by an internal redistribution of its initial holdings, make all of its members at least as well off, and some better off, than x makes them. However, $u_2(x_1) = 20/3 > 6 = u_2(x_2)$, so x is not fair. Therefore, barter exchange is apt to destroy fairness, even from a starting point of complete equality.

Our final example shows that fair trades themselves may destroy (allocative) fairness. In this sense the approaches of Schmeidler and Vind and Kolm are mu-

TABLE 1

	Utility Functions	ω_i	$u_i(\omega_i)$	x_i	$u_i(x_i)$
Trader 1	$u_1(x_1) = 3x_{11} + 2x_{12} + x_{13}$	(1, 1, 1)	6	(3, 2/3, 0)	10 1/3
Trader 2	$u_2(x_2) = 2x_{21} + x_{22} + 3x_{23}$	(1, 1, 1)	6	(0, 0, 2)	6
Trader 3	$u_3(x_3) = x_{31} + 3x_{32} + 2x_{33}$	(1, 1, 1)	6	(0, 7/3, 1)	9

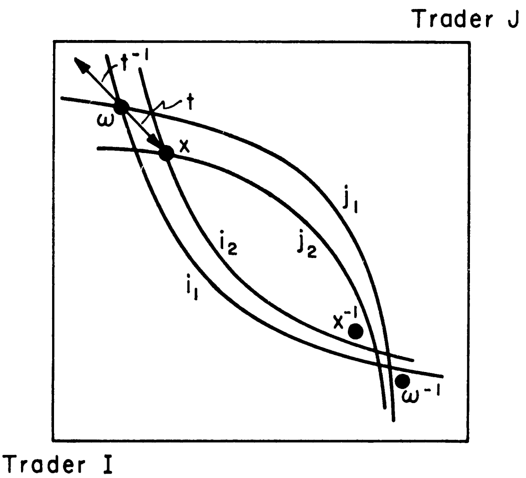

FIGURE 2

tually inconsistent. Moreover, the example best illustrates the delicacy of fairness; it shows that the preservation of complete allocative fairness requires more than just a monitoring of the fairness of the moves; it requires a constant monitoring of the *results* of the moves. Consider the Edgeworth box diagram of Figure 2.

Here we again have an economy of two traders and two goods: again i_1 and i_2 are two of trader I's indifference curves; j_1 and j_2 are two of trader J's; $\omega=(\omega_i, \omega_j)$ is the initial allocation; $t=(t_i, t_j)$ is a feasible trade, and $x=\omega+t$; ω^{-1} is again (ω_j, ω_i) and $x^{-1}\equiv(x_j, x_i)$. Also, $t^{-1}\equiv(t_j, t_i)$. Since both i_1 and j_1 pass "above" ω^{-1}, ω is *fair*. Since t makes I better off while t^{-1} would make him worse off, and a symmetrical argument applies to trader J, t is a *fair trade from* ω. But i_2 passes "under" x^{-1} as does j_2 and both I and J are envious at x. Therefore, x is *not* fair. A fair trade from a fair allocation can result in an unfair allocation.

When it's defined as the total absence of envy, fairness is a fragile condition. It is apt to disappear if people engage in trade for private benefit.

Having said this much, we will move on to less demanding criteria of fairness than the total absence of envy. Now let's measure the extent of envy, and consider the problem of minimizing it, without making anyone worse off. We are interested then in the qualitative implications, if there are any, of envy minimization.

III. When $C(x)$ Measures Envy

In this section we will analyze the problem of maximizing fairness in the sense of minimizing $C(x)$. The unconstrained problem is of course trivial, since it is solved by the equal allocation, among others. The interesting approach is to minimize $C(x)$ subject to a constraint, and the most obvious constraint is the requirement that no trader be made worse off by a fairness-increasing move. This is the natural constraint of voluntariness, natural because in a free society there is usually a governmental predisposition toward Pareto moves.

We are concerned, then, with minimizing $C(x)$ subject to $u_i(x_i)\geqq u_i(\omega_i)$, for all i. The problem clearly has a solution since $0\leqq C(x)\leqq n^2-n$ and C takes on only integer values. However, it cannot generally be solved by standard methods, so we will confine our analysis to a special case. We will assume that every trader has the *same* strictly quasi-concave, monotonic utility function u. We will also assume that u is homothetic: for $\lambda\geqq 0$, $u(\lambda x)=\phi(\lambda)u(x)$, where $\phi(\lambda)$ is some monotonic function of λ. Under these conditions it is possible to "reduce" the economy to one in which: 1) every trader has a bundle $\hat{x}_i$ which is proportional to $\sum_{i=1}^n \omega_i$; 2) there is a "social surplus" bundle $L\equiv \sum_{i=1}^n \omega_i - \sum_{i=1}^n \hat{x}_i$; 3) $\hat{x}\equiv(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is efficient in the economy with total resources $\sum_{i=1}^n \omega_i - L = \sum_{i=1}^n \hat{x}_i$; and 4) $u(\hat{x}_i)=u(\omega_i)$, for all i. Since all the $\hat{x}_i$ and L are proportional to ω, we can simply define $\sum_{i=1}^n \omega_i$ to be one unit of one *composite* good.

Now let us imagine that the economy

has been reduced, that is, that the allocation $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, L)$ satisfying 1)–4) above has been established, and that an enlightened ruler wants to distribute L in a way which will maximize fairness in the sense of minimizing $C(x)$. He can do so by distributing proportional bundles from the social surplus bundle L since the distribution of proportional bundles will preserve efficiency. Therefore, we can restrict our attention to the one-composite-good case. Let ΔL_i be the "fairness grant" of the composite good going to the ith individual. The ruler's problem is to choose $(\Delta L_1, \Delta L_2, \ldots, \Delta L_n) \geqq 0$ so that $\sum \Delta L_i \leqq L$, and so that the number of pairs $\{i, j\}$ for whom $u(\hat{x}_i + \Delta L_i) > u(\hat{x}_j + \Delta L_j)$ is minimized. By the monotonicity assumption, the inequality $u(\hat{x}_i + \Delta L_i) > u(\hat{x}_j + \Delta L_j)$ can be replaced by $\hat{x}_i + \Delta L_i > \hat{x}_j + \Delta L_j$.

We will partition the traders in the economy into $h \leqq n$ classes, $S_1, S_2, \ldots, S_h$, by putting traders with equal x_is into the same class. Let us suppose $\hat{x}_{i_1}$ is the $\hat{x}_i$ associated with class S_1, $\hat{x}_{i_2}$ is the $\hat{x}_i$ associated with S_2, and so on, and without loss of generality, we'll assume the classes are numbered from richest to poorest: $\hat{x}_{i_1} > \hat{x}_{i_2} > \ldots$. Now define $\delta_2 \equiv \hat{x}_{i_1} - \hat{x}_{i_2}$, the difference between the wealth of members of class S_1 and members of class S_2, $\delta_3 \equiv \hat{x}_{i_2} - \hat{x}_{i_3}, \ldots, \delta_h \equiv \hat{x}_{i_{h-1}} - \hat{x}_{i_h}$. Finally, suppose $n_1 \equiv$ the number of members of S_1, $n_2 \equiv$ the number of $S_2, \ldots, n_h \equiv$ the number of members of S_h.

The only way to eliminate instances of envy (without making anyone worse off) is to move groups of traders from lower classes to higher classes. It is clear that any total migration upward can be represented as a vector of one-step upward moves. Therefore, any movement upward can be represented by a vector $(k_2, k_3, \ldots, k_h)$, where $k_r \equiv$ the number of individuals who move from S_r to S_{r-1}. Since we cannot have negative numbers of individuals in any class, we must have

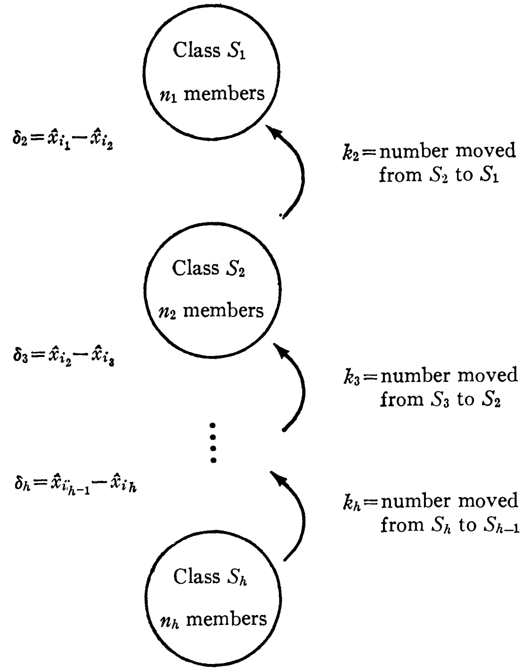

FIGURE 3

$n_r - k_r + k_{r+1} \geqq 0$ for $r = 2, \ldots, h-1$, and $n_h - k_h \geqq 0$. The total "cost" to the dictator of a movement represented by $(k_2, \ldots, k_h)$ is given by $\sum_{r=2}^{h} \delta_r k_r$. The process is illustrated in Figure 3.

It's easy to see that the number of instances of envy associated with the allocation $\hat{x}$ is given by

$$C(\hat{x}) = \sum_{i<j} n_i n_j, \text{ where } \quad 1 \leqq i \leqq j \leqq h$$

If the social surplus is disbursed in a way which brings about a vector of upward movements $(k_2, k_3, \ldots, k_n)$, the number of instances of envy becomes

$$\begin{aligned} &(n_2 - k_2 + k_3)(n_1 + k_2) + (n_3 - k_3 + k_4) \\ &\qquad \cdot (n_2 - k_2 + k_3 + n_1 + k_2) + \ldots \\ &\quad = \sum_{i<j} n_i n_j + \sum_{r=2}^{h} k_r (n_r - n_{r-1}) \\ &\qquad + \sum_{r=2}^{h} k_r (k_{r-1} - k_r) \end{aligned}$$

where we define $k_1 \equiv 0$.

We can finally reformulate the benevolent ruler's fairness maximization problem as follows: Having extracted the social surplus from the economy and brought it to an allocation $\hat{x}$, he wants to disburse that surplus, to establish a new allocation, say x, in a way that maximizes the reduction in envy (or minimizes the increase) brought about by his disbursement. That increase is given by

$$C(x) - C(\hat{x}) = \sum_{r=2}^{h} k_r(n_r - n_{r-1}) + \sum_{r=2}^{h} k_r(k_{r-1} - k_r)$$

so he wants to choose $(k_2, \ldots, k_h) \geqq 0$ to minimize $\sum_{r=2}^{h} k_r(n_r - n_{r-1}) + \sum_{r=2}^{h} k_r(k_{r-1} - k_r)$ subject to $k_1 = 0$, $n_r - k_r + k_{r+1} \geqq 0$ for $r = 1, \ldots, h-1$, $n_h - k_h \geqq 0$, and $\sum_{r=2}^{h} \delta_r k_r \leqq L$.

It is worth noting that a solution for this problem might be found for which

$$\sum_{r=2}^{h} \delta_r k_r < L$$

In such a case there is a leftover $L - \sum_{r=2}^{h} \delta_r k_r$ which cannot be used to reduce envy. This leftover can clearly be divided up and distributed in such a way that no one is moved out of his own class; the (fairness-maximizing) allocation which results is then efficient in the original (unreduced) economy.

The analysis of this fairness maximization problem is straightforward, providing we ignore the implicit integer constraints (only whole persons can be moved) on the k_is. First, we remark that the quadratic part of the objective function, $\sum_{r=2}^{h} k_r(k_{r-1} - k_r)$, is concave.[3] Therefore, the objective function is concave and the problem is one of minimizing a concave function on a closed and bounded convex set. It follows that it will have a solution at an extreme point of that feasible set. Therefore, the solution can be characterized in one of the following ways. (For notational simplicity we define $k_{h+1} \equiv 0$.)

PROPOSITION 1: *The solution to the fairness maximization (in the sense of $C(x)$ minimization) problem satisfies one of the following sets of conditions:*

$$(1) \qquad \sum_{r=2}^{h} \delta_r k_r < L$$

and for each $i = 1, 2, \ldots, h$, either $k_i = 0$ or $k_i = n_i + k_{i+1}$.

$$(2) \qquad \sum_{r=2}^{h} \delta_r k_r = L$$

and for all i but at most one, either $k_i = 0$ or $k_i = n_i + k_{i+1}$.

Let us interpret these conditions: (1) says that every class is either eliminated or has *no* out-migrants whatsoever. There are two degenerate subsolutions:

$$(1') \qquad k_r = n_r + k_{r+1}, \qquad \text{for } r = 2, \ldots, h$$

Now everyone moves up to the first class, which means that we must have had

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[3] If we let

$$M \equiv \begin{vmatrix} -1 & 1/2 & 0 & 0 & \ldots \\ 1/2 & -1 & 1/2 & 0 & \ldots \\ 0 & 1/2 & -1 & 1/2 & \ldots \\ \vdots & & & & \\ 0 & 0 & \ldots & 1/2 & -1 \end{vmatrix}$$

then

$$\sum_{r=2}^{h} k_r(k_{r-1} - k_r) = (k_2, \ldots, k_h) M \begin{bmatrix} k_2 \\ \vdots \\ k_h \end{bmatrix}$$

and the quadratic form is concave if M is negative definite (see, e.g., George Hadley), that is, if $-M$ is positive definite. Now $-M$ is positive definite if it has a positive dominant diagonal (see, e.g., Hukukane Nikaido, pp. 385–87), and it is a simple exercise, which we will not perform here, to show that $-M$ does in fact have a positive dominant diagonal.

$$\sum_{r=2}^{h} \delta_r k_r = \sum_{r=2}^{h} \delta_r \left[\sum_{i=r}^{h} n_i \right] < L$$

to begin with: the social surplus was large enough to establish complete equality.

$$(1'') \qquad k_2 = k_3 = \ldots = k_n = 0$$

Now no one moves. As an example, consider the case where $n_1=100$, $n_2=1000$, $\delta_2=1$, and $L=10$. Any choice of k_2 greater than zero (and necessarily smaller than 10), will cause envy to increase. It is clear that a necessary condition for the no-movement solution is that the class sizes be pyramidal; that is, $n_1<n_2< \ldots <n_h$.

For solutions in category (2), every class but one is either disappearing, or absorbing. The degenerate solution is again given by

$$(2') \qquad k_r = n_r + k_{r+1}, \qquad \text{for } r = 2, \ldots, h$$

Everyone moves up to the first class.

In *no* case do we have to be concerned with the possibility that some but not all individuals in class i move and some but not all individuals in class j move. Moreover, there is *no* presumption that the fairness disbursement need go first to the poorest classes.

Let us observe that the solution to the fairness maximization problem could be found by an exhaustive search of the extreme points of the feasible set. Is there a marginal algorithm which will also find it? The marginal gain in fairness (or reduction in envy) which results from the movement of one member of class j to class $j-1$ is given by

$$-n_j n_{j-1} + (n_j - 1)(n_{j-1} + 1) = n_j - 1 - n_{j-1}$$

Might we then not start out by moving that class j for which $(n_j-1-n_{j-1})/\delta_j$ is largest? Unfortunately, such local rules are unsatisfactory, because our problem is one of minimizing a concave, rather than convex, function over a convex set. For illustration, suppose

$$n_1 = 100 \quad n_2 = 50 \quad n_3 = 30 \quad n_4 = 6$$
$$\delta_2 = 2 \quad \delta_3 = 4 \quad \delta_4 = 1 \qquad L = 6$$

The marginal benefit-cost ratio is highest for a movement of a member of class 2 into class 1. Inspection reveals, however, that the maximum envy reduction is obtained by moving all the members of class 4 into class 3.

Let's summarize the above discussion. The solution to $C(x)$ minimization problem is an extreme solution, extreme in the sense that for all but at most one class, classes must be moved in their entirety, or they must be absorbing. Moreover, it is the *classes* that are crucial, since the strengths of individual feelings do not appear in $C(x)$. And, finally, there is no reason to believe that the poorest classes will be moved first.

What happens to this analysis if we relax our assumption that everyone has the same utility function u? If there is a *single* good in the economy, and if each person's utility function u_i increases monotonically in it, the analysis of the disbursement of a social surplus goes through exactly as it does above. (Where that surplus comes from, however, becomes problematical, since there is no inefficiency to begin with.) However, if there is *more* than one good, we run into difficulty. Given an allocation x, we can define a non-envy relation R on the set of individuals in the economy by saying iRj ("i doesn't envy j") whenever $u_i(x_i) \geqq u_i(x_j)$. Now if x is efficient, it can be shown that the relation R is complete and acyclic, and we can, therefore, given any subset of individuals, identify persons who envy no one in that subset (see, for example, Amartya Sen). But R may *not* be transitive, and it may therefore be impossible to define an envy class structure, as we've done above. The solution to the $C(x)$ minimization problem still exists, of course, but our discussion of how to find it becomes irrelevant.

IV. When $e_i(x)$ and $e^*(x)$ Measure Envy

Now we will consider the problem of maximizing fairness in the sense of minimizing our cardinal utility envy measures $E(\alpha, x)$ and $E^*(\alpha, x)$. We will again require that no one be made worse off, and we will again make the strong assumption that everyone has the same utility function $u=u_1=u_2= \ldots =u_n$. Rather than discussing the reduction of a many-goods economy to an efficient one-composite-good economy, we will presume at the outset that there is just one good. We will assume throughout this section that u is continuous and has a continuous first derivative u'. We will also suppose that u is nondecreasing and u' is nonincreasing: If $x_i \geqq x_j$, then $u(x_i) \geqq u(x_j)$ and $u'(x_i) \leqq u'(x_j)$. If we interpret x_i as i's income, then the marginal utility of income is nonnegative, and nonincreasing. We are now starting with a distribution of goods $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, L)$, where L is the social surplus which will be distributed by the benevolent ruler.

The constrained minimization of $E(\alpha, x)$ is trivial under these circumstances, for any set of weights $\alpha=(\alpha_1, \ldots, \alpha_n) \geqq 0$. For under the above assumptions,

$$\begin{aligned} E(\alpha, x) &= \sum_{i=1}^{n} \alpha_i e_i(x) \\ &= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{n} (u(x_j) - u(x_i)) \right] \\ &= \sum_{i=1}^{n} (A - n\alpha_i) u(x_i), \end{aligned}$$

where $A \equiv \sum_{i=1}^{n} \alpha_i$

To minimize $E(\alpha, x)$ the ruler can distribute all of the social surplus to an individual for whom α_i is largest, that is, an individual whose envy is given the greatest weight in the calculation of total social envy. If all individuals are weighted equally, then any and all allocations are $E(\alpha, x)$ minimizing, since $E((1/n, 1/n, \ldots, 1/n), x) \equiv 0$ for all x.

This rather amoral result stems from the fact that the measure $e_i(x)$ counts negative envy, or the psychic compensation i gets from the misfortune of those below him, as well as his jealousy of those above. When all the u_i are identical, and $\alpha=(1/n, \ldots, 1/n)$, the jealousies and the psychic compensations cancel out. This will clearly not be the case for $e^*(x)$ and, of course, $E^*(\alpha, x)$, since $e^*(x)$ does not share $e_i(x)$'s I-feel-better-when-others-are-hurt property.

Let us then turn to the minimization of $E^*(\alpha, x)$ via a distribution of the social surplus L. A simple example will illustrate the nature of this problem.

Suppose our economy has three members: $u(x) \equiv x$; $\hat{x}_1=10$; $\hat{x}_2=0$, and $\hat{x}_3=0$. Trader 1 is the rich man, and 2 and 3 are equally poor. Assume that $L=1$, $\alpha_1=1/10$, $\alpha_2=2/10$, and $\alpha_3=7/10$. Note that the envy weights are assigned to *persons*, not to positions in the hierarchy, so $\alpha_2 \neq \alpha_3$ although both 2 and 3 are equally poor. In this sense, $E^*(\alpha, x)$ is not "neutral" be-between persons or blind to individual identification. In our example the fairness maximizing disbursement of L is simply a grant of 1 to individual 3. Giving equal shares to 2 and 3 is not the way to minimize $E^*(\alpha, x)$. Moreover, the result would hold even if $\hat{x}_3$ were *greater* than zero. Therefore, fairness maximization (in the sense of $E^*(\alpha, x)$ minimization) may not only create instances of envy, it may also involve a policy of grants to the (relatively) rich, and one of benign neglect toward the poor.

However, this possibility disappears when we force some degree of neutrality on $E^*(\alpha, x)$. There are (at least) two ways to do so: The first is to assign weights to positions in the hierarchy rather than persons, and to assume that the envy of the poorer man is always weighted at least as heavily as the envy of the richer man. The second and simpler way is to assume that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$. Let's now take this latter approach. If $\alpha = (1/n, \ldots, 1/n)$, then

$$E^*(\alpha, x) = \sum_{i=1}^{n} \alpha_i \sum_{j: u(x_j) \geqq u(x_i)} [u(x_j) - u(x_i)]$$
$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j: u(x_j) \geqq u(x_i)} [u(x_j) - u(x_i)]$$

We will assume, without loss of generality, that our individuals are indexed so that $\hat{x}_1 \geqq \hat{x}_2 \geqq \ldots \geqq \hat{x}_n$. Since $\alpha_i = 1/n$ for all i, and $u_i = u$ for all i, $E^*(\alpha, x)$ can clearly be minimized in a way which does not affect the rank order of wealth. In other words, it can be minimized by an allocation x with the property that

$$(3) \qquad x_1 \geqq x_2 \geqq \ldots \geqq x_n$$

As long as (3) holds, however, we have

$$E^*\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right), x\right) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \leqq i} [u(x_j) - u(x_i)]$$

which gives, after some manipulation,

$$E^*\left(\left(\frac{1}{n}, \ldots, \frac{1}{n}\right), x\right) = \frac{n+1}{n} \sum_{i=1}^{n} u(x_i) - \frac{2}{n} \sum_{i=1}^{n} iu(x_i)$$

It follows that an incremental increase in x_i increases envy by an amount

$$\frac{n+1}{n} u'(x_i)dx_i - \frac{2}{n} iu'(x_i)dx_i = \left(\frac{n+1-2i}{n}\right) u'(x_i)dx_i$$

as long as (3) holds. Since we have assumed u' is nonnegative and nonincreasing, this term is smallest when $i = n$, so it's always best to bestow upon the humblest. When there are ties for last place (when, for example, $x_n = x_{n-1}$, etc.), equal shares must be given to all the poorest in order to preserve (3). This argument establishes:

PROPOSITION 2: *If $\alpha = (1/n, \ldots, 1/n)$, and marginal utility is nonnegative and nonincreasing, then the fairness maximization (or E^* minimization) problem is solved by a policy of pushing the poorest up from the bottom: The ruler gives to trader n until $x_n = x_{n-1}$, then he gives to n and $n-1$ until $x_n = x_{n-1} = \hat{x}_{n-2}$, then he gives to n, $n-1$, and $n-2$ until $x_n = x_{n-1} = x_{n-2} = \hat{x}_{n-3}$, and so on.*

Proposition 2 is the morally classical result, and seems almost obvious on its surface. We might remark that it is analogous to the utilitarian argument that social welfare (defined as a sum of identical individual utility functions) is maximized through an equal distribution of income. Like that argument, it depends on the assumptions of (i) identical and therefore comparable utility functions, and (ii) decreasing marginal utility of income. Both assumptions are perhaps more plausible from the philosopher's viewpoint than from the economist's.

V. Conclusion

This paper has three major points. The first is that standard voluntary economic transactions have little apparent connection with the fairness, or lack of fairness, of allocations. In general, even if economic transactions are fair in themselves, like trades to competitive equilibria, they can be expected neither to establish nor to preserve allocative fairness. Fairness, unlike efficiency, has no automatic enforcers.

Second, an envy measure which simply counts instances of envy imposes certain types of solutions on a benevolent dictator's constrained fairness maximization problem. The discontinuity of the counting measure forces the dictator to look at classes, rather than individuals, since the measure is a function of numbers in classes, rather than intensities of individual envies. Moreover, the problem is such that it will have extreme solutions: with one possible exception, classes will be moved up in their entireties, or not moved up at all. And, finally, there is no assurance that it is the poorest classes which will be moved.

Third, envy measures which assume cardinal utility, or which depend on intensities of individual envies, lead the benevolent dictator down different paths. If an envy measure includes psychic compensation that the rich receive from the poverty of those poorer than themselves, the fairness optimizing policy may be to do nothing. If the rich are assumed to get no satisfaction from the poverty of the poor, an enlightened ruler may, under certain conditions, maximize fairness by giving society's excess to the poorest. This is, of course, the most intuitive solution, but it is a solution which depends on rather stringent assumptions.

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