Puzzles and Problems

Answer

1973-*Q*734.—Harl Ryder and Allan M. Feldman, Brown University. The following is in response to question Q734 (Ellen Hertzmark and Richard Zeckhauser, *J.P.E.* 81 [May/June 1973]: 796–97).

1. It is indeed the case that an allocation that is Pareto optimal from the standpoint of the participants on one side of the market must give at least one of these participants his most preferred outcome. *Proof*: Let there be n candidates and n jobs. Suppose each candidate can rank the n jobs in order of his preference, with no ties. Assume that the initial assignment is such that no candidate has his most preferred job. We shall show that it is possible to construct a new allocation that improves the welfare of at least two candidates and makes no one worse off. We shall form a list of jobs by the following procedure: Start with an arbitrary job. Put that job on the list and consider transferring the candidate assigned to that job to another job he prefers. This can be done since no candidate is assigned to his most preferred job. Add this new job to the list and consider a similar transfer for the candidate assigned to it. Continue in this manner until we find a candidate who prefers one of the jobs already on the list. We must eventually find such a candidate, since the nth candidate is assigned to the nth job and must prefer one of the $n - 1$ jobs already on the list. When we find such a candidate, we will have constructed a closed circuit of welfare-improving transfers that will benefit everyone in the circuit.

2. The "deferred-acceptance" procedure has been analyzed by David Gale and Lloyd S. Shapley in "College Admissions and the Stability of Marriage" (*American Mathematical Monthly* 57, no. 1 [January 1962]: 9–15). Let's consider the simple marriage-assignment variant of the problem, when there are an equal number of men and women and no man (woman) is indifferent between any two women (men). An assignment which is Pareto-optimal-for-the-whole-group is one with the characteristic that no other assignment can be found which would make everyone (men *and* women) at least as well off, and some people better

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off, than they are under it. Any outcome of the deferred-acceptance procedure is Pareto optimal in this sense. For if X were such an outcome, and it weren't Pareto-optimal-for-the-whole-group, there would be a man, say α, who is married to a woman, say A, under X, but to a different woman whom he prefers, say B, under an assignment Y which is Pareto superior to X. Now, since α prefers B to A, the deferred-acceptance procedure requires that he propose to B before A, but since she rejected him and ultimately married another man, β, under X, she must prefer β to α, because a man is rejected only when a woman has accepted or conditionally accepted a better man. However, this contradicts the assumption that Y is Pareto superior to X since B is better off under X than under Y.

The question has a number of very interesting ramifications which should be pointed out. Let's call an assignment *unstable* if it includes a man-woman pair who would rather be married to each other than to their respective marriage partners and *stable* if it includes no such pair. Gale and Shapley prove that the deferred-acceptance procedure leads to an assignment which is stable in this sense. The sort of argument we make above establishes that any stable assignment is Pareto-optimal-for-the-whole-group; but there may be *no* stable assignments which are Pareto-optimal-for-the-men, or Pareto-optimal-for-the-women.

Gale and Shapely also prove a very subtle theorem about the "optimality" of the deferred-acceptance procedure in which the men propose and the women dispose. It turns out that any assignment which results from this procedure is not only stable, but all the men like it as well *as any other stable assignment*. So it is "optimal" for the men in this very special sense, although it might not be optimal-for-the-men in the crude sense of optimality which implicitly defines women as chattel.