Nonmanipulable multi-valued social decision functions

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Introduction

An individual can 'manipulate' a social decision function if, by misrepresenting his preferences, he can secure a social outcome which he *prefers* to the outcome he secures when he is honest. A social decision function is 'nonmanipulable or 'cheatproof' if it can never be manipulated. Gibbard (1973), Satterthwaite (1975) and Schmeidler and Sonnenschein (forthcoming) have established that any social decision function which is single-valued, nonimposed and nonmanipulable must be dictatorial. This distressing 'impossibility' theorem indicates there is no entirely satisfactory way to aggregate individual preferences into unique social choices – if one's notion of 'satisfactory' precludes manipulation. The result clearly reinforces Arrow's (1963) theorem for social welfare functions, according to which there is no entirely satisfactory way to aggregate individual preferences into unique social preference relations – if one's notion of 'satisfactory' incorporates certain other attractive criteria.

Efforts have been made, however, to escape the dictatorship-manipulation dilemma by examining *multi-valued* social decision functions (Barberá, 1977; Feldman, 1977; Gardenfors, 1978; Kelly, 1977). The idea is that single-valuedness asks too much. Obviously, if alternatives are correctly defined, in the end only one can be chosen. But many decision rules, and indeed many criteria from welfare economics, simply identify desirable *sets* of alternatives. In fact, any economist who is completely serious about the noncomparability of individuals' utility levels will recognize that there are *many Pareto optima*, but generally *no unique best* distribution of goods. If the goal is to identify best sets of alternatives, then we ought to ask: what are the properties of the rules which transform collections of individual preference orders into best sets? That is, what are the properties of multi-valued social decision functions? In particular, are they necessarily either dictatorial (in some sense) or liable to manipulation? This is a central

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question in Kelly (1977), and Barberá (1977), Gärdenfors (1976) asks a similar question: if multi-valued social decision functions are anonymous and neutral (that is, unbiased among persons and alternatives), and if they satisfy the Condorcet criterion, are they necessarily liable to manipulation? The dictatorship-manipulation dilemmas of Kelly and Barberá are explored further in this paper, and although Gärdenfors' exploration of the Condorcet criterion isn't pursued here, further implications of anonymity and neutrality are.

Barberá and Kelly derive two very similar theorems which say that, under certainty regularity conditions, any nonmanipulable social decision function must give substantial power to some group of individuals ('weak dictators' in Kelly an 'oligarchy' in Barberá. These individuals are especially powerful in the sense that if one prefers x to y, and the social agenda is x vs. y, the social choice cannot be y alone. That is, they have vetoes. (It is interesting to note that they interpret their results quite differently: Barberá sees a 'conclusive negative answer to the question whether one could find satisfactory relational collective choice rules', whereas Kelly allows that the theorem might simply be 'part of a critique of the regularity conditions'.) Theorem 2 below is another result along the same lines, although it differs from Kelly's and Barberá's because of a distinctly different proof, a proof like that of the veto theorem of Blau and Deb (1977).

However, I will go beyond the veto theorem result in this paper because, in my view, *weak dictatorship, or oligarchy, or veto power, are not per se objectionable*. The social decision function that maps preference profiles into sets of Pareto optimal alternatives makes everyone a weak dictator and makes the set of all individuals an oligarchy. Yet, the only problem the Pareto rule has is that it is multi-valued. It is, for some tastes, too 'indeterminate'. Also, it is unfortunate if a social decision function makes person A a weak dictator, and does not do the same for person B. But the difficulty in this case is a bias among individuals, or the violation of another reasonable criterion for social decision functions, namely anonymity. The point of this paper, then, is to characterize social decision functions that meet *several* reasonable criteria, including immunity to manipulation, rather than to construct 'impossibility' theorems, as is done in Kelly, Gärdenfors and Barberá.

The reasonable criteria for social decision functions used here are, in addition to nonmanipulability, (1) neutrality, or unbiasedness among alternatives, (2) anonymity, or unbiasedness among individuals, (3) nonimposition, and (4) consistency for social choice under contraction and expansion of agendas. The paper has four main results, or four characterizations of reasonable social decision functions.

First, a neutral, nonmanipulable, contraction consistent social decision function gives someone a veto (Theorem 2). Thus nonmanipulation becomes a substitute for the monotonicity part of Blau-Deb's 'neutrality-independence-monotonicity' assumption. Intuitively, *if a social decision function is unbiased among alternatives and immune to individuals' misrepresenting their preferences, it must give someone substantial power over choices between pairs of alternatives.*

Second, a neutral, anonymous, nonmanipulable, contraction and expansion consistent social decision function always produces choice sets which include at least one favorite alternative for every individual (Theorem 5). *When a nonmanipulable decision function is unbiased among the alternatives and unbiased among the individuals, every person must always find a plum in the social choice basket.*

Third, a non-imposed, nonmanipulable, contraction consistent social decision function always produces choice sets which are entirely contained in the broadly-defined sets of Pareto optima (Theorem 8). *Thus minimal assumptions on a nonmanipulable multi-valued social decision function ensure that all the chosen alternatives are optimal.* In my view, this is an encouraging result for those of us who believe in Pareto optimality as the principal criterion of welfare economics.

Fourth, if no individual is ever indifferent between two alternatives, a social decision function that is neutral, anonymous, non-imposed, nonmanipulable, and contraction and expansion consistent, must always produce choice sets that are *bracketed* between the sets of all individuals' favorites, and the broadly-defined Pareto optimal sets (Theorem 9). *That is, under plausible conditions, for any preference profile, every person's favorite alternative must be in the choice set, and every alternative in the choice set must be optimal.*

Finally, these are all 'possibility' results. There do exist multi-valued social decision functions which satisfy all the desirable criteria. The view of this paper is that there are good multi-valued social decision functions, and that they can be characterized reasonably closely.

The model

There are n individuals, indexed by $i = 1, 2, \ldots, n$, who make choices from a finite set X of alternatives. Each individual has a preference order R_i on the alternatives. I_i is i's indifference relation and P_i is i's strict preference relation; each is defined from R_i in the usual way. A *preference profile* $R = (R_1, R_2, \ldots, R_n)$ is a specification of all individuals' preference orders. An *agenda* is a non-empty subset S of X. A *social decision function*, or SDF, is a mapping which assigns to every agenda S and preference profile R a non-empty choice set contained in S. I will use the following notation:

$C(\cdot)$ represents an SDF.

$C(S, R)$ represents the choice set produced by the SDF $C(\cdot)$ when the agenda is S and the preference profile is R.

The definitions that pertain to SDFs are as follows:

1. Neutrality. An SDF is *neutral* if the following holds: For any four alternatives $\{x, y, z, w\}$ and any preference profiles R and R', if $xR_iy \Leftrightarrow zR'_iw$ and $wR'_iz \Leftrightarrow yR_ix$ for all i, then $C(\{x, y\}, R) = \{x\}$ (or $\{y\}$ or $\{x, y\}$, respectively) $\Rightarrow C(\{z, w\}, R') = \{z\}$ (or $\{w\}$ or $\{z, w\}$, respectively).

Intuitively, what holds for the pair of alternatives $\{x, y\}$ ought to hold for the pair $\{z, w\}$, if people's preferences regarding x and y are analogous to their preferences regarding z and w. One consequence of neutrality is the binary version of Arrow's independence of irrelevant alternatives condition: Let $z = x$ and $w = y$. If $xR_iy \Leftrightarrow xR'_iy$ and $yR'_ix \Leftrightarrow yR_ix$ for all i, then $C(\{x, y\}, R) = C(\{x, y\}, R')$.

2. Anonymity. Let $\theta: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ be a permutation of the individuals. An SDF is *anonymous* iff for any permutation θ, any preference profile R, and any agenda S, if $R'_i = R_{\theta(i)}$ for all i, then $C(S, R') = C(S, R)$.

Intuitively, if the preference orders are simply switched around among members of society, the choice sets must remain fixed.

3. Nonmanipulability. An individual might want to misrepresent his true feelings in order to bring about a preferred choice set. If R_i and P_i are i's 'true' preference and strict preference relations, let $\bar{R}_i$ represent a 'false' preference relation. For notational brevity, I will write $C(S, R_i)$ for $C(S, (R_1, R_2, \ldots, R_i, \ldots, R_n))$, or $C(S, R)$, and $C(S, \bar{R}_i)$ for $C(S, (R_1, R_2, \ldots, \bar{R}_i, \ldots, R_n))$.

Individual i can *manipulate* the SDF $C(\cdot)$ if there exists a preference profile R, a false preference relation $\bar{R}_i$, and a pair of alternatives $\{x, y\}$ such that:

(i) $C(\{x, y\}, R_i) = \{y\}, C(\{x, y\}, \bar{R}_i) = \{x\}$ and xP_iy, or
(ii) $C(\{x, y\}, R_i) = \{y\}, C(\{x, y\}, \bar{R}_i) = \{x, y\}$ and xP_iy, or
(iii) $C(\{x, y\}, R_i) = \{x, y\}, C(\{x, y\}, \bar{R}_i) = \{x\}$ and xP_iy.

If no individual can manipulate $C(\cdot)$, it is *nonmanipulable* or *cheatproof.*

Intuitively, if i prefers x to y, and the true choice set is $\{y\}$, he should not be able to manipulate the choice of either $\{x\}$, or $\{x, y\}$. On the other hand, if the true choice set is $\{x, y\}$, he should not be able to manipulate the choice of $\{x\}$. This intuitively clear definition of manipulation is inspired by Brams and Fishburn (forthcoming, p. 10); it is also similar to Barberá's note of uniform manipulation.

Note that manipulation is defined here only in terms of pairs, or two-alternative agendas. If $C(\{x, y, z\}, R_i) = \{z\}$, $C(\{x, y, x\}, \bar{R}_i) = \{x\}$, and xP_iz, person i is *not* manipulating $C(\cdot)$ according to the narrow usage of this paper. Defining manipulation for agendas of arbitrary size is a complicated matter because of ambiguities over when one *set* of alternatives ought to be preferred to another *set*. Possible arbitrary-set definitions of manipulation are surveyed in Feldman (1977) and Gärdenfors. The narrow pairwise definition of manipulation is sufficient for the purposes of this paper because it is coupled with the assumptions of contraction and expansion consistency, defined below.

4. Non-imposition. An SDF is *non-imposed* iff for any pair $\{x, y\} \subset X$, there exists a preference profile R such that $C(\{x, y\}, R) = \{x\}$.

If an SDF is non-imposed, for each pair of alternatives, there are circumstances under which each of the alternatives is the unique social choice.

5. Correction consistency, or property $\alpha 2$. An SDF satisfies *propery* $\alpha 2$, or is *contraction consistent*, iff for any preference profile R and any agenda S, if $x \in C(S, R)$, then $x \in C(\{x, y\}, R)$ for all $y \in S$.

The definition is adapted from Sen (1977), as is the next.

6. Expansion consistency, or property $\gamma 2$. An SDF satisfies *property* $\gamma 2$, or is *expansion consistent*, iff for any preference profile R and any agenda S, if $x \in S$ and $x \in C(\{x, y\}, R)$ for all $y \in S$, then $x \in C(S, R)$.

If an SDF satisfies $\alpha 2$ and $\gamma 2$, it will be called *normal*. Given a preference profile R, any SDF generates a choice function $C(\cdot, R)$. From this choice function one can obtain a social preference relation $\bar{R}_C$, defined by $x\bar{R}_C y$ iff $x \in C(\{x, y\}, R)$. Normality for an SDF is then equivalent to the requirement that the choice function $C(\cdot, R)$ and the social preference relation $\bar{R}_C$ have the same information content. Each can be used to generate the other (see Sen, 1977, pp. 63-65).

Several other more or less standard definitions are used in this paper:

If $x \in S$ and xR_iy for all $y \in S$, x is *maximal for i in S*. The set of alternatives maximal for i in S is $M_i(S, R)$. Also, let $M(S, R) \equiv \cup M_i(S, R)$. If $x \in M(S, R)$, then there is at least one i for whom x is maximal. $M(S, R)$ will be called the *maximal set*. The SDF that transforms (S, R) into $M(S, R)$ will be called the *maximal rule*. This notion of maximality is the same as Kelly's.

If xR_iy for all i, x is *Pareto-as-good-as* y. If xP_iy for all i, x is *Pareto superior* to x. If $x \in S$ and there is no $y \in S$ such that y is Pareto superior to x, then x is *Pareto optimal* in S. Let $P(S, R)$ stand for the set of Pareto optima in S. The SDF that transforms (S, R) into $P(S, R)$ will be called the *Pareto rule*. The definition of Pareto superiority used here is a 'strict' one and the definition of Pareto optimality is consequently 'weak'. That is, the set of Pareto optima defined in this paper is broader, or more inclusive, than it would be if we said 'x is Pareto optimal in S if there is no $y \in S$ such that yR_ix for all i and yP_ix for some i'. Of course, the distinction between broadly and narrowly defined Pareto optima disappears when indifference is not permitted, as in Theorem 9.

Individual i has a *veto* if for all $\{x, y\} \subset X$ and all preference profiles R, $xP_iy \Rightarrow x \in C(\{x, y\}, R)$. That is, if i prefers x to y, the social choice can be x, or x and y, but it cannot be y alone. Note that, in the absence of any contraction or expansion consistency assumptions, the notion of i's having a veto, like the notion of i's manipulating, pertains only to pairwise choice. If i has a veto and prefers x to y to z, and if the agenda is $\{x, y, z\}$, the choice might be z.

The results

Blau and Deb (1977) provide a definition for what they call *neutrality-independence-monotonicity* or *NIM*: An SDF satisfies NIM providing the following holds: For any four alternatives $x, y, z, w \in X$, and any preference profiles R and R', if $xR_iy \Rightarrow zR_i'w$ and $wR_i'z \Rightarrow yR_ix$ for all i, then $x \in C(\{x, y\}, R) \Rightarrow z \in C(\{z, w\}, R')$.

Intuitively, if z has as much support as x, and w has no more support than y, then if x is chosen given $\{x, y\}$ and R, z must be chosen given $\{z, w\}$ and R'.

Our definition of neutrality is obviously related to NIM, although it is probably closer to earlier definitions of neutrality, e.g. May (1952). Both our neutrality and NIM imply independence of irrelevant alternatives. But NIM has a weak monotonicity aspect that is absent in neutrality.

Suppose, for example, that y is Pareto-as-good-as x, and that $x \in C(\{x,y\}, R)$. Referring to the definition of NIM, let $z = y$, $w = x$, and $R' = R$. Then $xR_iy \Rightarrow yR_ix$ is tautological, since yR_ix is true for all i. It follows from NIM that $y \in C(\{x, y\}, R)$. If x is in the choice set, and y is Pareto-as-good-as x, then y is in the choice set, a monotonicity result. This cannot be inferred from neutrality alone.

Using NIM, in conjunction with SDFs that map preference profiles into acyclic social preference relations, Blau and Deb prove:

> *Veto Theorem 1:* If $C(\cdot)$ satisfies NIM and if $|X| \geqslant n$, then someone has a veto.

I propose to prove a similar theorem for neutral nonmanipulable SDFs that map agendas and preference profiles into choice sets:

> *Veto Theorem 2:* If $C(\cdot)$ is neutral, nonmanipulable, and satisfies $\alpha 2$, and if $|X| \geqslant n$, then someone has a veto.

Proof: Suppose individual 1 does not have a veto. Then there exists a pair of alternatives $\{x, y\}$ and a preference profile R such that xP_1y and $C(\{x,y\}, R) = \{y\}$.

By the independence of irrelevant alternatives part of the neutrality assumption, the choice set for $S = \{x, y\}$ is contingent only on individual preferences vis-a-vis x and y. I now examine only these.

Assume without loss of generality that for $i = 2, \ldots, k$, xP_iy; for $i = k+1, \ldots, \ell$, xI_iy, and for $i = \ell+1, \ldots, n$, yP_ix. The preference profile R on $\{x, y\}$ is then as follows:

1	*2*	. . .	*k*	*k+1*	. . .	ℓ	ℓ*+1*	. . .	*n*
x	x		x				y		y
		. . .		xy	. . .	xy		. . .	
y	y		y				x		x

(Here $\begin{smallmatrix}x\\y\end{smallmatrix}$ means x is preferred to y, xy means x is indifferent to y, and so on.)

Again, $C(\{x, y\}, R) = \{y\}$.

Now suppose individual 2 misrepresents his preference as $\begin{smallmatrix}y\\x\end{smallmatrix}$. If this misrepresentation causes the choice set to change to either $\{x\}$ or $\{x, y\}$, the SDF is manipulable. Similar arguments are applied in sequence to $3, \ldots, k$. It follows that if the SDF is nonmanipulable, and if R' on $\{x, y\}$ is given by:

1	*2*	. . .	*k*	*k+1*	. . .	ℓ	ℓ*+1*	. . .	*n*
x	y		y				y	. . .	y
		. . .		xy	. . .	xy			
y	x		x				x		x

then $C(\{x, y\}, R') = \{y\}$.

Now consider individual $k+1$. Let R'' on $\{x, y\}$ be:

1	*2*	. . .	*k*	*k+1*	. . .	ℓ	ℓ*+1*	. . .	*n*
x	y		y	y			y		y
		. . .			. . .	xy		. . .	
y	x		x	x			x		x

Suppose $C(\{x, y\}, R'') = \{x\}$ or $\{x, y\}$. If $k+1$'s real preference is $\begin{smallmatrix}y\\x\end{smallmatrix}$, he can manipulate the SDF by falsely declaring xy. Therefore, if the SDF is nonmanipulable, $C(\{x, y\}, R'') = \{y\}$.

Similar arguments are applied in sequence to $k+2, k+3, \ldots, \ell$. The result is that for the preference profile R''' on $\{x, y\}$ given by:

1	*2*	. . .	*n*
x	y		y
y	x		x

one must have $C(\{x, y\}, R''') = \{y\}$.

To this point I have assumed that individual 1 does not have a veto, and I have shown that there is a pair of alternatives $\{x, y\}$, such that when 1 prefers x to y and everyone else prefers y to x, the *social choice is* y. Now I will use the full force of the neutrality assumption. By neutrality, what is true for $\{x, y\}$ must be true for *any* pair of alternatives. That is, when individual 1 is unanimously opposed by all the others in *any* pairwise choice, the choice set must be the single option preferred by the rest of society.

Now I will proceed as in Blau-Deb (1977): That is, I will assume no one has a veto, and this assumption will lead to a contradiction.

Suppose no one has a veto. Then the arguments for person 1 above apply to everyone. Therefore, when *any* individual is unanimously opposed by all the others in any pairwise choice, the choice set must be the single option preferred by all the others. Consider the latin square profile $\bar{R}$ below, which can be constructed since $|X| \geqslant n$:

1	*2*	*3*	...	*n*
x_1	x_2	x_3		x_n
x_2	x_3	x_4		x_1
.	.	.		.
.	.	.		.
.	.	.		.
x_{n-1}	x_n	x_1		x_{n-2}
x_n	x_1	x_2		x_{n-1}
.	.	.		.
.	.	.		.
.	.	.		.

Since person 2 prefers x_2 to x_1, but all the others prefer x_1 to x_2, $C(\{x_1, x_2\}, \bar{R}) = x_1$. Since person 3 prefers x_3 to x_2, but all the others prefer x_2 to x_3, $C(\{x_2, x_3\}, \bar{R}) = x_2$. Similar arguments establish $C(\{x_3, x_4\}, \bar{R}) = x_3$, $\ldots$ $C(\{x_{n-1}, x_n\}, \bar{R}) = x_{n-1}$, and $C(\{x_n, x_1\}, \bar{R}) = \{x_n\}$. Now take $x_j \in C(\{x_1, x_2, \ldots, x_n\}, \bar{R})$. By the above, $x_j \notin C(\{x_{j-1}, x_j\}, \bar{R})$. (For $j = 1$, let $x_{j-1} \equiv x_n$.) But this violates property $\alpha 2$.

Assuming that no one has a veto thus leads to a contraction. Q.E.D.

Next I turn to the relationships among i's holding a veto, the alternatives maximal for i, and the choice set. If i holds a veto and the agenda is $\{x, y\}$, then $M_i(\{x, y\}, R) \cap C(\{x, y\}, R) \neq \phi$, clearly. For an analogous result with arbitrary agendas, one assumes normality for the SDF, that is, both expansion and contraction consistency.

Theorem 3: If $C(\cdot)$ is normal and i holds a veto, then, for all S and R, $M_i(S, R) \cap C(S, R) \neq \phi$.

Proof: $C(\cdot)$ satisfies properties $\alpha 2$ and $\gamma 2$. Let $x \in C(M_i(S, R), R) \subset M_i(S, R) \subset S$. I will show $x \in C(S, R)$. By $\alpha 2$, $x \in C(\{x, y\} R)$ for all $y \in M_i(S, R)$. On the other hand, by the definition of $M_i(S, R)$, xP_iy for all $y \in S - M_i(S, R)$. Since i holds a veto, $x \in C(\{x, y\}, R)$ for all $y \in S - M_i(S, R)$. Thus $x \in C(\{x, y\}, R)$ for all $y \in S$, and by $\gamma 2$, $x \in C(S, R)$.

Q.E.D.

Now, combining Theorems 2 and 3, we have:

Theorem 4: If $C(\cdot)$ is normal, neutral and nonmanipulable, and if $|X| \geqslant n$, then there exists an i such that, for all S and R, $M_i(S, R) \cap C(S, R) \neq \phi$.

Anonymity has not been used to this point. However, it is obvious that for an anonymous SDF, if one individual holds a veto, all most hold a veto. From this and Theorem 4 follows:

Theorem 5: If $C(\cdot)$ is normal, neutral, anonymous and nonmanipulable, and if $|X| \geqslant n$, then for all S and R, and all i,

$$M_i(S, R) \cap C(S, R) \neq \phi .$$

Finally, in the case where all individual preferences are *antisymmetric* (for all $x, y \in X$, xR_iy and $yR_ix \Rightarrow x = y$), that is, where no individual is ever indifferent between two distinct alternatives, $M_i(S, R)$ must be a singleton. With Theorem 5, this implies:

Theorem 6: Suppose indifference is disallowed. If $C(\cdot)$ is normal, neutral, anonymous, and nonmanipulable, and if $|X| \geqslant n$, then, for all S and R,

$$M(S, R) \subset C(S, R) .$$

Theorem 6 provides conditions under which the maximal set is always contained in the choice set. This establishes a 'lower bound' for the choice set; the choice set must include at least $M(S, R)$. Now I turn to establishing an 'upper bound' for the choice set. The general idea is that under some set of conditions on the SDF, the choice set must always be contained in a very important set of alternatives, namely the Pareto optima.

First, I will show that non-imposition and nonmanipulability combine to give a strong form of monotonicity for the SDF.

Theorem 7: If $C(\cdot)$ is non-imposed and nonmanipulable, and if x is Pareto superior to y, then $C(\{x, y\}, R) = \{x\}$.

Proof: Suppose to the contrary. Then there is some preference profile R for which xP_iy for all i, and for which $C(\{x, y\}, R) = \{y\}$ or $\{x, y\}$.

By non-imposition, there exists a preference profile R' such that $C(\{x, y\}, R') = \{x\}$.

Let

$$R^0 \equiv R = (R_1, R_2, \ldots, R_n)\,;$$

$$R^1 \equiv (R_1', R_2, \ldots, R_n)\,;$$

$$R^2 \equiv (R_1', R_2', \ldots, R_n)\,;$$

$$R^n \equiv R' = (R_1', R_2', \ldots, R_n')\,.$$

Let j be the smallest number such that:

$$C(\{x, y\}, R^j) = \{x\}.$$

Now we have:

$$C(\{x, y\}, (R_1', \ldots, R_{j-1}', R_j', R_{j+1}, \ldots, R_n)) = \{x\},$$

while

$$C(\{x, y\}, (R_1', \ldots, R_{j-1}', R_j, R_{j+1}, \ldots, R_n)) = \{y\} \text{ or } \{x, y\}.$$

Since xP_jy, this allows individual j to manipulate, a contradiction. Q.E.D.

This result leads very quickly to a major containment theorem:

> *Theorem 8:* If $C(\cdot)$ satisfies $\alpha 2$, is non-imposed and nonmanipulable, then for all S and R,
>
> $$C(S, R) \subset P(S, R)\,.$$

Proof: Suppose to the contrary that $x \in C(S, R)$ but x is not Pareto optimal in S. Then there is a $y \in S$ which is Pareto superior to x. By Theorem 7, $C(\{x, y\}, R) = \{y\}$. We have $x \in C(S, R)$, $y \in S$, and $x \notin C(\{x, y\}, R)$, which contradicts $\alpha 2$. Q.E.D.

Theorems 6 and 8 show that, under the indicated conditions, the choice set must always be bracketed between the maximal set and the Pareto optimal set. That is, each person's favorite alternative must be in the choice set, and each element in the choice set must be Pareto optimal. Formally, we have:

Theorem 9: Suppose indifference is disallowed. If $C(\cdot)$ is normal, neutral, anonymous, non-imposed and nonmanipulable, and if $|X| \geqslant n$, then for all S and R,

$$M(S, R) \subset C(S, R) \subset P(S, R) .$$

If indifference is allowed, then the first inclusion, $M(S, R) \subset C(S, R)$, must be modified to $M_i(S, R) \cap C(S, R) \neq \phi$ for all i. At least one of each person's favorite alternatives must be in the choice set. It is still the case that each element in the choice set must be Pareto optimal, under our broad definition of Pareto optimality.

Concluding remarks

Logically satisfactory methods for narrowing the range of social choice can be designed. For example, the SDF which transforms (S, R) into $P(S, R)$, that is, the Pareto rule, satisfies all the conditions imposed on $C(\cdot)$ in Theorem 9: The Pareto rule is normal. For if $x \in P(S, R)$ and $y \in S$, then y cannot be Pareto superior to x; so $x \in P(\{x, y\}, R)$, and $\alpha 2$ is satisfied. If $x \in S$ and $x \in P(\{x, y\}, R)$ for all $y \in S$, no y in S is Pareto superior to x, and therefore, $x \in P(S, R)$; so $\gamma 2$ is satisfied. The Pareto rule is clearly neutral and anonymous; it is unbiased among the alternatives and among the individuals. The Pareto rule is obviously non-imposed. The Pareto rule is nonmanipulable. For if xP_iy, $P(\{x, y\}, R)$ must be either $\{x\}$or $\{x, y\}$. If $P(\{x, y\}, R) = \{x, y\}$, there is an individual $j \neq i$ for whom yR_jx. Consequently no misrepresentation by i can force y out of the set of optima, and therefore the rule is cheatproof.

The significance of this paper is that *any* rule which is logically satisfactory (in the sense of the conditions of Theorem 9) *must* be bracketed between the maximal and Pareto rules. So those rules are especially important: they provide lower and upper bounds for completely satisfactory multi-valued SDFs.

References

Arrow, K.J. *Social Choice and Individual Values*, 2nd ed. New York: John Wiley and Sons, 1963.

Barberá, S. 'Manipulation of Social Decision Functions.' *Journal of Economic Theory* 15 (1977), pp. 262-278.

Blau, J.H., and Deb, R. 'Social Decision Functions and the Veto.' *Econometrica* 45 (1977), pp. 871-879.

Brams, S.J., and P.C. Fishburn. 'Approval Voting.' *American Political Science Review* (forthcoming).

Feldman, A.M. 'Optimality and Manipulability of Collective Choice Rules.' Working Paper No. 77-24, Brown University, Department of Economics, 1977.

Gärdenfors, P. 'Manipulation of Social Choice Functions.' *Journal of Economic Theory* 13 (1976), pp. 217-228.

______ 'On Definitions, of Manipulation of Social Choice Functions,' in J.J. Laffont, ed.,, *Aggregation and Revelation of Preferences*. North-Holland, 1978.

Gibbard, A. 'Manipulation of Voting Schemes: A General Result.' *Econometrica* 41 (July 1973), pp. 587-601.

Kelly, J.S. 'Strategy-Proofness and Social Choice Functions Without Singlevaluedness.' *Econometrica* 45, no. 2 (1977), pp. 439-446.

May, K.O. 'A Set of Independent, Necessary and Sufficient Conditions for Simple Majority Decisions.' *Econometrica* 20 (1952), pp. 680-684.

Satterthwaite, M.A. 'Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions.' *Journal of Economic Theory* 10 (1975), pp. 187-217.

Schmeidler, D., and Sonnenschein, H. 'Two Proofs of the Gibbard-Satterthwaite Theorem on the Possibility of a Strategy-Proof Social Choice Function,' in *Proceedings of a Conference on Decision Theory and Social Ethics*. Reidel Publishing Co., forthcoming.

Sen, A.K. 'Social Choice Theory, A Re-examination.' *Econometrica* 45 (1977), pp. 53-89.