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UTILITY FUNCTIONS FOR PUBLIC OUTPUTS AND MAJORITY VOTING

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This paper studies the existence and stability of majority voting equilibria over sequences of one-dimensional choice problems, in the public sector. Voters' public sector preferences are derived from their 'primitive' preferences over public outputs and private consumption by maximizing the latter out. If agents ignore the potential effects of public sector states on private sector prices, we find that majority voting equilibria will exist under the usual assumptions. But these equilibria are not stable unless the primitive utility functions are separable *and* admit no wealth effects on demand for public goods. If agents do take account of the effects of public sector states on private sector prices, voting cycles may arise even when the choice space is one dimensional.

1. Introduction

It is traditional in public choice theory to model the prevailing levels of output of goods and services in the public sector as majority voting equilibria [see Bowen (1943), Barr and Davis (1966)]. If the choice is over one dimension, and voters have public sector utility functions which are strictly quasiconcave, an equilibrium occurs at the median voter's optimum. If the choice is over two or more dimensions, majority voting equilibria will normally not exist [Plott (1967)]. To explain the prevailing levels of output of several public goods in terms of majority voting equilibria, one strategy is to impose agenda restrictions so that voters face sequences of one-dimensional choice problems [see Kramer (1972), Slutsky (1977)]. Equilibria will exist if voters' public sector utility functions are strictly quasi-concave. However, the stability properties of these equilibria are unclear, unless utility functions are also separable.

The purpose of this paper is to see whether public sector utility functions are likely to be strictly quasi-concave and separable, as is often assumed. Utility functions defined over levels of public goods are derived from 'primitive' utility functions defined over bundles of public *and* private goods.

We analyze two methods for deriving public sector utility functions from primitive utility functions. The first method, discussed in section 3, assumes that voters ignore the effects of public sector production on private sector prices. This method is equivalent to Barr and Davis' (1966) loss function procedure, and has been used by Denzau and Parks (1977, 1979) and Slutsky (1977). Strict quasi-concavity of the derived utility functions is assured, under this procedure, by the usual regularity assumptions on primitive utility functions. However, we show in section 3 that separability for the derived utility functions has severe implications for the admissible forms of the primitive functions.

The second method assumes that voters take account of the effects of public sector production on private sector prices. In section 4 we argue that this is a more sensible method than the one used in section 3. When a worker in a defense plant votes for a Congressman who votes on a defense appropriation bill, that worker and his Congressman are well aware that public sector choices will affect the worker's wages. This consideration affects his feelings about the provision of national defense. In section 4 we find that public sector utility functions may lose their quasi-concavity when they incorporate general equilibrium prices. We conclude by giving an example of a classical voting cycle in a *one-dimensional* choice problem, an example in which voters have strictly quasi-concave primitive utility functions. Since we show a voting cycle, we raise the question of whether or not a majority voting equilibrium exists.

2. The model

We assume there are L voter-taxpayers, indexed by $l=1,2\dots, L$. Individual l's *primitive utility function* depends on the outputs of the public sector, the public projects, and on his consumption of privately provided goods and services. We call the publicly provided goods 'public goods' although they may not have the usual public good features like non-exclusivity in use. For example, in terms of our model, government-provided medical care is a 'public good'. Let $x \in R^m_+$ represent the output levels of the public goods, and let $y_l = R^n_+$ represent l's consumption of private (that is, privately provided) goods. The competitive market prices of the private goods are given by a price vector $p=(p_1,\dots,p_n)$.

We assume throughout this paper that individual l's preferences are representable by a twice continuously differentiable utility function:

$$u_l(x, y_l).$$

We also assume $u_l(\cdot)$ is strictly quasi-concave. Individual l's initial endowment of private goods is $\bar{y}_l \in R^n_+$. He can consume or sell any or all of this endowment, and with the proceeds of such sales he can buy private goods and pay taxes.

Public goods are produced from private goods by the government. The cost of producing x, in units of account or 'money', is given by a function $C(x)$. We assume $C(\cdot)$ is strictly monotone increasing, convex, and twice continuously differentiable.

Individual l's tax share, t_l, depends on the endowments of the L taxpayers and on private sector prices. Since the endowments are fixed, we can write $t_l = t_l(p)$. We require that $0 \leqq t_l \leqq 1$, for $l = 1, 2, \ldots, L$, and $\sum_{l=1}^{L} t_l = 1$. Individual l's total tax liability is

$$T_l(x, p) = t_l\, C(x).$$

Since $\sum_{l=1}^{L} t_l = 1$, the government's budget is balanced. It is important that the tax share t_l does not depend directly on x (although the total liability T_l does). Dependence of t_l on x would destroy the convexity of the individual's budget set, and Proposition 1 in section 3 would not hold. [See also Slutsky (1977, p. 307, fn. 3).]

Given a vector x of public sector outputs, individual l chooses a private good consumption vector y_l to solve

$$\text{maximize} \quad u_l(x, y_l)$$

$$\text{subject to}$$

$$p \cdot y_l + t_l\, C(x) \leqq p \cdot \bar{y}_l \quad \text{and} \quad y_l \geqq 0.$$

Under the strict quasi-concavity assumption, this maximization problem has a unique solution, y_l^*. We define the *induced or derived utility* function $v_l(\cdot)$ by

$$v_l(x) \equiv u_l(x, y_l^*). \tag{2}$$

Our assumptions on $u(\cdot)$ and $C(\cdot)$ guarantee that $v_l(\cdot)$ is twice continuously differentiable.

3. Partial equilibrium analysis

In this section we assume that the prices of privately provided goods are fixed, and independent of levels of output of the public goods. That is, $p = (p_1, \ldots, p_n)$ is a constant. Our purpose here is to see if strict quasi-concavity and separability of the individual's primitive utility function, $u_l(x, y_l)$, carry over to his derived utility function, $v_l(x)$.

In this section we concentrate on a single individual, so we now drop the subscript l.

Proposition 1 gives conditions under which strict quasi-concavity carries over into the derived utility function [see also Denzau and Parks (1979, theorem 4)]. The proof is left to the reader:

Proposition 1. If $u(x, y)$ is strictly quasi-concave (respectively, quasi-concave) and $C(x)$ is convex (respectively, strictly convex), then $v(x)$ is strictly quasi-concave.

Strict quasi-concavity for $v(\cdot)$ is important because it is directly related to questions of existence of a majority voting equilibrium. If x is one-dimensional, and if all voters have strictly quasi-concave preferences over x, then preferences are single-peaked, and Black's theorems [see, for example, Feldman (1980)] can be applied. If x is multi-dimensional, strict quasi-concavity for $v(\cdot)$ will ensure the existence of a majority voting equilibrium provided there is some mechanism which can appropriately reduce the dimensionality of choice.

Separability is another appealing property of the induced utility function $v(\cdot)$. Its virtue is that if the levels of $x_1, x_2, \ldots, x_n$ are voted sequentially, and $v(\cdot)$ is separable, the outcome is independent of the sequence of votes, and it is stable.

For the purpose of this discussion, we define separability for $v(\cdot)$ as follows: $v(x)$ is *separable* if it can be written

$$v(x) = v_1(x_1) + v_2(x_2) + \ldots + v_m(x_m).$$

[See also Blackorby, Primont and Russell (1977) and Denzau and Parks (1977).] We assume in this treatment of separability that y is a *scalar* (i.e. $n = 1$). Without loss of generality, we set $p = 1$. Our aim is to show that separability for $v(\cdot)$ places very stringent requirements on the underlying utility function $u(\cdot)$.

If $v(\cdot)$ is separable, then

$$\frac{\partial^2 v}{\partial x_i \, \partial x_j} = 0 \text{ everywhere,} \quad \text{for all } i \text{ and } j. \tag{3}$$

By the non-satiation assumption, the budget constraint will be binding. Therefore, we have

$$v(x) = u(x, y^*) = u(x, \bar{y} - tC(x)).$$

With (3), this implies:

$$\frac{\partial^2 v}{\partial x_i\,\partial x_j}=\frac{\partial^2 u}{\partial x_i\,\partial x_j}-t\frac{\partial C}{\partial x_j}\frac{\partial^2 u}{\partial x_i\,\partial_y}-t\frac{\partial^2 C}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial y}$$

$$-t\frac{\partial C}{\partial x_i}\left[\frac{\partial^2 u}{\partial y\partial x_j}-t\frac{\partial C}{\partial x_i}\frac{\partial^2 u}{\partial y^2}\right]=0 \text{ everywhere.} \tag{4}$$

To avoid the $u(\cdot)$'s being dependent on parameters of the cost function, we must require that eq. (4) hold for all admissible cost functions. Suppose we restrict our attention to separable cost functions. Then $\partial^2 C/\partial x_i\,\partial x_j=0$ everywhere. Also, since $C(\cdot)$ is monotone increasing, we have $\partial C/\partial x_i>0$ and $\partial C/\partial x_j>0$ everywhere. Therefore, for (4) to hold for all separable cost functions, we must have

$$\frac{\partial^2 u}{\partial x_i\,\partial x_j}=\frac{\partial^2 u}{\partial x_i\,\partial y}=\frac{\partial^2 u}{\partial x_j\partial y}=\frac{\partial^2 u}{\partial y^2}=0 \text{ everywhere.} \tag{5}$$

By eq. (5), $u(\cdot)$ must be of the following form:

$$u(x,y)=\sum_{i=1}^{m} u^i(x_i)+\gamma_1 y+\gamma_2, \tag{6}$$

where u^i, $i=1,2,\ldots,m$, are arbitrary differentiable functions, and γ_1 and γ_2 are constants.

Now if there are admissible cost functions which are *not* additively separable, eq. (4) can hold only if $\partial u/\partial y=0$, in addition to (5). So, to get separable $u(\cdot)$'s, We must restrict ourselves to separable cost functions.

The above arguments establish this proposition:

Proposition 2. *Suppose $n=1$. Then $v(\cdot)$ is separable under all separable cost functions if and only if*

$$u(x,y)=\sum_{i=1}^{m} u^i(x_i)+\gamma_1 y+\gamma_2,$$

where u^i, $i=1,2,\ldots,n$, are arbitrary differentiable functions, and γ_1 and γ_2 are constants.

Proposition 2 shows how restrictive an assumption separability for $v(\cdot)$ really is. It requires that the primitive utility function be separable and that there be no income effects on the demands for x_i's.

4. General equilibrium analysis

In section 3 we assumed that prices in the private sector were fixed and independent of levels of output of public goods. In this section we assume that prices are variable.

Denzau and Parks (1979) studied the relationship between p and public sector equilibria. They looked at conditions under which public sector preferences are independent of private sector prices, and concluded that such 'price independence' requires rather strong assumptions. Our purpose at this point is to show that price independence is extremely unlikely.

We now modify the model of sections 1 and 2 by letting $v(\cdot)$ depend explicitly on the vector of private sector prices, p. Therefore, we define $v_l(x,p) \equiv u_l(x_l, y^*)$.

At this point we drop the subscript l unless we need to distinguish among different individuals.

Price independence is formally defined as follows: $v(x,p)$ is *price-independent* if for any x^1 and x^2, $v(x^1,\bar{p}) \geqq v(x^2,\bar{p})$ for some $\bar{p} \Rightarrow v(x^1,p) \geqq v(x^2,p)$, for all p.

We can immediately think of plausible reasons why price independence might not hold. For example: (i) an individual's wealth will generally depend on private sector prices, and this will affect his public sector preferences; (ii) his tax liability may depend on private sector prices, and this will affect his public sector preferences; (iii) some public goods might be complements or substitutes of some private goods.

Example 1 below shows that price independence is even less likely than reasons (i)–(iii) indicate. That is, even of (i)–(iii) are not operative, price independence may fail. The example also refutes a claim made by Denzau and Parks (1979, p. 350, discussion following theorem 10), that if taxes are independent of p and if $u(\cdot)$ is separable, $v(\cdot)$ will be price independent.

Example 1 Let $m=1$, $n=2$

$$u(x, y_1, y_2) = x + y_1^2 y_2,$$

$$\bar{y} = (1,1), \qquad t = 1/10, \qquad C(x) = x.$$

The individual's budget constraint is

$$p_1 y_1 + p_2 y_2 + x/10 \leqq p_1 + p_2.$$

It follows that

$$v(x,p) = x + \frac{4(p_1 + p_2 - x/10)^3}{27 p_1^2 p_2}.$$

Now let $x^1=10$, $x^2=1$, $\bar{p}=(\frac{1}{10},\frac{9}{10})$, and $\hat{p}=(\frac{9}{10},\frac{1}{10})$. Then $v(x^1,\bar{p})=10<v(x^2,\bar{p})=13$, but $v(x^1,\hat{p})=10>v(x^2,\hat{p})=7/3$.

In this example the switch from $\bar{p}$ to $\hat{p}$ leaves $p\cdot\bar{y}$ unaffected, so (i) is not operating; $tC(x)$ is independent of p, so (ii) is not operating; and $u(\cdot)$ is separable between private goods and the public good, so (iii) is not operating. What is happening is that relative prices affect the opportunity cost of public goods in terms of forgone utility from private goods.

We conclude from this example that price independence is most unlikely. In general, derived preferences for public goods should depend on private sector prices, and we should expect majority voting equilibria for public goods to depend on private goods prices.

At this point we turn to a full-blown general equilibrium model in which private goods prices are determined endogenously. Slutsky (1977) analyzes the existence of equilibria for an economy with public and private sectors. His equilibria are competitive (Arrow–Debreu) ones in the private sector, and majority voting ones in the public sector. But in Slutsky's model voters decide on preferred public goods outputs without paying attention to the potential effects of public good choices on private good prices. This assumption is commonly made in the literature, but seems quite unreasonable. A road worker who votes on road construction is likely to take the effects of his actions on road builders' wages into account.

In the example that follows, individuals incorporate correct general equilibrium prices in their derived public sector preferences. The example shows that this (intelligent) behavior can destroy the quasi-concavity of $v(\cdot)$, and create voting cycles. In order to make the algebra tractable, we have used utility functions that allow satiation in private goods consumption. The functions are standard in every other respect.

Example 2 Let $m=1$, $n=2$, and suppose there are three individuals. Their utility functions are

$$u_l(x,y_{l1},y_{l2})=6x^{1/2}-(3-y_{l1}-x)^2-(8-y_{l2})^2, \qquad l=1,2,3.$$

Note that, for a given x, individual l is satiated in private good consumption at $y_{l1}=3-x$ and $y_{l2}=8$. The reader can check to see that $u_l(\cdot)$ is strictly concave.

We assume the initial endowments are $\bar{y}_1=(5,1)$, $\bar{y}_2=(0,11)$ and $\bar{y}_3=(0,8)$. We also assume the tax shares are $t_1=0$, $t_2=1$, and $t_3=0$; this means that individual 2 pays the full cost of the public good.

The public good is produced from the second private good according to a simple technology: one unit of public good output requires two units of input of the second private good. Without loss of generality we let the second private good be the numeraire, and so $p=(p_1,1)$. It follows that the cost function for the public good is $C(x)=2x$. Note that $C(\cdot)$ is monotone increasing and convex.

The budget constraints for the three individuals are:

Person 1: $p_1y_{11}+y_{12}\leqq 5p_1+1$.

Person 2: $p_1y_{21}+y_{22}+2x\leqq 11$.

Person 3: $p_1y_{31}+y_{32}\leqq 8$.

In this example p_1 is determined endogenously. We find p_1 and $v_l(x)$ for $l=1,2,3$, at three levels of x: $x=0$, 3 and 4. We assume throughout that all individuals are price takers.

(a) $x=0$. Utility maximization and market clearing for private good 1 gives
$p_1=1$. Solving for y_1^*, y_2^* and y_3^*, and substituting back in $u_l(\cdot)$, gives $v_1(0)=-12.5$, $v_2(0)=0$, and $v_3(0)=-4.5$.

(b) $x=3$. For $x\geqq 3$, no individual wants any of private good 1. Therefore, $p_1=0$. Using the budget constraints to solve for y_1^*, y_2^*, and y_3^*, and substituting back in $u_l(\cdot)$, gives $v_1(3)=-38.6$, $v_2(3)=1.4$, and $v_3(3)=10.4$.

(c) $x=4$. Proceeding as in (b) gives $v_1(4)=-38$, $v_2(4)=-14$, and $v_3(4)=11$.

We now have the following table of utility levels:

	$x=0$	$x=3$	$x=4$
$v_1(x)$	-12.5	-38.6	-38
$v_2(x)$	0	$+1.4$	-14
$v_3(x)$	-4.5	$+10.4$	$+11$

This example has two interesting features. First, $v_1(\cdot)$ is *not quasi-concave*: it falls between $x=0$ and $x=3$, but it rises between $x=3$ and $x=4$. Proposition 1 fails here because of the general equilibrium setting, in which the price vector p is endogenous and dependent on levels of x.

Second, there is a *voting cycle*. That is, $x=3$ defeats $x=0$; $x=4$ defeats $x=3$; and $x=0$ defeats $x=4$. So majority voting fails in a one-dimensional issue space, and it fails because in a general equilibrium context, levels of x affect private good prices.

The assumption that agents incorporate correct equilibrium prices in their induced preferences for public goods is a strong one. But it makes more sense, in our view, than the assumption that agents ignore the effects on private goods prices of public good production. It would be wrong to rationalize the latter assumption by arguing that the individual is 'small' relative to the market. In a voting situation an individual often faces a choice between two very different proposals. The equilibrium private sector prices associated with the two proposals may differ significantly. There is no reason why an intelligent voter should ignore these significant differences.

We conclude by observing that the problematic feedback from public outputs to private sector prices would not arise in a production and exchange model with constant returns to scale and no commodity taxation. In more general settings, however, in addition to their direct costs and benefits, public sector states will affect both the marginal utilities of private income and the distribution of income among individuals. The first effect has received attention in the literature on optimal provision of public goods under commodity taxation [see Atkinson and Stern (1974)]. In a majority voting framework, this effect implies that 'price independence' is highly unlikely. The second, distributional, effect poses a potential problem of non-existence of equilibria.

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