

## INFERENCE IN DYNAMIC DISCRETE CHOICE MODELS WITH SERIALLY CORRELATED UNOBSERVED STATE VARIABLES

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This paper develops a method for inference in dynamic discrete choice models with serially correlated unobserved state variables. Estimation of these models involves computing high-dimensional integrals that are present in the solution to the dynamic program and in the likelihood function. First, the paper proposes a Bayesian Markov chain Monte Carlo estimation procedure that can handle the problem of multidimensional integration in the likelihood function. Second, the paper presents an efficient algorithm for solving the dynamic program suitable for use in conjunction with the proposed estimation procedure.

**KEYWORDS:** Dynamic discrete choice models, Bayesian estimation, MCMC, nearest neighbors, random grids.

### 1. INTRODUCTION

DYNAMIC DISCRETE CHOICE MODELS (DDCMs) describe the behavior of a forward-looking economic agent who chooses between several alternatives repeatedly over time. Estimation of the deep structural parameters of these models is a theoretically appealing and promising area in empirical economics. One important feature of DDCMs that was often assumed away in the literature due to computational difficulties is serial correlation in unobserved state variables. Ability, productivity, health status, taste idiosyncrasies, and many other unobservables are, however, likely to be persistent over time. This paper develops a computationally attractive method for inference in DDCMs with serially correlated unobservables.

Advances in simulation methods and computing speed over the last two decades made the Bayesian approach to statistical inference practical. Bayesian methods are now applied to many problems in statistics and econometrics that are difficult to tackle by the classical approach. Static discrete choice models and, more generally, models with latent variables, are one of those areas where the Bayesian approach was particularly fruitful; see for example [Albert and Chib \(1993\)](#), [McCulloch and Rossi \(1994\)](#), and [Geweke, Keane, and Runkle \(1994\)](#). Similarly to the static case, the likelihood function for a DDCM can be thought of as an integral over latent variables (the unobserved state

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variables). If the unobservables are serially correlated, computing this integral is very hard. A Markov chain Monte Carlo (MCMC) algorithm is employed in this paper to handle this issue.

An important obstacle for Bayesian estimation of DDCMs is the computational burden of solving the dynamic program (DP) at each iteration of the estimation procedure. Imai, Jain, and Ching (2005), from now on IJC, were the first to attack this problem and consider application of Bayesian methods for estimation of DDCMs. Their method uses an MCMC algorithm that solves the DP and estimates the parameters at the same time. The Bellman equation is iterated only once for each draw of the parameters. To obtain the approximations of the expected value functions for the current MCMC draw of the parameters, the authors used kernel smoothing over the approximations of the value functions from the previous MCMC iterations.

This paper extends the work of IJC in several dimensions. IJC employed MCMC to “solve the DP problem and estimate the parameters simultaneously” rather than handle more flexible specifications for unobservables. Their theory does not apply to a Gibbs sampler that includes blocks for simulating unobservables. In contrast, I develop an algorithm that applies MCMC to handle serially correlated unobservables and possibly other interesting forms of heterogeneity that would lead to hard integration problems in computing the likelihood function. Second, the algorithm developed in this paper can be applied to more general DDCMs: models with infinite state space and random state transitions (IJC’s algorithm works for finite state space and deterministic transitions for all state variables except independent and identically distributed (i.i.d.) errors). I achieve this more general applicability of the algorithm in part by using nearest neighbors instead of the kernel smoothing used by IJC. Also, in addition to approximating the value function in the parameter space, an algorithm for solving the DP has to deal with an integration problem for computing the expectations of the value functions. My prescriptions for handling this integration problem differ from IJC’s. Finally, this paper develops theory that justifies statistical inference made on the basis of the algorithm’s output. In the Bayesian framework, most inference exercises involve computing posterior expectations of some functions. IJC showed that the last draw from their algorithm will converge in distribution to the posterior. I show that sample averages from my algorithm can be used to approximate posterior expectations, and this is exactly how MCMC output is used in practice.

The proposed method was experimentally evaluated on two different DDCMs: a binary choice model of optimal bus engine replacement (Rust (1987)) and a model of medical care use and work absence (Gilleskie (1998)). Experiments are excluded from this paper for brevity. They can be found in Norets (2007, 2008). In summary, experiments demonstrate that ignoring serial correlation in unobservables of DDCMs can lead to serious misspecification errors and that the proposed method for handling serially correlated unobservables is feasible, accurate, and reliable.

The paper is organized as follows. Section 2 describes setup and estimation of a general DDCM. The algorithm for solving the DP and corresponding convergence results are presented in Sections 3 and 4. Proofs of the theoretical results can be found in the Supplemental Material (Norets (2009b)).

2. SETUP AND ESTIMATION OF DDCMS

Eckstein and Wolpin (1989), Rust (1994), and Aguirregabiria and Mira (2007) surveyed the literature on estimation of DDCMs. Below, I introduce a general model setup and emphasize possible advantages of the Bayesian approach to the estimation of these models, especially in treating the time dependence in unobservables. I also briefly discuss most relevant previous research.

Under weak regularity conditions (see, e.g., Rust (1994)), a DDCM can be described by the Bellman equation

$$(1) \quad V(s_t; \theta) = \max_{d_t \in D} \mathcal{V}(s_t, d_t; \theta),$$

where  $\mathcal{V}(s_t, d_t; \theta) = u(s_t, d_t; \theta) + \beta E\{V(s_{t+1}; \theta) | s_t, d_t; \theta\}$  is an alternative-specific value function,  $u(s_t, d_t; \theta)$  is a per-period utility function,  $s_t \in S$  is a vector of state variables,  $d_t$  is a control from a finite set  $D$ ,  $\theta \in \Theta$  is a vector of parameters,  $\beta$  is a time discount factor, and  $V(s_t; \theta)$  is a value function or lifetime utility of the agent. The state variables are assumed to evolve according to a controlled first order Markov process with a transition law denoted by  $f(s_{t+1} | s_t, d_t; \theta)$  for  $t \geq 1$ ; the distribution of the initial state is denoted by  $f(s_1 | \theta)$ . This formulation embraces a finite horizon case if time  $t$  is included in the vector of the state variables.

In estimable DDCMs, some state variables, denoted here by  $y_t$ , are assumed to be unobserved by econometricians. The observed states are denoted by  $x_t$ . All the state variables  $s_t = (x_t, y_t)$  are known to the agent at time  $t$ . Examples of the unobserved state variables include taste idiosyncrasy, health status, ability, and returns to patents. The unobservables play an important role in the estimation. The likelihood function is a product of integrals over the unobservables

$$(2) \quad p(x, d | \theta) = \prod_{i=1}^I \int p(y_{T_i,i}, x_{T_i,i}, d_{T_i,i}, \dots, y_{1,i}, x_{1,i}, d_{1,i} | \theta) d(y_{T_i,i} \cdots y_{1,i}),$$

where  $(x, y, d) = \{x_{t,i}, y_{t,i}, d_{t,i}\}_{t=1}^{T_i}$ ,  $i \in \{1, \dots, I\}$ ,  $I$  is the number of the observed individuals,  $T_i$  is the number of time periods individual  $i$  is observed,

$$\begin{aligned} & p(y_{T_i,i}, x_{T_i,i}, d_{T_i,i}, \dots, y_{1,i}, x_{1,i}, d_{1,i} | \theta) \\ &= \prod_{t=1}^{T_i} p(d_{t,i} | y_{t,i}, x_{t,i}; \theta) f(x_{t,i}, y_{t,i} | x_{t-1,i}, y_{t-1,i}, d_{t-1,i}; \theta), \end{aligned}$$

$f(\cdot|\cdot; \theta)$  is the state transition density,  $\{x_{0,i}, y_{0,i}, d_{0,i}\} = \emptyset$ , and  $p(d_{t,i}|y_{t,i}, x_{t,i}; \theta)$  is the choice probability conditional on all state variables.

In general, evaluation of the likelihood function in (2) involves computing multidimensional integrals of an order equal to  $T_i$  times the number of components in  $y_i$ , which becomes very difficult for large  $T_i$  and/or multidimensional unobservables  $y_i$ . That is why in previous literature the unobservables were often assumed to be i.i.d. In a series of papers, Rust developed a dynamic multinomial logit model, where he assumed that the utility function of the agents is additively separable in the unobservables and that the unobservables are extreme value i.i.d. In this case, the integration in (2) can be performed analytically. Pakes (1986) used Monte Carlo simulations to approximate the likelihood function in a model of binary choice with a serially correlated one-dimensional unobservable. More recently, several authors estimated models with particular forms of serial correlation in unobservables by adopting the method of Keane and Wolpin (1994), which uses Monte Carlo simulations to compute the likelihood and interpolating regressions to speed up the solution to the DP.<sup>2</sup> Even for DDCMs with special forms of serial correlation that reduce the dimension of integration in (2), estimation is still very hard. In this paper, I propose a computationally attractive Bayesian approach to estimation of DDCMs with serial correlation in unobservables.

In the Bayesian framework, the high-dimensional integration over  $y_i$  for each parameter value can be circumvented by employing Gibbs sampling and data augmentation. In models with latent variables, the Gibbs sampler typically has two types of blocks: (a) parameters conditional on other parameters, latent variables, and the data; (b) latent variables conditional on other latent variables, parameters, and the data (this step is called data augmentation). Draws from this Gibbs sampler form a Markov chain with the stationary distribution equal to the joint distribution of the parameters and the latent variables conditional on the data. The densities for both types of blocks are proportional to the joint density of the data, the latent variables, and the parameters. Therefore, to construct the Gibbs sampler, we need to be able to evaluate the joint density of the data, the latent variables, and the parameters. For a textbook treatment of these ideas, see Chapter 6 in Geweke (2005).

It is straightforward to obtain an analytical expression for the joint density of the data, the latent variables, and the parameters under the parameterization

<sup>2</sup>For example, Erdem and Keane (1996) estimated a model in which consumer perceptions of products are modelled by a sum of a parameter and an i.i.d. component, and thus are serially correlated. Consumer product usage requirements are modelled similarly in Erdem, Imai, and Keane (2003). In Sullivan (2006), a job match-specific wage draw persists for the duration of a match. In Keane and Wolpin (2006), women draw from husbands earnings distribution and the draw stays fixed for the duration of the match. It is also common to allow for serial correlation in unobservables induced by latent types (see, for example, Keane and Wolpin (1997)). I thank an anonymous referee for bringing these references to my attention.

of the Gibbs sampler in which the unobserved state variables are directly used as the latent variables in the sampler

$$(3) \quad p(\theta, x, d, y) = p(\theta) \prod_{i=1}^I \prod_{t=1}^{T_i} p(d_{t,i} | x_{t,i}, y_{t,i}; \theta) \times f(x_{t,i}, y_{t,i} | x_{t-1,i}, y_{t-1,i}, d_{t-1,i}; \theta),$$

where  $p(d_{t,i} | x_{t,i}, y_{t,i}; \theta) = 1_{\{\mathcal{V}(y_{t,i}, x_{t,i}, d_{t,i}; \theta) \geq \mathcal{V}(y_{t,i}, x_{t,i}, d; \theta), \forall d \in D\}}(y_{t,i}, x_{t,i}, d_{t,i}; \theta)$  is an indicator function and  $p(\theta)$  is a prior density for the parameters. In this Gibbs sampler, the conditional density of a parameter given the data, the rest of the parameters, and the latent variables will be proportional to (3). Since (3) includes a product of indicator functions  $p(d_{t,i} | y_{t,i}, x_{t,i}; \theta)$ , in this Gibbs sampler, the distributions for parameter blocks will be truncated to a region defined by inequality constraints that are nonlinear in  $\theta$ :

$$(4) \quad \mathcal{V}(y_{t,i}, x_{t,i}, d_{t,i}; \theta) \geq \mathcal{V}(y_{t,i}, x_{t,i}, d; \theta) \quad \forall d \in D, \forall t \in \{1, \dots, T_i\}, \forall i \in \{1, \dots, I\}.$$

For realistic sample sizes, the number of these constraints is very large and the algorithm is impractical; for example, parameter draws from an acceptance sampling algorithm never got accepted in experiments with a sample size of more than 100 observations. The same situation occurs under the parameterization in which  $u_{t,d,i} = u(y_{t,i}, x_{t,i}, d_{t,i}; \theta)$  are used as the latent variables in the sampler instead of some or all of the components of  $y_{t,i}$ .

The complicated truncation region (4) in drawing the parameter blocks could be avoided if we use  $\mathcal{V}_{t,i} = \{\mathcal{V}_{t,d,i} = \mathcal{V}(s_{t,i}, d; \theta), d \in D\}$  as latent variables in the sampler. Under this parameterization, the joint density of the data, the latent variables, and the parameters (needed for construction of the Gibbs sampler) does not have a convenient analytical form because  $\mathcal{V}_{t,d,i}$  depends on other unobservables through the expected value function, which can only be approximated numerically. In general, even evaluation of a kernel of this distribution is not easy. However, under some reasonable assumptions on the unobservables, a feasible Gibbs sampler can be constructed. In particular, let us assume that the unobserved part of the state vector includes some components that do not affect the distribution of the future state. Let us denote them by  $\nu_t$  and denote the other (possibly serially correlated) components by  $\epsilon_t$ ; so,  $y_t = (\nu_t, \epsilon_t)$ . This assumption means that the transition law  $f(x_{t+1}, \nu_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t, d; \theta)$  and thus the expected value function  $E\{V(s_{t+1}; \theta) | s_t, d; \theta\}$  do not depend on  $\nu_t$ .

The presence of  $\nu_t$  is well justified in an estimable model. If the support of these unobservables is sufficiently large and if they enter the utility function in a particular way, then the econometric model will be consistent with any possible sequence of observed choices (specification for unobservables is then

called saturated (Rust (1994, p. 3102))). If, in contrast, all the unobservables do affect the expected value function  $E\{V(s_{t+1}; \theta)|s_t, d; \theta\}$ , then the desirable saturation property might not hold or be very difficult to establish.

Since the expected value function  $E\{V(s_{t+1}; \theta)|s_t, d; \theta\}$  does not depend on  $v_t$ , the alternative specific value functions  $\mathcal{V}_{t,i} = \{u(v_{t,i}, \epsilon_{t,i}, x_{t,i}, d; \theta) + \beta E[V(s_{t+1}; \theta)|\epsilon_{t,i}, x_{t,i}, d; \theta], d \in D\}$  will depend on  $v_{t,i}$  only through  $u(v_{t,i}, \epsilon_{t,i}, x_{t,i}, d; \theta)$ . The per-period utility  $u(\cdot)$  and the distribution for  $v_{t,i}$  can be specified in such a way that  $p(\mathcal{V}_{t,i}|\theta, x_{t,i}, \epsilon_{t,i})$  has a convenient analytical expression (or at least a quickly computable density kernel). In this case, a marginal conditional decomposition of the joint distribution of the data, the parameters, and the latent variables will consist of parts with analytical or easily computable expressions. Construction of the Gibbs sampler in this case is illustrated by the following example.

**EXAMPLE 1—A Model of Optimal Bus Engine Replacement (Rust (1987)):** In this model, a maintenance superintendent of a transportation company decides every time period whether to replace an engine for each bus in the company's fleet. The observed state variable is bus mileage  $x_t$  since the last engine replacement. The per-period utility is the negative of per-period costs. If the engine is not replaced at time  $t$ , then  $u(x_t, \epsilon_t, v_t, d_t = 1; \alpha) = \alpha_1 x_t + \epsilon_t$ ; otherwise,  $u(x_t, \epsilon_t, v_t, d_t = 2; \alpha) = \alpha_2 + v_t$ , where  $\epsilon_t$  and  $v_t$  are the unobserved state variables,  $\alpha_1$  is the negative of per-period maintenance costs per unit of mileage, and  $\alpha_2$  is the negative of the costs of engine replacement. The bus mileage is discretized into  $M = 90$  intervals  $X = \{1, \dots, M\}$ . The change in the mileage ( $x_{t+1} - x_t$ ) evolves according to a multinomial distribution on  $\{0, 1, 2\}$  with parameters  $\eta = (\eta_1, \eta_2, \eta_3)$ .

Rust assumed that  $\epsilon_t$  and  $v_t$  are extreme value i.i.d. Under this assumption, the integrals over  $y_t = (\epsilon_t, v_t)$  in the Bellman equation (1) and in the likelihood function (2) can be computed analytically. Rust used the maximum likelihood method to estimate the model. Since the expression for the likelihood function involves the expected value functions, Rust's algorithm solves the DP numerically on each iteration of the estimation procedure. Rust's assumptions on unobservables considerably reduce computational burden. However, it is reasonable to expect that engine-specific maintenance costs represented by  $\epsilon_t$  are serially correlated. Thus, one could assume  $v_t$  is i.i.d.  $N(0, h_v^{-1})$  truncated to an interval  $[-\bar{v}, \bar{v}]$ ,  $\epsilon_t$  is  $N(\rho\epsilon_{t-1}, h_\epsilon^{-1})$  truncated to  $E = [-\bar{\epsilon}, \bar{\epsilon}]$ , and  $\epsilon_0 = 0$ . When  $\epsilon_t$  is serially correlated, the dimension of integration in the likelihood function can exceed 200 for Rust's data. It would be very hard to compute these integrals on each iteration of an estimation procedure. The Gibbs sampler with data augmentation described below can handle this problem.

Each bus/engine  $i$  is observed for  $T_i$  time periods:  $\{x_{t,i}, d_{t,i}\}_{t=1}^{T_i}$  for  $i = 1, \dots, I$ . When the engine is replaced, the state is reinitialized:  $x_{t-1} = 1$ ,  $\epsilon_{t-1} = 0$ . Therefore, a bus with a replaced engine can be treated as a separate

observation. The parameters are  $\theta = (\alpha, \eta, \rho, h_\epsilon)$ ;  $h_\nu$  is fixed for normalization. The latent variables are  $\{\mathcal{V}_{t,i}, \epsilon_{t,i}\}_{t=1}^{T_i}$ ,  $i = 1, \dots, I$ , where  $\mathcal{V}_{t,i} = x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} - \nu_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})$  and  $F_{t,i}(\theta, \epsilon) = \beta(E[V(x', \epsilon', \nu'; \theta)|\epsilon, x_{t,i}, d_{t,i} = 1; \theta] - E[V(x', \epsilon', \nu'; \theta)|\epsilon, x_{t,i}, d_{t,i} = 2; \theta])$ . A compact space for parameters (required by the theory in the following sections)  $\Theta$  is defined as  $\alpha_i \in [-\bar{\alpha}, \bar{\alpha}]$ ,  $\rho \in [-\bar{\rho}, \bar{\rho}]$ ,  $h_\epsilon \in [\bar{h}'_\epsilon, \bar{h}''_\epsilon]$ , and  $\eta$  belongs to a two-dimensional simplex.

The joint distribution of the data, the parameters, and the latent variables is

$$\begin{aligned}
 (5) \quad & p(\theta; \{x_{t,i}, d_{t,i}; \mathcal{V}_{t,i}, \epsilon_{t,i}\}_{t=1}^{T_i}; i = 1, \dots, I) \\
 &= p(\theta) \prod_{i=1}^I \prod_{t=1}^{T_i} [p(d_{t,i}|\mathcal{V}_{t,i}) p(\mathcal{V}_{t,i}|x_{t,i}, \epsilon_{t,i}; \theta) \\
 &\quad \times p(x_{t,i}|x_{t-1,i}; d_{t-1,i}; \eta) p(\epsilon_{t,i}|\epsilon_{t-1,i}, \rho, h_\epsilon)],
 \end{aligned}$$

where  $p(\theta)$  is a prior,

$$\begin{aligned}
 & p(x_{t,i}|x_{t-1,i}; d_{t-1,i}; \eta) = \eta_{x_{t,i}-x_{t-1,i}+1}, \\
 & p(d_{t,i}|\mathcal{V}_{t,i}) = 1_{\{d_{t,i}=1, \mathcal{V}_{t,i} \geq 0 \text{ or } d_{t,i}=2, \mathcal{V}_{t,i} < 0\}}, \\
 & p(\epsilon_{t,i}|\epsilon_{t-1,i}, \theta) \\
 &= \frac{h_\epsilon^{1/2} \exp\{-0.5h_\epsilon(\epsilon_{t,i} - \rho\epsilon_{t-1,i})^2\}}{\sqrt{2\pi}[\Phi(\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5}) - \Phi(-\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5})} 1_E(\epsilon_{t,i}),
 \end{aligned}$$

and

$$\begin{aligned}
 & p(\mathcal{V}_{t,i}|x_{t,i}, \epsilon_{t,i}; \theta) \\
 (6) \quad &= \exp\{-0.5h_\nu(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})])^2\} \\
 (7) \quad &\cdot 1_{[-\bar{\nu}, \bar{\nu}]}(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})]) \\
 &\cdot \frac{h_\nu^{0.5}}{\sqrt{2\pi}[\Phi(\bar{\nu}h_\nu^{0.5}) - \Phi(-\bar{\nu}h_\nu^{0.5})]}.
 \end{aligned}$$

Densities for Gibbs sampler blocks will be proportional to the joint distribution in (5). In this Gibbs sampler the observed choice optimality constraints do not involve parameters and affect only blocks for simulating  $\mathcal{V}_{t,i}|\dots$ , which will have a normal truncated distribution proportional to (6) and (7), and also truncated to  $R^+$  if  $d_{t,i} = 1$  or to  $R^-$  otherwise. Efficient algorithms for simulating from truncated normal distributions are readily available; see, for example, Geweke (1991).

The density for  $\epsilon_{t,i}|\dots$  is

$$\begin{aligned}
 & p(\epsilon_{t,i}|\dots) \\
 & \propto \frac{\exp\{-0.5h_\nu(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})])^2\}}{\Phi([\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5}) - \Phi(-\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5})} \\
 & \cdot 1_{[-\bar{\nu}, \bar{\nu}]}(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})]) \\
 (8) \quad & \cdot \exp\{-0.5h_\epsilon(\epsilon_{t+1,i} - \rho\epsilon_{t,i})^2 - 0.5h_\epsilon(\epsilon_{t,i} - \rho\epsilon_{t-1,i})^2\} \cdot 1_E(\epsilon_{t,i}).
 \end{aligned}$$

Direct simulation from  $\epsilon_{t,i}|\dots$  could be difficult. However, the kernel of this density can be evaluated numerically (approximations to  $F_{t,i}(\theta, \epsilon_{t,i})$  are discussed in the next section). Therefore, a Metropolis-within-Gibbs<sup>3</sup> algorithm can be used for this Gibbs sampler block. A convenient transition density for this Metropolis-within-Gibbs step is a truncated normal density proportional to (8).

Assuming a normal prior  $N(\underline{\rho}, \underline{h}_\rho^{-1})$  truncated to  $[-\bar{\rho}, \bar{\rho}]$ ,

$$\begin{aligned}
 & p(\rho|\dots) \propto \frac{\exp\left\{-0.5h_\nu \sum_{i,t} (\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})])^2\right\}}{\prod_{i,t} \Phi([\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5}) - \Phi(-\bar{\epsilon} - \rho\epsilon_{t-1,i}]h_\epsilon^{0.5})} \\
 & \cdot \prod_{i,t} 1_{[-\bar{\nu}, \bar{\nu}]}(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})]) \\
 (9) \quad & \cdot \exp\{-0.5\bar{h}_\rho(\rho - \bar{\rho})^2\} \cdot 1_{[-\bar{\rho}, \bar{\rho}]}(\rho),
 \end{aligned}$$

where  $\bar{h}_\rho = \underline{h}_\rho + \sum_i \sum_{t=2}^{T_i} \epsilon_{t-1,i}^2$  and  $\bar{\rho} = \bar{h}_\rho^{-1}(\underline{h}_\rho \underline{\rho} + h_\epsilon \sum_i \sum_{t=2}^{T_i} \epsilon_{t,i} \epsilon_{t-1,i})$ . A Metropolis-within-Gibbs algorithm with truncated normal transition density proportional to (9) can be used for this Gibbs sampler block. Blocks for other parameters can be constructed in a similar way; see Norets (2007).

The Gibbs sampler presented in this example can be generalized and applied to different models with other interesting forms of heterogeneity such as individual-specific parameters. Also, components of  $\nu_t$  do not have to enter the utility function linearly. The essential requirement is the ability to evaluate a kernel of  $p(\mathcal{V}_{t,i}|\theta, x_{t,i}, \epsilon_{t,i})$  quickly. The Gibbs sampler outlined above requires computing the expected value functions for each new parameter draw  $\theta^m$  from

<sup>3</sup>To produce draws from some target distribution, the Metropolis or Metropolis-Hastings MCMC algorithm only needs values of a kernel of the target density. The draws are simulated from a transition density and they are accepted with probability that depends on the values of the target density kernel and the transition density. For more details, see, for example, Chib and Greenberg (1995).

the MCMC iteration  $m$  and each observation in the sample. The following section describes how the approximations of the expected value functions can be efficiently obtained.

### 3. ALGORITHM FOR SOLVING THE DP

For a discussion of methods for solving the DP for a given parameter vector  $\theta$ , see Rust (1996). Below, I introduce a method of solving the DP suitable for use in conjunction with the Bayesian estimation of a general DDCM. This method uses an idea from Imai, Jain, and Ching (2005): to iterate the Bellman equation only once at each step of the estimation procedure and use information from previous steps to approximate the expectations in the Bellman equation. However, the way the previous information is used differs for the two methods. A detailed comparison is given in Section 3.2.

#### 3.1. Algorithm Description

In contrast to conventional value function iteration, this algorithm iterates the Bellman equation only once for each parameter draw. First, I will describe how the DP solving algorithm works and then how the output of the DP solving algorithm is used to approximate the expected value functions in the Gibbs sampler.

The DP solving algorithm takes a sequence of parameter draws  $\theta^m$ ,  $m = 1, 2, \dots$ , as an input from the Gibbs sampler, where  $m$  denotes the Gibbs sampler iteration. For each  $\theta^m$ , the algorithm generates random states  $s^{m,j} \in S$ ,  $j = 1, \dots, \hat{N}(m)$ . At each random state, the approximations of the value functions  $V^m(s^{m,j}; \theta^m)$  are computed by iterating the Bellman equation once. At this one iteration of the Bellman equation, the future expected value functions are computed by importance sampling over value functions  $V^k(s^{k,j}; \theta^k)$  from previous iterations  $k < m$ .

The random states  $s^{m,j}$  are generated from a density  $g(\cdot) > 0$  on  $S$ . This density  $g(\cdot)$  is used as an importance sampling source density in approximating the expected value functions. The collection of the random states  $\{s^{m,j}\}_{j=1}^{\hat{N}(m)}$  will be referred to below as the random grid. (Rust (1997) showed that value function iteration on random grids from a uniform distribution breaks the curse of dimensionality for DDCMs.) The number of points in the random grid at iteration  $m$  is denoted by  $\hat{N}(m)$  and will be referred to below as the size of the random grid (at iteration  $m$ ).

For each point in the current random grid  $s^{m,j}$ ,  $j = 1, \dots, \hat{N}(m)$ , the approximation of the value function  $V^m(s^{m,j}; \theta^m)$  is computed according to

$$(10) \quad V^m(s; \theta) = \max_{d \in D} \{u(s, d; \theta) + \beta \hat{E}^{(m)}[V(s'; \theta) | s, d; \theta]\}.$$

Not all of the previously computed value functions  $V^k(s^{k,j}; \theta^k)$ ,  $k < m$ , are used in importance sampling for computing  $\hat{E}^{(m)}[V(s'; \theta)|s, d; \theta]$  in (10). To converge, the algorithm has to forget the remote past. Thus, at each iteration  $m$ , I keep track only of the history of length  $N(m)$ :  $\{\theta^k; s^{k,j}, V^k(s^{k,j}; \theta^k), j = 1, \dots, \hat{N}(k)\}_{k=m-N(m)}^{m-1}$ . In this history, I find  $\tilde{N}(m)$  closest to  $\theta$  parameter draws. Only the value functions corresponding to these nearest neighbors are used in importance sampling. Formally, let  $\{k_1, \dots, k_{\tilde{N}(m)}\}$  be the iteration numbers of the nearest neighbors of  $\theta$  in the current history:  $k_1 = \operatorname{arg\,min}_{i \in \{m-N(m), \dots, m-1\}} \|\theta - \theta^i\|$  and

$$(11) \quad k_j = \operatorname{arg\,min}_{i \in \{m-N(m), \dots, m-1\} \setminus \{k_1, \dots, k_{j-1}\}} \|\theta - \theta^i\|, \quad j = 2, \dots, \tilde{N}(m).$$

If the  $\operatorname{arg\,min}$  returns a multivalued result, I use the lexicographic order for  $(\theta^i - \theta)$  to decide which  $\theta^i$  is chosen first. If the result of the lexicographic selection is also multivalued,  $\theta^i = \theta^j$ , then I choose  $\theta^i$  over  $\theta^j$  if  $i > j$ . This particular way to resolve the multivaluedness of the  $\operatorname{arg\,min}$  might seem irrelevant for implementing the method in practice; however, it is used in the proof of the measurability of the supremum of the approximation error, which is necessary for the uniform convergence results. A reasonable choice for the norm in (11) would be  $\|\theta\| = \sqrt{\theta^T \underline{H}_\theta \theta}$ , where  $\underline{H}_\theta$  is the prior precision for the parameters. Importance sampling is performed as

$$(12) \quad \hat{E}^{(m)}[V(s'; \theta)|s, d; \theta] = \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} \frac{V^{k_i}(s^{k_i,j}; \theta^{k_i}) \cdot f(s^{k_i,j} | s, d; \theta) / g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\hat{N}(k_r)} f(s^{k_r,q} | s, d; \theta) / g(s^{k_r,q})}$$

$$(13) \quad = \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} V^{k_i}(s^{k_i,j}; \theta^{k_i}) W_{k_i,j,m}(s, d, \theta).$$

The target density for importance sampling is the state transition density  $f(\cdot|s, d; \theta)$ . The source density is the density  $g(\cdot)$  from which the random grid on the state space is generated. In general,  $g(\cdot)$  should give reasonably high probabilities to all parts of the state space that are likely under  $f(\cdot|s, d; \theta)$  with reasonable values of the parameter  $\theta$ . To reduce the variance of the approximation of expectations produced by importance sampling, one should make  $g(\cdot)$  relatively high for the states that result in large absolute values for value functions ( $g(s')$  that minimizes the variance of the importance sampling approximation to the expectation is proportional to  $|V(s'; \theta)f(s'|s, d; \theta)|$ ).

Section 3.3 formally presents the assumptions on model primitives and restrictions on  $g(\cdot)$ ,  $\hat{N}(m)$ ,  $N(m)$ , and  $\tilde{N}(m)$  that are sufficient for algorithm convergence.

After  $V^m(s^{m,j}; \theta^m)$  are computed from (10) and (12), they can be used in a formula similar to (12) to obtain the approximations of the expectations  $E[V(s_{t+1}; \theta^m) | x_{t,i}, \epsilon_{t,i}^m, d; \theta^m]$  on iteration  $m$  of the Gibbs sampler.

### 3.2. Comparison With Imai, Jain, and Ching (2005)

An algorithm for solving the DP has to deal with an integration problem for computing the expectations of the value functions in addition to approximating the value function in the parameter space. My prescriptions for handling this integration problem differ from IJC's. IJC used kernel smoothing over all  $N(m)$  previously computed value functions to approximate the expected value functions. They also generated only one new state at each iteration,  $\hat{N}(m) = 1 \forall m$ . For a finite observed state space, deterministic transitions for the observed states, and i.i.d. unobservables IJC proved convergence of their DP solution approximations. To handle compact state space and random state transitions I introduce growing random grids:  $\hat{N}(m)$  increases with  $m$ . A fixed random grid size that works for IJC's i.i.d. errors does not seem to be enough for general random transitions. When the size of the random grid grows, the nearest neighbor (NN) algorithm that I use to approximate value functions in the parameter space is computationally much more efficient than the kernel smoothing used by IJC. The computational advantage of using the NN algorithm in this case stems from the fact that importance sampling over the random grids has to be performed only for a few nearest neighbors and not for the whole tracked history of length  $N(m)$ . The convergence results I obtain are also stronger. IJC proved uniform convergence in probability for their DP solution approximations. For the NN algorithm, I establish complete uniform convergence, which implies uniform a.s. convergence. Furthermore, the NN algorithm easily accommodates more than one iteration of the Bellman equation for each parameter draw to improve the approximation precision in practice. Overall, the nearest neighbors method is not just a substitute for kernel smoothing that might work better in higher dimensions (see, e.g., Scott (1992, pp. 189–190)), but an essential part of the algorithm that, in conjunction with random grids, makes it computationally efficient and applicable to more general model specifications.

### 3.3. Theoretical Results

The following assumptions on the model primitives and the algorithm parameters are made:

ASSUMPTION 1:  $\Theta \subset R^{J_\theta}$  and  $S \subset R^{J_s}$  are compact, and  $\beta \in (0, 1)$  is known.

The assumption of compactness of the parameter space is standard in econometrics. Fixing  $\beta$  is also a usual practice in the literature on estimation of DDCMs.

ASSUMPTION 2:  $u(s, d; \theta)$  is continuous in  $(\theta, s)$  (and thus bounded on compacts).

ASSUMPTION 3:  $f(s'|s, d; \theta)$  is continuous in  $(\theta, s, s')$  and  $g(s)$  is continuous in  $s$ . Discrete state variables can be accommodated by defining densities with respect to the counting measure.

Assumptions 1–3 imply continuity of  $V(s; \theta)$  in  $(\theta, s)$  (see Proposition 4 in the Supplemental Material (Norets (2009b)) or Norets (2009a) for more general results).

ASSUMPTION 4: The density of the state transition  $f(\cdot|\cdot)$  and the importance sampling density  $g(\cdot)$  are bounded above and away from zero, which gives

$$\inf_{\theta, s', s, d} f(s'|s, d; \theta)/g(s') \geq \underline{f} > 0 \quad \text{and}$$

$$\sup_{\theta, s', s, d} f(s'|s, d; \theta)/g(s') \leq \bar{f} < \infty.$$

Assumption 4 can be relaxed. The support of the transition density can be allowed to depend on the decision  $d$  and the discrete state variables if they take a finite number of values. Deterministic transitions for discrete state variables and, in some cases, for continuous state variables (e.g., setting  $\epsilon_t = 0$  when  $d_t = 2$  in Rust's engine replacement model) can be accommodated. Corollaries 1 and 2 below describe changes in the DP solving algorithm required to relax Assumption 4.

ASSUMPTION 5:  $\exists \hat{\delta} > 0$  such that  $P(\theta^{m+1} \in A | \omega^m) \geq \hat{\delta} \lambda(A)$  for any Borel measurable  $A \subset \Theta$ , any  $m$ , and any feasible history  $\omega^m = \{\omega_1, \dots, \omega_m\}$ , where  $\lambda$  is the Lebesgue measure. The history includes all the parameter and latent variable draws from the Gibbs sampler and all the random grids from the DP solving algorithm:  $\omega_t = \{\theta^t, \mathcal{V}^t, \epsilon^t; s^{t,j}, j = 1, \dots, \hat{N}(t)\}$ .

Assumption 5 means that at each iteration of the algorithm, the parameter draw can get into any part of  $\Theta$ . This assumption should be verified for each specific DDCM and the corresponding parameterization of the Gibbs sampler. The assumption is only a little stronger than standard conditions for convergence of the Gibbs sampler; see Corollary 4.5.1 in Geweke (2005). Since a careful practitioner of MCMC would have to establish convergence of the Gibbs sampler, a verification of Assumption 5 should not require much extra effort.

ASSUMPTION 6: Let  $1 > \gamma_0 > \gamma_1 > \gamma_2 \geq 0$  and  $N(t) = [t^{\gamma_1}]$ ,  $\tilde{N}(t) = [t^{\gamma_2}]$ ,  $\hat{N}(t) = [t^{\gamma_1 - \gamma_2}]$ , and  $\hat{N}(0) = 1$ , where  $[x]$  is the integer part of  $x$ .

Multiplying the functions of  $t$  in Assumption 6 by positive constants will not affect any of the theoretical results below.

THEOREM 1: Under Assumptions 1–6, the approximation to the expected value function in (12) converges uniformly and completely to the exact value: that is, the following statements hold:

- (i)  $\sup_{s, \theta, d} |\hat{E}^{(t)}[V(s'; \theta) | s, d; \theta] - E[V(s'; \theta) | s, d; \theta]|$  is measurable.
- (ii) For any  $\tilde{\epsilon} > 0$  there exists a sequence  $\{z_t\}$  such that  $\sum_{t=0}^{\infty} z_t < \infty$  and

$$P\left(\sup_{s, \theta, d} |\hat{E}^{(t)}[V(s'; \theta) | s, d; \theta] - E[V(s'; \theta) | s, d; \theta]| > \tilde{\epsilon}\right) \leq z_t.$$

COROLLARY 1: Let the state space be a product of a finite set  $S_f$  and a bounded rectangle  $S_c \in R^{J_{S_c}}$ ,  $S = S_f \times S_c$ . Let  $f(s'_f, s'_c | s_f, s_c; \theta)$  be the state transition density with respect to the product of the counting measure on  $S_f$  and the Lebesgue measure on  $S_c$ . Assume for any  $s_f \in S_f$  and  $d \in D$ , we can define  $S(s_f, d) \subset S$  such that  $f(s'_f, s'_c | s_f, s_c, d; \theta) > 0$  for any  $(s'_f, s'_c) \in S(s_f, d)$  and any  $s_c \in S_c$  and  $f(s'_f, s'_c | s_f, s_c, d; \theta) = 0$  for any  $(s'_f, s'_c) \notin S(s_f, d)$  and any  $s_c \in S_c$ . For each  $s_f \in S_f$  and  $d \in D$ , let density  $g_{s_f, d}(\cdot)$  be such that  $\inf_{\theta \in \Theta, (s'_f, s'_c) \in S(s_f, d), s_c \in S_c} f(s'_f, s'_c | s_f, s_c, d; \theta) / g_{s_f, d}(s'_f, s'_c) \geq \underline{f} > 0$  and  $\sup_{\theta \in \Theta, (s'_f, s'_c) \in S(s_f, d), s_c \in S_c} f(s'_f, s'_c | s_f, s_c, d; \theta) / g_{s_f, d}(s'_f, s'_c) \leq \bar{f} < \infty$ . In the DP solving algorithm, generate the random grid over the state space for each discrete state  $s_f \in S_f$  and decision  $d \in D$ :  $s_{s_f, d}^{m, j} \sim g_{s_f, d}(\cdot)$ , and use these grids to compute the approximations of the expectations  $E(V(s'; \theta) | s_f, s_c, d; \theta)$ . Then the conclusions of Theorem 1 hold.

COROLLARY 2: If the transition for the discrete states is independent from the other states, then a more efficient alternative would also work. Let us denote the transition probability for the discrete states by  $f(s'_f | s_f, d; \theta)$ . Suppose that for  $f(s'_c | s_c, d; \theta)$  and some  $g(\cdot)$  defined on  $S_c$ , Assumption 4 holds and the random grid  $s_c^{m, j}$  is generated only on  $S_c$  from  $g(\cdot)$ . Consider the following approximation of the expectations,  $\hat{E}^{(m)}[V(s'; \theta) | s_f, s_c, d; \theta]$ , in the DP solving algorithm:

$$(14) \quad \sum_{s'_f \in S_f(s_f, d)} f(s'_f | s_f, d; \theta) \times \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\tilde{N}(k_i)} \frac{V^{k_i}(s'_f, s^{k_i, j}; \theta^{k_i}) f(s^{k_i, j} | s, d; \theta) / g(s^{k_i, j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\tilde{N}(k_r)} f(s^{k_r, q} | s, d; \theta) / g(s^{k_r, q})},$$

where  $S_f(s_f, d)$  denotes the set of possible future discrete states given the current state  $s_f$  and decision  $d$ . Then the conclusions of Theorem 1 hold.

#### 4. CONVERGENCE OF POSTERIOR EXPECTATIONS

In Bayesian analysis, most inference exercises involve computing posterior expectations of some functions. For example, the posterior mean and the posterior standard deviation of a parameter and the posterior probability that a parameter belongs to a set can all be expressed in terms of posterior expectations. More importantly, the answers to the policy questions that DDCMs address also take this form. Using the uniform complete convergence of the approximations of the expected value functions, I prove the complete convergence of the approximated posterior expectations under mild assumptions on a kernel of the posterior distribution.

*ASSUMPTION 7: Assume that  $\epsilon_{t,i}$ ,  $\theta$ , and  $v_{t,k,i}$  have compact supports  $E$ ,  $\Theta$ , and  $[-\bar{v}, \bar{v}]$  correspondingly, where  $v_{t,k,i}$  denotes the  $k$ th component of  $v_{t,i}$ . Let the joint posterior distribution of the parameters and the latent variables be proportional to a product of a continuous function and indicator functions,*

$$(15) \quad p(\theta, \mathcal{V}, \epsilon; F|d, x) \propto r(\theta, \mathcal{V}, \epsilon; F(\theta, \epsilon)) \cdot 1_\Theta(\theta) \cdot \left( \prod_{i,t} 1_E(\epsilon_{t,i}) p(d_{t,i}|\mathcal{V}_{t,i}) \right) \cdot \left( \prod_{i,t,k} 1_{[-\bar{v}, \bar{v}]}(q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}(\theta, \epsilon_{t,i}))) \right),$$

where  $r(\theta, \mathcal{V}, \epsilon; F)$  and  $q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i})$  are continuous in  $(\theta, \mathcal{V}, \epsilon, F)$ ,  $F = \{F_{t,d,i}, \forall i, t, d\}$  stands for a vector of the expected value functions, and  $F_{t,i}$  are the corresponding subvectors. Also assume that the level curves of  $q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i})$  corresponding to  $\bar{v}$  and  $-\bar{v}$  have zero Lebesgue measure,

$$(16) \quad \lambda[(\theta, \mathcal{V}, \epsilon) : q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = \bar{v}] = \lambda[(\theta, \mathcal{V}, \epsilon) : q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = -\bar{v}] = 0.$$

This assumption is likely to be satisfied for most models formulated on a bounded state space, in which distributions are truncated to bounded regions required by the theory. The kernel of the joint distribution for the engine replacement example from Section 2 has the form in (15). Condition (16) is also easy to verify. In Rust’s model,  $q_d(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = \Delta u(x_{t,i}, d) + \epsilon_{t,d,i} + F_{t,d,i}(\theta, \epsilon_{t,i}) - \mathcal{V}_{t,d,i} = \bar{v}$  defines a continuous function  $\mathcal{V}_{t,d,i} = \Delta u(x_{t,i}, d) + \epsilon_{t,d,i} + F_{t,d,i}(\theta, \epsilon_{t,i}) - \bar{v}$ . Since the Lebesgue measure of the graph of a continuous function is zero, (16) will be satisfied.

**THEOREM 2:** *Let  $h(\theta, \mathcal{V}, \epsilon)$  be a bounded function. Under Assumptions 1–7, the expectation of  $h(\theta, \mathcal{V}, \epsilon)$  with respect to the approximated posterior that uses the DP solution approximations  $\hat{F}^n$  from step  $n$  of the DP solving algorithm converges completely (and thus a.s.) to the true posterior expectation of  $h(\theta, \mathcal{V}, \epsilon)$  as  $n \rightarrow \infty$ . In particular, for any  $\epsilon > 0$ , there exists a sequence  $\{z_n\}$  such that  $\sum_{n=0}^{\infty} z_n < \infty$  and the probability*

$$P\left(\left| \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F|d, x) d(\theta, \mathcal{V}, \epsilon) - \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; \hat{F}^n|d, x) d(\theta, \mathcal{V}, \epsilon) \right| > \epsilon \right)$$

is bounded above by  $z_n$ .

One way to apply Theorem 2 is to stop the DP solving algorithm at an iteration  $m$  and run the Gibbs sampler for extra  $n$  iterations using the DP solution  $\hat{E}^{(m)}[V(s'; \theta)|s, d; \theta]$  from iteration  $m$ . If the Gibbs sampler is uniformly ergodic (see Tierney (1994)) for any fixed approximation  $\hat{E}^{(m)}[V(s'; \theta)|s, d; \theta]$ , then for any  $\delta > 0$  and  $\epsilon > 0$  there exist  $m$  and  $N$  such that for all  $n \geq N$ ,

$$P\left(\left| \frac{1}{n} \sum_{i=m+1}^{m+n} h(\theta^i, \mathcal{V}^i, \epsilon^i) - \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F|d, x) d(\theta, \mathcal{V}, \epsilon) \right| > \epsilon \right) \leq \delta.$$

If we do not stop the DP solving algorithm and run it together with the Gibbs sampler, then the stochastic process for  $(\theta^i, \mathcal{V}^i, \epsilon^i)$  will not be a Markov chain. In this case, results from the adaptive MCMC literature (e.g., Roberts and Rosenthal (2006)) can be adapted to prove laws of large numbers and convergence in distribution for  $(\theta^i, \mathcal{V}^i, \epsilon^i)$ .

**THEOREM 3:** *Let us define the following two conditions. (a) The MCMC algorithm that uses the exact DP solutions is uniformly ergodic: for any  $\epsilon > 0$  there is  $N$  such that*

$$\|P^N((\theta, \mathcal{V}, \epsilon), \cdot; F) - P(\cdot; F|d, x)\| \leq \epsilon$$

for any  $(\theta, \mathcal{V}, \epsilon)$ , where  $P^N(\cdot, \cdot)$  is the Markov transition kernel implied by  $N$  iterations of the MCMC algorithm,  $P(\cdot; F|d, x)$  is the posterior probability measure, and  $\|\cdot\|$  is the bounded variation norm.

(b) *The transition kernel that uses the approximate DP solutions converges uniformly in probability to the transition kernel that uses the exact solutions*

$$\sup_{\theta, \mathcal{V}, \epsilon} \|P((\theta, \mathcal{V}, \epsilon), \cdot; F) - P((\theta, \mathcal{V}, \epsilon), \cdot; F^n)\| \xrightarrow{P} 0, \quad \text{as } n \rightarrow \infty.$$

Conditions (a) and (b) imply the following two results:

(i) *The MCMC algorithm that uses the approximate DP solutions is ergodic: for any  $(\theta^0, \mathcal{V}^0, \epsilon^0)$  and any  $\varepsilon > 0$  there exists  $M$  such that for any  $i \geq M$ ,*

$$\sup_A |P((\theta^i, \mathcal{V}^i, \epsilon^i) \in A | \theta^0, \mathcal{V}^0, \epsilon^0) - P(A; F | d, x)| \leq \varepsilon.$$

(ii) *A weak law of large numbers (WLLN) holds: for any  $(\theta^0, \mathcal{V}^0, \epsilon^0)$  and a bounded function  $h(\cdot)$ ,*

$$\sum_{i=1}^n h(\theta^i, \mathcal{V}^i, \epsilon^i) / n \xrightarrow{P} \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F | d, x) d(\theta, \mathcal{V}, \epsilon).$$

Norets (2007) showed how to establish condition (a) for the MCMC algorithm used for inference in the engine replacement example. A verification of condition (b) involves arguments and assumptions similar to those employed in the statement and proof of Theorem 2. (Theorem 2 implies strong convergence for the approximated posterior probability, while here we need a similar result for the approximated Markov transition probability.)

## 5. CONCLUSION

This paper presents a feasible method for Bayesian inference in dynamic discrete choice models with serially correlated unobserved state variables. I construct the Gibbs sampler, employing data augmentation and Metropolis steps, that can successfully handle multidimensional integration in the likelihood function of these models. The computational burden of solving the DP at each iteration of the estimation algorithm can be reduced by efficient use of the information obtained on previous iterations. Serially correlated unobservables are not the only possible source of intractable integrals in the likelihood function of DDCMs. The Gibbs sampler algorithm can be extended to allow for other interesting features in DDCMs such as individual-specific coefficients, missing data, macroshocks, and cohort effects. The proposed theoretical framework is flexible and leaves room for experimentation. For details on implementation and experiments, the interested reader is referred to Norets (2007, 2008). Overall, combined with efficient DP solution strategies, standard computational tools of Bayesian analysis seem to be very promising in making more elaborate DDCMs estimable.

## REFERENCES

- AGUIRREGABIRIA, V., AND P. MIRA (2007): "Dynamic Discrete Choice Structural Models: A Survey," Working Paper 297, Department of Economics, University of Toronto. [1667]
- ALBERT, J. H., AND S. CHIB (1993): "Bayesian Analysis of Binary and Polychotomous Response Data," *Journal of the American Statistical Association*, 88, 669–679. [1665]
- CHIB, S., AND E. GREENBERG (1995): "Understanding the Metropolis–Hastings Algorithm," *The American Statistician*, 49, 327–335. [1672]
- ECKSTEIN, Z., AND K. WOLPIN (1989): "The Specification and Estimation of Dynamic Stochastic Discrete Choice Models: A Survey," *Journal of Human Resources*, 24, 562–598. [1667]
- ERDEM, T., AND M. KEANE (1996): "Decision-Making Under Uncertainty: Capturing Dynamic Brand Choice Processes in Turbulent Consumer Goods Markets," *Marketing Science*, 15, 1–20. [1668]
- ERDEM, T., S. IMAI, AND M. KEANE (2003): "A Model of Consumer Brand and Quantity Choice Dynamics Under Price Uncertainty," *Quantitative Marketing and Economics*, 1, 5–64. [1668]
- GEWEKE, J. (1991): "Efficient Simulation From the Multivariate Normal and Student-*t* Distributions Subject to Linear Constraints and the Evaluation of Constraint Probabilities," in *Computing Science and Statistics: Proceedings of the Twenty-Third Symposium on the Interface*, ed. by E. M. Keramidas. Fairfax: Interface Foundation of North America, 571–578. [1671]
- (2005): *Contemporary Bayesian Econometrics and Statistics*. Hoboken, NJ: Wiley-Interscience. [1668,1676]
- GEWEKE, J., M. KEANE, AND D. RUNKLE (1994): "Alternative Computational Approaches to Inference in the Multinomial Probit Model," *The Review of Economics and Statistics*, 76, 609–632. [1665]
- GILLESKIE, D. (1998): "A Dynamic Stochastic Model of Medical Care Use and Work Absence," *Econometrica*, 6, 1–45. [1666]
- IMAI, S., N. JAIN, AND A. CHING (2005): "Bayesian Estimation of Dynamic Discrete Choice Models," Unpublished Manuscript, Queens University. [1666,1673,1675]
- KEANE, M., AND K. WOLPIN (1994): "The Solution and Estimation of Discrete Choice Dynamic Programming Models by Simulation and Interpolation: Monte Carlo Evidence," *Review of Economics and Statistics*, 76, 648–672. [1668]
- (1997): "The Career Decisions of Young Men," *Journal of Political Economy*, 105, 417–522. [1668]
- (2006): "The Role of Labor and Marriage Markets, Preference Heterogeneity and the Welfare System in the Life Cycle Decisions of Black, Hispanic and White Women." Working Paper 06-004, PIER. Available at <http://ssrn.com/abstract=889550>. [1668]
- MCCULLOCH, R., AND P. ROSSI (1994): "An Exact Likelihood Analysis of the Multinomial Probit Model," *Journal of Econometrics*, 64, 207–240. [1665]
- NORETS, A. (2007): "Bayesian Inference in Dynamic Discrete Choice Models," Ph.D. Dissertation, The University of Iowa. [1666,1672,1680]
- (2008): "Implementation of Bayesian Inference in Dynamic Discrete Choice Models," Unpublished Manuscript, Princeton University. [1666,1680]
- (2009a): "Continuity and Differentiability of Expected Value Functions in Dynamic Discrete Choice Models," Unpublished Manuscript, Princeton University. [1676]
- (2009b): "Supplement to 'Inference in Dynamic Discrete Choice Models With Serially Correlated Unobserved State Variables'," *Econometrica Supplemental Material*, 77, [http://www.econometricsociety.org/ecta/Supmat/7292 proofs.pdf](http://www.econometricsociety.org/ecta/Supmat/7292%20proofs.pdf). [1667,1676]
- PAKES, A. (1986): "Patents as Options: Some Estimates of the Value of Holding European Patent Stocks," *Econometrica*, 54, 755–784. [1668]
- ROBERTS, G., AND J. ROSENTHAL (2006): "Coupling and Ergodicity of Adaptive MCMC," available at <http://www.probability.ca/jeff/research.html>. [1679]
- RUST, J. (1987): "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher," *Econometrica*, 55, 999–1033. [1666,1670]

- (1994): “Structural Estimation of Markov Decision Processes,” in *Handbook of Econometrics*, ed. by R. Engle and D. McFadden. Amsterdam: North-Holland. [1667,1670]
- (1996): “Numerical Dynamic Programming in Economics,” in *Handbook of Computational Economics*, ed. by H. Amman, D. Kendrick, and J. Rust. Amsterdam: North-Holland. Available at <http://gemini.econ.umd.edu/jrust/sdp/ndp.pdf>. [1673]
- (1997): “Using Randomization to Break the Curse of Dimensionality,” *Econometrica*, 65, 487–516. [1673]
- SCOTT, D. (1992): *Multivariate Density Estimation*. New York: Wiley-Interscience. [1675]
- SULLIVAN, P. (2006): “A Dynamic Analysis of Educational Attainment, Occupational Choices, and Job Search,” Paper 861, MPRA, University Library of Munich, Germany. [1668]
- TIERNEY, L. (1994): “Markov Chains for Exploring Posterior Distributions,” *The Annals of Statistics*, 22, 1758–1762. [1679]

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