Affirmative Action in Winner-Take-All Markets*

Roland G. Fryer, Jr.
Harvard Society of Fellows and NBER

Glenn C. Loury
Boston University

January 2005

Revised: March 2005

Abstract

Whom to hire, promote, admit into elite universities, elect, or issue government contracts to are all determined in a tournament-like (winner-take-all) structure. This paper constructs a simple model of pair-wise tournament competition to investigate affirmative action in these markets. We consider two forms of affirmative action: group-sighted, where employers are allowed to use group identity in pursuit of their diversity goals; and group-blind, where they are not. It is shown that the equilibrium group-sighted affirmative action scheme involves a constant handicap given to agents in the disadvantaged group. Equilibrium group-blind affirmative action creates a unique semi-separating equilibrium in which a large pool of contestants exerts zero effort, and this pool is increasing in the aggresiveness of the affirmative action mandate.

*We are grateful to Edward Glaeser, Jerry Green, and Bart Lipman for helpful comments and suggestions. Correspondence can be addressed to Fryer at Littauer Center, Harvard University, Cambridge MA, 02138 (e-mail: rfryer@fas.harvard.edu); or Loury at Department of Economics, Boston University, 270 Bay State Road, Boston MA, 02215 (email: gloury@bu.edu).

1 Introduction

Winner-take-all markets are pervasive in our society. Whom to hire, promote, admit, elect, or contract with are all determined in a tournament-like (winner-take-all) structure. In general, these markets tend to emerge when there is quality variation, but little price flexibility. As a result, prizes tend to be awarded on the basis of relative, not absolute, performance. Employment hiring and promotion, admissions into elite universities, government contracts, or clerks to a supreme court justice all have this feature in common.

Because of their structure, winner-take-all markets have the feature that small differences in quality can be associated with large differences in rewards. This, coupled with the fact that many of the controversial applications of affirmative action (mentioned above) involve winner-take-all markets, makes it surprising that there has been no theoretical analysis of affirmative action in these environments. Understanding the theoretical trade-offs involved when applying affirmative action in winner-take-all markets is of great importance for public policy.

In this paper, we analyze affirmative action in winner-take-all markets. A short synopsis of our approach is as follows. There are many positions and many more individuals seeking these positions. Nature moves first and assigns a marginal cost of investment to each individual. Individuals observe their cost and choose a level of effort to exert in the contest. Nature, then, randomly assigns individuals to firms, where they are randomly matched to compete in pair-wise contests. Absent any affirmative action target, the individual in each match with the higher level of effort wins an exogenously determined prize.

We establish three main results. First, we solve explicitly for the equilibrium of a heterogeneous tournament model. In the unique equilibrium, an individual's behavioral strategy involves emitting effort as a function of the distribution of individuals with cost above his. This leads naturally to highly unequal outcomes between social groups endowed with different cost distributions and, hence, a demand for affirmative action. We go on to analyze group-sighted and group-blind affirmative action policies. The distinction between these two forms of redistribution turns on how group information is used in the implementation of affirmative action policies. We show that the unique equilibrium under group-sighted affirmative action involves a constant handicap for agents in the disadvantaged group. This is similar, in particular, to employing a lower minimum standardized test threshold for admission into elite colleges and universities for applicants belonging to disadvantaged groups. Finally, when we impose the group-blind constraint, the nature of equilibrium handicapping changes drastically. Under this formulation, a non-trivial measure of individuals pool on a common low effort level and the prize is randomly distributed to members of the pool whenever they are matched against one another.

This paper lies at the intersection of two literatures: tournament theory (Lazear and Rosen 1981, Green and Stokey 1983, Rosen 1986) and income redistribution (Mirrlees 1971, Akerlof 1978). Our approach has little in common with the existing models in the tournament literature. These models were developed to investigate the economic efficiency of tournament play and to analyze tournaments as optimum labor contracts. In contrast, we focus on the implications of redistribution in tournament-like environments.

The paper is related to the well studied problem of income redistribution. In the traditional optimal tax literature, individuals essentially pool their incomes and a central authority redistributes the pool back to individuals in an incentive compatible manner. This approach is quite different from our proposed framework. Affirmative action in winner-take-all contests imposes an additional restriction on redistribution by constraining the employer to redistribute utility (i.e. the probability of winning) using functionally irrelevant group identifiers in each match, but allowing employers to reach their redistribution goals by aggregating outcomes across all of their matches. In other words, when making a particular hiring decision, an employer must choose between a given slate of candidates, but she evaluates those candidates with an eye toward achieving sufficient diversity across all of her hiring decisions.

The exposition proceeds as follows: section 2 presents and solves our pair-wise tournament model; section 3 analyzes group-sighted and group-blind affirmative action in our tournament setup; and section 4 concludes.

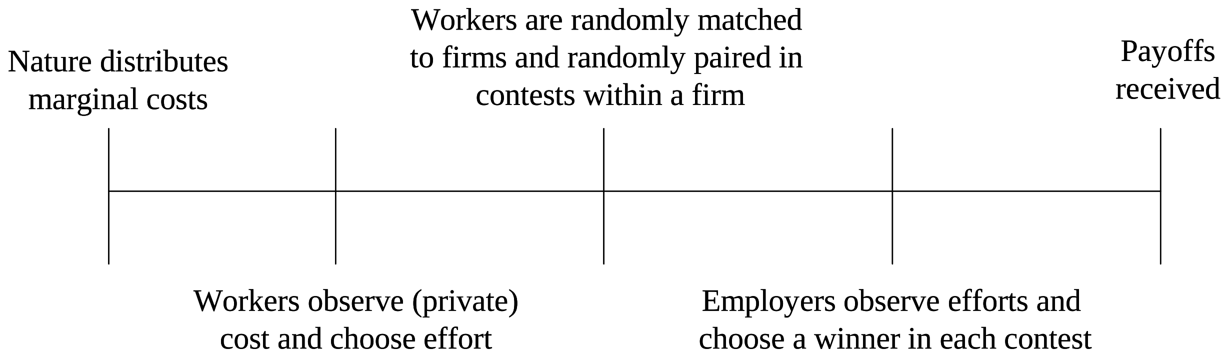

Figure 1: Sequence of Actions

2 A Model of Pair-wise Tournament Competition

Consider a simple model of tournament competition. There is a continuum of workers with unit measure, and a large but finite number, N, of identical employers. Nature moves first and distributes a constant marginal cost of effort to each worker. This cost is distributed according to an atomless cumulative distribution function $F(c)$, where $c \in [c_{\min}, c^{\max}]$, $c_{\min} \equiv \inf\{c \mid F(c) > 0\} \geq 0$, and $c^{\max} \equiv \sup\{c \mid F(c) < 1\} \leq \infty$. Let $f : [c_{\min}, c^{\max}] \to \Re_+$ denote the associated probability density function.

After observing their cost, each worker chooses an effort level e. Each employer is then randomly matched with a continuum of workers of measure $\frac{1}{N}$, who in turn are randomly matched with one another to compete in a continuum of pair-wise contests for a measure $\frac{1}{2N}$ of positions. Thus, each employer faces a pool of workers that is a statistical replica of the overall worker population. Moreover, employers and workers anticipate that each worker will be paired for competition against an opponent drawn randomly from the overall population. We assume that each employer takes the workers' effort distribution as given, independent of her hiring policy, and chooses winners across these pair-wise contests so as to maximize the expected effort of those hired.[1] The worker hired from each pair receives an exogenously given wage ω, while the one not hired receives zero.

A strategy for workers is a function, $e : [c_{\min}, c^{\max}] \to \Re_+$, that maps their costs into an

[1] This, of course, means that employers simply hire the contestant with the greater effort, unless otherwise constrained. Section 3 discusses employer behavior under a group representation constraint.

effort decision. A strategy for an employer is an assignment function in each pair-wise match, $A : \Re^2_+ \to [0,1]$, that maps the effort levels she observes in each contest to a probability of winning for each worker in that contest. A worker presenting effort e wins against a worker presenting effort $\widehat{e}$ with a probability $A(e, \widehat{e})$. Thus, the payoff to a worker if he wins the contest is $\omega - ce$, while the payoff is $-ce$ if he loses.

A. Equilibrium

An equilibrium consists of functions $e^*(c)$ and $A^*(e, \widehat{e})$ such that each is a best response to the other.

Proposition 1 *The equilibrium consists of an assignment function satisfying*

$$A^*(e, \widehat{e}) = \begin{cases} 1 \text{ } if \text{ } e > \widehat{e} \\ 0 \text{ } if \text{ } e < \widehat{e} \end{cases}$$

and an effort supply function

$$e^*(c) = \omega \int_c^{c^{\max}} \frac{dF(y)}{y}, \text{ } for \text{ } all \text{ } c \in [c_{\min}, c^{\max}] \tag{1}$$

Proposition 1 provides an equilibrium behavioral strategy for workers that holds for general cost distributions, and (trivially) an equilibrium assignment function for employers. In essence, the equilibrium effort supply for any worker depends (in the manner illustrated in equation 1) on the distribution of workers who have cost higher than his. Naturally, employers hire the worker in each match with the higher effort.

Let $G(e)$ denote the population cumulative distribution of effort in equilibrium, and let $v(c)$ denote the equilibrium net benefit to an agent with cost c. It follows directly that this equilibrium net benefit is given by

$$v(c) = \omega \int_c^{c^{\max}} \left[1 - \frac{c}{y}\right] dF(y); \tag{2}$$

and the equilibrium effort distribution satisfies

$$G(e) = 1 - F\left(e^{*^{-1}}(e)\right). \tag{3}$$

Notice that, taken together, equations (1) and (3) define a mapping from the exogenous cumulative distribution of costs, $F(c)$, to the cumulative distribution of effort in equilibrium, $G(e)$, in the natural way, which provides an explicit solution to the model.

To establish the proposition notice that, because workers choose effort to maximize their net benefit, we have the first-order condition: $\omega \frac{d}{de} G(e)_{e=e^*(c)} = c$. A simple revealed preference argument establishes that $e^*(c)$ is non-increasing in c. Hence, equilibrium behavior implies that $G(e^*(c)) \equiv [1 - F(c)]$. Putting these equations together, we have:

$$c = \omega \frac{d}{de}[G(e)]_{e=e^*(c)} = \omega \frac{d}{dc}[1 - F(c)] \cdot \frac{dc}{de} \mid_{\{(e,c)|e=e^*(c)\}} .$$

Therefore, $e^*(c)$ satisfies the differential equation $\frac{de^*}{dc} = -\left(\frac{\omega}{c}\right)\left(\frac{dF(c)}{dc}\right)$. Integrating this yields:

$$e^*(c) - e^*(c^{\max}) = \omega \int_c^{c^{\max}} \frac{dF(y)}{y}.$$

Finally, $e^*(c^{\max}) = 0$ since a worker with cost $c = c^{\max}$ loses with probability one. This establishes the result.

B. Introducing Groups

Suppose now that employers can divide the population of workers into two identifiable groups.[2] Let $i \in \{1, 2\}$ index a worker's group, and let $\pi_i > 0$ denote the fraction of the worker population belonging to group i, where $\pi_1 + \pi_2 = 1$. Hereafter, we use the subscript, i, to indicate group identity. Thus, $e_i(c)$ denotes the effort exerted by a member of group i with cost c, F_i denotes the cumulative distribution function of cost for group i workers, and G_i denotes the cumulative distribution of effort (in equilibrium) among group i workers. We will assume throughout that the cost distributions for the two groups have a common support.

Given our random matching assumptions, π_i is also the probability that any worker, regardless of his group, is paired to compete with a worker from group i. Since workers choose their effort prior to being paired against an opponent, and because they are assigned randomly to firms and then paired with each other at random for competition within firms, every worker faces the same

[2] These groups can be defined in terms of race, gender, social class, so on and so forth.

distribution of opponents (i.e., a statistical replica of the overall worker population.) Thus, despite any asymmetry between groups that arises when $F_1(c) \neq F_2(c)$, the workers paired to compete with one another are playing a symmetric game. So, absent any group-based policy intervention, the equilibrium behavior of firms and of workers (whatever their group) will be as described in Proposition 1, with $F(c) \equiv \pi_1 F_1(c) + \pi_2 F_2(c)$.

Now, with two distinct groups, three types of matches are possible: a group 2 worker can be matched with a group 2 worker, which occurs with probability π_2^2; a group 1 worker can be matched with a group 1 worker, which occurs with probability π_1^2; and a mixed match (between group 1 and group 2 workers) occurs with probability $2\pi_1\pi_2$. We will assume that group 1 is "advantaged" relative to group 2, in that the group 1 cost distribution monotonically first-order stochastically dominates that of group 2.

Definition 1 *Group* 1 *is said to be advantaged relative to group* 2, *if* $\frac{f_1(c)}{f_2(c)}$ *is a strictly decreasing function on* $(c_{\min}, c^{\max})$.

This definition implies $F_1(c) > F_2(c)$ for all $c \in (c_{\min}, c^{\max})$. Let γ_0 denote the probability that a group 1 worker wins when matched with a group 2 worker, in the laissez-faire equilibrium described in Proposition 1. We refer to γ_0 as the "natural win rate" of group 1 agents over group 2 agents. Given Definition 1, it is intuitively obvious and straightforward to show that $\gamma_0 \equiv \int_{c_{\min}}^{c^{\max}} dF_1(c)\,[1 - F_2(c)] > \frac{1}{2}$.[3]

Because of the asymmetries in the cost distributions there will be a short-fall in the share of group 2 agents hired by the employer, relative to their share of the worker population. Specifically, the proportion of contests won by group 2 workers is

$$\pi_2^2 + 2\pi_2(1 - \pi_2)(1 - \gamma_0) < \pi_2,$$

[3] To make this transparent, notice that:

$$\int_{c_{\min}}^{c^{\max}} dF_1(c)\,[1 - F_2(c)] = \int_{c_{\min}}^{c^{\max}} dF_1(c)\,[1 - F_1(c)] + \int_{c_{\min}}^{c^{\max}} dF_1(c)\,[F_1(c) - F_2(c)] \tag{4}$$

and

$$\int_{c_{\min}}^{c^{\max}} dF_1(c)\,[1 - F_1(c)] = -\frac{1}{2}\int_{c_{\min}}^{c^{\max}} \frac{d}{dc}\{[1 - F_1(c)]^2\} = \frac{1}{2}. \tag{5}$$

Further, under the assumption that group 2 is disadvantaged, $\int_{c_{\min}}^{c^{\max}} dF_1(c)\,[F_1(c) - F_2(c)]$ is necessarily a positive number. Thus, we have the desired inequality.

in view of the fact that $\gamma_0 > \frac{1}{2}$. Thus, a demand for affirmative action could naturally arise here. This is the subject of the next section.

3 Affirmative Action

Affirmative action involves equilibrium handicapping of workers in the disadvantaged group by employers who take the distribution of worker effort as exogenous when setting the handicap.[4] For instance, a set of universities designing their admissions policies to ensure sufficient diversity, each of which thinks its policies are unlikely to affect the effort distribution in the pool of college bound seniors from which its applicants are drawn, is a case of affirmative action.

We will also employ the distinction between *blind* and *sighted* affirmative action. Blind versus sighted refers to what markers employers are allowed to use in the pursuit of their affirmative action policies. Group-sighted handicapping allows employers to use group identification directly in achieving their redistributive target. Group-blind handicapping forbids the use of group information in achieving diversity goals.

A. Group-Sighted Affirmative Action

Suppose a regulator wants to decrease the win rate of the advantaged group, and let $\gamma \in [\frac{1}{2}, \gamma_0)$ denote the target level of diversity.[5] Notice that, as $\gamma \to \gamma_0$, the laissez-faire equilibrium described in Proposition 1 obtains, while as $\gamma \to \frac{1}{2}$, the employer is forced to achieve group parity.

It is straightforward to show that if an employer desires to maximize the expected effort of the contestants, subject to the constraint that agents from the advantaged group only win at the rate $\gamma \in \left[\frac{1}{2}, \gamma_0\right)$ when matched with an agent from the disadvantaged group, then the best way to do so is to give group 2 workers a constant effort handicap, $\lambda^* (\gamma)$. In particular, the employers' optimization problem implies the maximization of a Lagrangian form as follows:

$$\max_A \left\{ \int_0^\infty \int_0^\infty \left[A(e_1, e_2) e_1 + (1 - A(e_1, e_2)) e_2 + \lambda (\gamma - A(e_1, e_2)) \right] dG_1(e_1) dG_2(e_2) \right\} \quad (6)$$

[4] For concreteness, one can think about affirmative action in this context as coming about because all employers are required by some external authority to increase their hiring rate for workers in the disadvantaged category, though each employer acts independently of the others to achieve this goal.

[5] That is, γ denotes the target win rate of group 1 workers when matched against opponents from group 2. Notice that, under sighted affirmative action, these are the only matches that a regulator would seek to influence.

Obviously, the solution must takes the form:

$$A(e_1, e_2) = \begin{cases} 1 \text{ if } e_1^*(c) > e_2^*(c) + \lambda^*(\gamma) \\ 0 \text{ if } e_1^*(c) < e_2^*(c) + \lambda^*(\gamma) \end{cases}, \tag{7}$$

where $\lambda^*(\gamma)$ is the Lagrangian multiplier on the redistribution constraint. (In effect, $\lambda^*(\gamma)$ is the "shadow price of diversity" when the redistribution target is γ.) Thus, the equilibrium handicap is independent of the effort levels of the contestants, and varies positively with the aggressiveness of the diversity goal.

It follows that, if $e_i^*(c)$ denotes the equilibrium effort supply of workers in group $i \in \{1, 2\}$, then when an agent from group 1 is matched with an agent from group 2, the agent in group 1 wins the contests if

$$e_1^*(c) > e_2^*(c) + \lambda^*(\gamma).$$

Notice that a group 1 worker who exerts low effort, $e_1 \in (0, \lambda^*)$, must lose if matched with any group 2 worker.

We will now derive the equilibrium in this model under group-sighted affirmative action with target $\gamma \in [\frac{1}{2}, \gamma_0)$. Suppose there is an effort cost threshold, $c^*(\gamma)$, such that group 1 agents with cost $c_1 \geq c^*(\gamma)$ lose when matched with any group 2 agents, and group 1 agents with cost $c_1 < c^*(\gamma)$, win when matched with any agent with higher cost.[6] Then, by the affirmative action constraint:

$$c^*(\gamma) \text{ solves } \int_{c_{\min}}^{c^*} dF_1(c)\,[1 - F_2(c)] = \gamma. \tag{8}$$

Notice, $c^*(\gamma)$ is increasing in γ, and $c^*(\gamma)$ tends toward $c^{\max}$, as γ tends toward γ_0; the natural win rate of group1 agents over group 2's.

To solve the model for any desired level of affirmative action, $\gamma \in \left[\frac{1}{2}, \gamma_0\right)$, we must solve for the equilibrium affirmative action handicap, λ, and the associated equilibrium effort levels. This is the subject of our next result.

Proposition 2 *Given* $\gamma \in \left[\frac{1}{2}, \gamma_0\right)$ *and* $c^*(\gamma)$ *defined in equation* (8), *the equilibrium group-sighted*

[6] We shall show momentarily that in the presence of constant effort handicapping for category 2 agents the equilibrium effort supply functions imply this property.

affirmative action handicap is given by

$$\lambda^*(\gamma) = \omega\left[\pi_2 \int_{c^*(\gamma)}^{c^{\max}} \left[\frac{1}{c^*(\gamma)} - \frac{1}{y}\right] dF_2(y) + \pi_1 \int_{c^*(\gamma)}^{c^{\max}} \frac{dF_1(y)}{y}\right], \tag{9}$$

the associated effort levels are given by

$$e_1^*(c) = e_2^*(c) + \lambda^*(\gamma) = \omega\left[\pi_1 \int_c^{c^{\max}} \frac{dF_1(y)}{y} + \pi_2\left[\int_c^{c^*(\gamma)} \frac{dF_2(y)}{y} + \int_{c^*(\gamma)}^{c^{\max}} \frac{dF_2(y)}{c^*(\gamma)}\right]\right],$$

for all $c \in [c_{\min}, c^*(\gamma))$, *and*

$$e_1^*(c) = \omega\pi_1 \int_c^{c^{\max}} \frac{dF_1(y)}{y}; \qquad e_2^*(c) = \omega\pi_2 \int_c^{c^{\max}} \frac{dF_2(y)}{y}$$

for all $c \in [c^*(\gamma), c^{\max}]$.

Proposition 2 provides a solution to the group-sighted affirmative action handicapping problem. The result depends critically on three factors: (1) constant marginal cost of effort; (2) setting of handicaps by many independent employers facing affirmative action regulation; and (3) random matching. The solution implies a partition of agents into four classes: {group 1 or 2}×{high cost ($c \geq c^*(\gamma)$) or low cost ($c < c^*(\gamma)$)}. For convenience of exposition, let H_i (resp. L_i) denote the set of high (resp. low) cost types of group $i \in \{1, 2\}$. The equilibrium behavior of the agents under group-sighted affirmative action handicapping can be summarized in the following concise manner. $H_i's$ only compete at the margin against other $H_i's$ in their same group, and lose to $L_i's$ in either group. Further, $L_i's$ compete at the margin against anyone with whom they are matched, prevailing if and only if they encounter a contestant with higher cost. However, $L_2's$ receive the effort subsidy $\lambda^*(\gamma)$, such that $e_1^*(c) = e_2^*(c) + \lambda^*(\gamma)$ for all $c \in [c_{\min}, c^*(\gamma))$. Figure 2 provides a graphical illustration of Proposition 2.

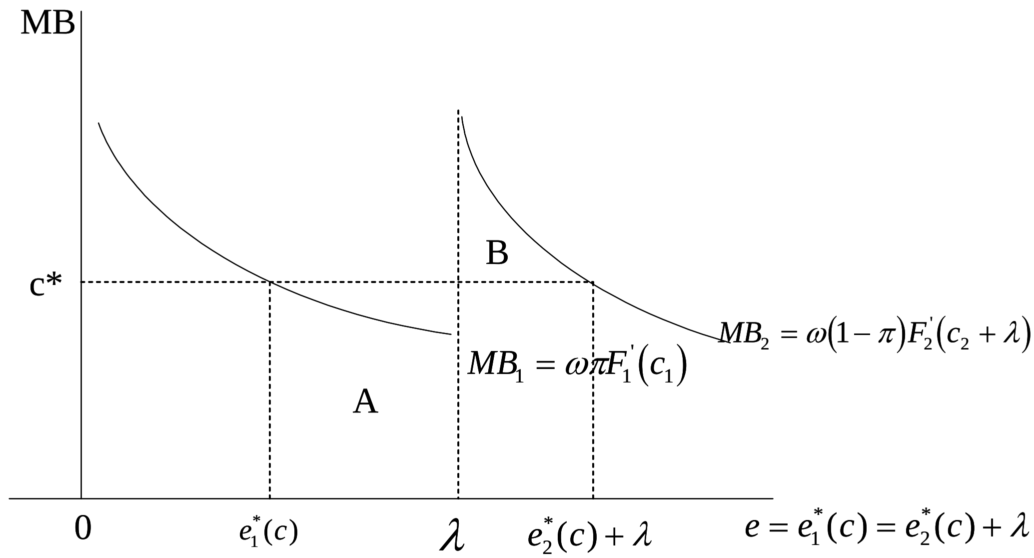

Figure 2: Group-Sighted Affirmative Action Equilibrium

To establish the proposition, let $e = e_1 = e_2 + \lambda$ denote effective effort supply of agents, when group 2 workers receive the effort handicap λ. Given that, in equilibrium, $e_i \geq 0$ and $e_i\left(c^{\max}\right) = 0, i = 1, 2$, there exist a set of high costs agents in both groups (i.e., those with c "close to" $c^{\max}$) who supply relatively low effort. For group 1 agents this implies $e = e_1 \approx 0$, and for group 2 agents this implies that $e = e_2 + \lambda \approx \lambda > 0$. Thus, group 1 agents will face a non-convex decision problem, in equilibrium, since at very low effort levels ($e_1 < \lambda$) they compete only against other high cost group 1 agents. At levels $e_1 > \lambda$, they discontinuously encounter additional benefits from marginal increases in effort, due to the presence of some high cost group 2 agents who are choosing $e_2 \approx 0$, so $e_2 + \lambda \approx \lambda$. Figure 2 captures this intuition. Because group 1's marginal benefit curve jumps upward at λ, it must be the case that the optimal effort function $e_1^*(c)$ is discontinuous. At cost $c = c^*$ the area (in Figure 2) $A \equiv \left[\lambda - e_1\left(c^*\right)\right] \cdot c^*$ is exactly equal (by construction) to the area $B \equiv \int_{\lambda}^{e_2(c^*)+\lambda}\left[MB_2(e) - c^*\right] de$. Therefore, $H_1's$ prefer $e_1^*(c) < \lambda$, along the MB_1 curve, whereas $L_1's$ prefer to "jump" past λ to some $e_1^*(c) > e_2\left(c^*\right) + \lambda$. In other words, all $H_1's$ choose to lose to all group 2 agents, should they end-up paired with one, and compete at the margin only against other $H_1's$; and all $L_i's$ lose only if they are matched with a worker who has lower cost. When λ has been set such that the constraint that workers in group 1 win against workers in group 2 with probability $\gamma \in \left[\frac{1}{2}, \gamma_0\right)$ holds, then it is straightforward to verify that we must have $c^* = c^*(\gamma)$

defined in equation (8).

Using equation (1) and noting that $H_i's$ only compete at the margin against other $H_i's$ in the same group, we deduce that the marginal benefit for group 1 and group 2 agents can be expressed as:

$$
\begin{aligned}
MB_1(e) &= c \text{ if and only if } e = \omega\pi_1 \int_c^{c^{\max}} \frac{dF_1(y)}{y} \equiv e_1^*(c) \\
MB_2(e) &= c \text{ if and only if } e = \omega\pi_2 \int_c^{c^{\max}} \frac{dF_2(y)}{y} \equiv e_2^*(c)
\end{aligned}
$$

Recall, however, $H_1's$ always have the option of boosting their effort to compete with group 2 workers. This is a break-even proposition when $c = c^*$ and strictly pays if $c < c^*$. Hence, for all $c < c^*$, the effort supply function looks much like that in Proposition 1. Both groups supply the same effective effort, given their cost, and $e^*(c)$ solves the differential equation:

$$
\frac{de^*}{dc} = -\frac{\omega}{c}\frac{dF}{dc} = -\frac{\omega}{c}\left[\pi_1 F_1'(c) + \pi_2 F_2'(c)\right], \tag{10}
$$

as in Proposition 1, but with the boundary condition: $e^*(c^*) = e_2^*(c^*) + \lambda$. Finally, λ must satisfy

$$
c^*\left[\lambda - e_1^*(c^*)\right] = v_2(c^*)
$$

where, using equation (2), $v_2(c) = \omega\pi_2 \int_c^{c^{\max}} \left[1 - \frac{c}{y}\right] dF_2(y)$. By integrating equation (10), using the relevant boundary conditions and definitions derived thus far, the conditions of the Proposition can be easily verified by algebraic manipulation.

B. Group-Blind Affirmative Action

The employer's problem is more complicated when she is not allowed to use group information in the pursuit of her affirmative action goals. Employers observe two effort levels in each match, e and $\widehat{e}$, but under the blindness assumption they do not know the group identity of the workers. Accordingly, to achieve their diversity objectives, employers need to estimate the likelihood that each effort level was emitted from a worker in the disadvantaged group. Thus, we overlay a signalling model on top of our pairwise competitive framework. This is a point worth further emphasis. Under group-sighted affirmative action an employer is allowed to narrowly tailor her policies for agents in disadvantaged groups in order to achieve her diversity goal – focusing exclusively on handicapping in mixed contests. When constrained to be group-blind, however, she has to implement her policy across all contests. This is especially inefficient when the disadvantaged group is a small share of the population.

Let $\xi(e)$ denote an employer's belief about the probability that a worker with observed effort level e is from group 2. Then the employer's problem can be written as:

$$\max_{A}\left\{\int_{0}^{\infty}\int_{0}^{\infty}\{A(e,\widehat{e})\,e+(1-A(e,\widehat{e}))\,\widehat{e}+\phi\,[\gamma-A(e,\widehat{e})\,\xi(e)-(1-A(e,\widehat{e}))\xi(\widehat{e})]\}dG(e)dG(\widehat{e})\right\} \tag{11}$$

The solution takes on the form:

$$A(e,\widehat{e})=\begin{cases}1 \text{ if } e+\phi\xi(e)>\widehat{e}+\phi\xi(\widehat{e})\\ 0 \text{ if } e+\phi\xi(e)<\widehat{e}+\phi\xi(\widehat{e})\end{cases}, \tag{12}$$

where ϕ is a Lagrangian multiplier associated with the affirmative action constraint – the shadow price on group 2 membership. Let $V(e)=e+\phi\xi(e)$ denote the value to an employer of a worker with observed effort e. In any pair-wise contest, the employer hires the worker with the higher value. If two matched workers have equal value, the employer chooses either with probability 1/2. This value is comprised of two parts: the direct benefit to the employer of effort, and the expected benefit of diversity, which equals the product of the likelihood of an individual with effort e belonging to the disadvantaged group, times the shadow price of diversity.

To characterize equilibrium with group-blind affirmative action, it is helpful to think about the qualitative properties of $V(e)$. As in the model under Laissez-faire with no diversity mandate, a revealed preference argument establishes that the equilibrium effort supply of workers, $e^*(c)$, must be non-increasing. If this function is strictly decreasing on $[c_{\min}, c^{\max}]$ then we have a separating equilibrium: employers, observing a worker's effort, can invert the equilibrium effort supply function to learn the worker's cost, and to infer the likelihood that the worker belongs to group 2. If $e^*(c)$ is constant on some range of costs, then we have pooling in equilibrium. Now, let E denote the set of efforts reached by some worker in equilibrium: $E=\{e=e^*(c)$, for some $c\in[c_{\min}, c^{\max}]\}$. It is obvious that if $e^*(c)$ is an equilibrium effort schedule for workers, then $V(e)$ must be strictly increasing on E. For if there were two levels of effort, $e,\tilde{e}\in E$ with $e<\tilde{e}$ and $V(e)\geq V(\tilde{e})$, then any worker choosing $\tilde{e}$ could gain by reducing his level of effort to e, which lowers his cost incurred without lowering his chances of winning the contest. But, from this it follows that a separating equilibrium cannot obtain here. For, if $e^*(c)$ were strictly monotonic on $(c_{\min}, c^{\max})$, and if $V(e)$ were strictly increasing on E, then a worker would not be hired whenever matched against another worker with lower cost[7], in which case the affirmative action constraint could not be satisfied. We conclude that there must be some pooling in equilibrium. That is, the equilibrium effort supply function, $e^*(c)$, must be constant over some non-empty interval(s) of costs. Moreover, a worker in the pool would be hired (not hired) with probability one when paired with a worker whose effort level is lower (higher) than that of the pool, and would be hired with probability 1/2 when paired with another worker in the pool. The size of the pool will increase with the aggressiveness of the diversity goal.

This situation is captured in Figures 3, which show a pooling equilibrium where all worker types c in the closed interval $[\widehat{c}, \widehat{\widehat{c}}]$ select the common effort level, e_{pool}. Figure 3A depicts a worker's value to the employer as a function of his effort. Figure 3B shows the worker's best response to the employer strategy "hire that worker with the greater value," as a function of the worker's cost.

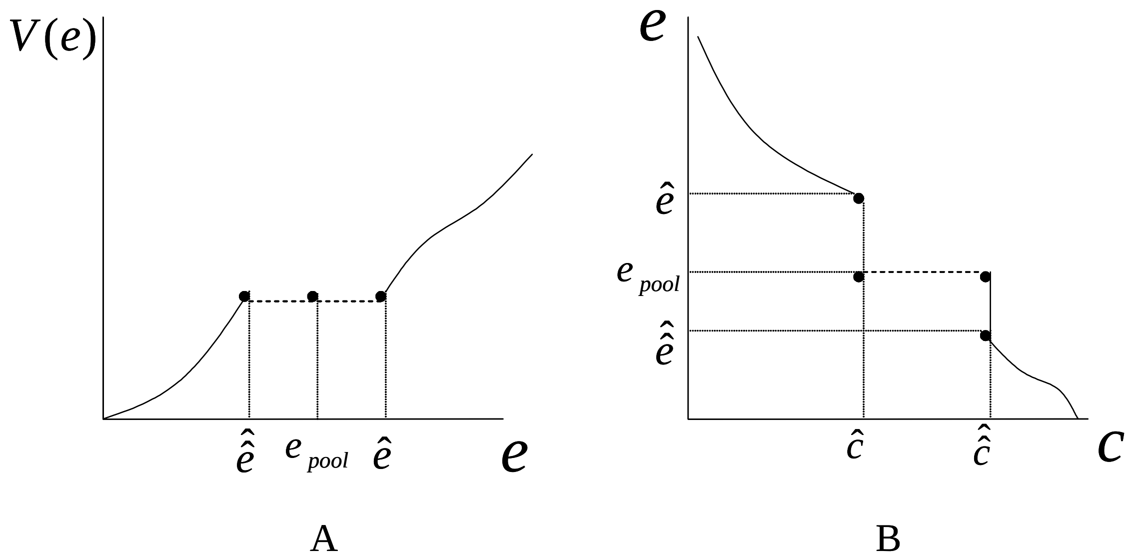

Figure 3: Group-Blind Affirmative Action Equilibrium

[7] Ties will occur with probability zero when $e^*(c)$ is strictly monotonic, since we have assumed the cost distribution to be non-atomic.

In any equilibrium under group-blind affirmative action several conditions need to be satisfied, all of which can be illustrated in Figures 3.[8] First, the worker with marginal cost $\widehat{c}$ must be indifferent between leaving the pool by putting in effort $\widehat{e}$, which would imply that he wins for sure against all workers in the pool, and exerting the effort e_{pool}, which has him winning with probability $\frac{1}{2}$ when matched with anyone in the pool. Similarly, the worker with cost $\widehat{\widehat{c}}$ must be indifferent between staying with the pool, and reducing his effort to $\widehat{\widehat{e}}$. Also, firms must be indifferent between hiring from the pool, and hiring a worker with effort level $\widehat{e}$ (resp., $\widehat{\widehat{e}}$) when such a worker is known to have a cost [and associated probability of belonging to group 2] of $\widehat{c}$ (resp., $\widehat{\widehat{c}}$).[9] Finally, we must specify an employer's beliefs in the event that she were to observe an effort $e \in [\widehat{\widehat{e}}, e_{pool}) \cup (e_{pool}, \widehat{e}]$, which is off the equilibrium path. To support the candidate pooling equilibrium, employers' beliefs must be such that they would strictly prefer a worker in the pool when matched against a hypothetical worker with effort in (the interior of) this region. Here we will appeal to a large literature on equilibrium refinements and select a natural one, the D1 refinement (Cho and Kreps, 1987).

Loosely speaking, the D1 refinement requires out-of-equilibrium actions by informed agents (workers) to be interpreted by uninformed agents (firms) as having been taken by the worker type who would gain most [lose least] from the deviation, relative to his payoff in the candidate equilibrium when this gain [loss] is calculated under the supposition that firms, while adopting this interpretation, will respond to the deviant action in a manner that is best for themselves. (Even more loosely speaking, the D1 refinement requires firms to believe that the deviator is the type who gains most from taking the deviant action [in a set inclusion sense] when he knows that firms will discover his type when he deviates, and then, based on knowing his type, will respond optimally to his deviant action.) Equilibria supported by such out-of-equilibrium beliefs are called

[8] These figures depict a single pooled effort level, whereas in principle there could be many pools in equilibrium. However, we will soon introduce a natural restriction on employers' out-of-equilibrium beliefs that implies the existence of a unique equilibrium in this model, with a single pool consisting of the highest cost worker types exerting the minimal effort level. Accordingly, the exposition proceeds from this point onward under the supposition that there is but one pool in equilibrium.

[9] This indifference condition for firms is required because, were it to fail, then for any plausible off-equilibrium-path beliefs that firms might hold, they would want to respond to some deviation from the pool in such a way as to make that deviation pay for some workers in the pool.

D1 equilibria.[10]

Under D1, if an employer observes an effort $e \in (e_{pool}, \widehat{e}]$, she believes that the deviator is the lowest cost type in the pool, $\widehat{c}$. (This type, compared to others either inside or outside of the pool, gains most [loses least] from such a deviation.) Hence (using Bayes's Rule), employer beliefs must satisfy:

$$\xi(e) = \frac{\pi_2 f_2(\widehat{c})}{f(\widehat{c})} \equiv \widehat{\xi}, \quad \text{for all } e \in (e_{pool}, \widehat{e}].$$

In light of the indifference conditions mentioned above, no workers inside or outside of the pool have an incentive to deviate by choosing e in this interval. On the other hand, if $e_{pool} > 0$, and if an employer observes an effort $e \in [\widehat{\widehat{e}}, e_{pool})$, then under D1 she must believe that the deviator is the highest cost type in the pool, $\widehat{\widehat{c}}$. (This type, compared to all of the others, gains most [loses least] from such a deviation.) Accordingly, under D1 employer beliefs must satisfy:

$$\xi(e) = \frac{\pi_2 f_2(\widehat{\widehat{c}})}{f(\widehat{\widehat{c}})} \equiv \widehat{\widehat{\xi}}, \quad \text{for all } e \in [\widehat{\widehat{e}}, e_{pool}).$$

But now, some workers will have an incentive to deviate. To see this, let ξ_{pool} be the probability that a worker belongs to group 2, conditional on the worker being in the pool. Then, in light of the assumption that group 1 is advantaged, the monotone first-order stochastic dominance property implies:

$$\widehat{\xi} < \xi_{pool} \equiv \frac{\pi_2[F_2(\widehat{c}) - F_2(\widehat{\widehat{c}})]}{[F(\widehat{c}) - F(\widehat{\widehat{c}})]} < \widehat{\widehat{\xi}}.$$

Now, it was an implication of the firm's constrained maximization problem that the value of a worker to the firm is $V(e) = e + \phi\xi(e)$, with the Lagrangian multiplier (the shadow price of diversity) $\phi > 0$. Hence, a worker in the pool can anticipate that his value will increase if he deviates by lowering his effort slightly below e_{pool} (since this marginal reduction induces firms to believe he is strictly more likely than someone drawn from the pool to belong to group 2.) It follows that in all D1 pooling equilibria, e_{pool} must equal zero. But then, since $e^*(c)$ is non-increasing in any equilibrium, we must have that $\widehat{\widehat{c}} = c^{\max}$. Figures 4 replicate Figures 3 after the application of

[10] It has been shown that the only D1 equilibrium to the canonical Spence job signaling model is the Riley (separating) equilibrium (Riley, 1979). This is the unique efficient, separating equilibrium defined by the initial condition wherein the infimal separating ability type adopts its complete information best educational investment level, while all higher types choose the lowest educational levels consistent with separation (which strictly exceed their respective complete information decisions.)

the D1 refinement, showing the unique D1 equilibrium in this model.

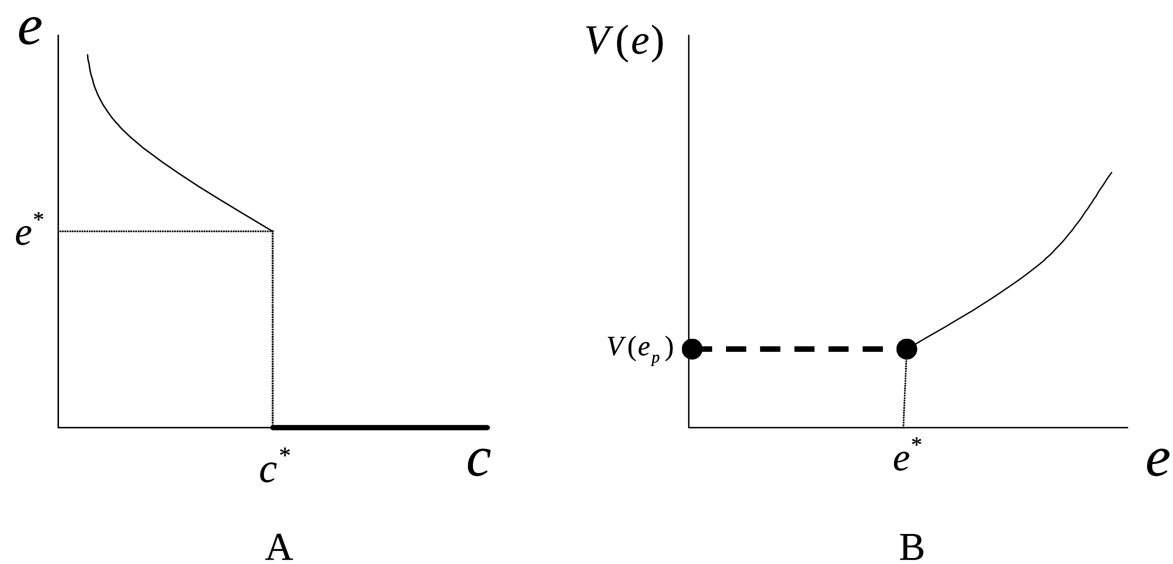

Figure 4: Group-Blind Affirmative Action Equilibrium After Application of D1

We can summarize the discussion to this point as follows:

Proposition 3 *Given* $\gamma \in \left[\frac{1}{2}, \gamma_0\right)$, *the equilibrium group-blind affirmative action handicap, after application of the D1 refinement, is given by*

$$\phi = \frac{\widehat{e}}{\xi_{pool} - \widehat{\xi}}, \quad \text{where } \widehat{e} = \frac{\frac{\omega}{2}\left[1 - F\left(\widehat{c}\right)\right]}{\widehat{c}}; \xi_{pool} = \frac{\pi_2[1 - F_2(\widehat{c})]}{[1 - F(\widehat{c})]}; \ \widehat{\xi} = \frac{\pi_2 f_2(\widehat{c})}{f(\widehat{c})};$$

and where $\widehat{c}$ *is the unique solution for*

$$\int_{c_{\min}}^{\widehat{c}} dF_1\left(c\right)\left[1 - F_2\left(c\right)\right] + \frac{1}{2}\left[1 - F_1\left(\widehat{c}\right)\right] \cdot \left[1 - F_2\left(\widehat{c}\right)\right] = \gamma.$$

The equilibrium effort supply function for both groups is given by

$$e^*\left(c\right) = \widehat{e} + \omega \int_c^{\widehat{c}} \frac{dF\left(y\right)}{y}, \ c \in \left[c_{\min}, \widehat{c}\right),$$

and

$$e^*\left(c\right) = 0, \ \textit{for all } c \in \left[\widehat{c}, c^{\max}\right].$$

Proposition 3 highlights an important feature of group-blind affirmative action; there is a non-trivial measure of workers that supply zero effort. And, this pool increases with the aggressiveness of the diversity goal. In the extreme case where the employer strives for group parity, the only *D1-equilibrium* involves all workers supplying zero effort and the employer picking at random between them![11]

To establish the proposition, consider figures 4. In order to derive the equilibrium, we must

[11] This result has a curious implication that warrants mention. If the cost distributions are identical for the two groups, then each group prevails in half the mixed contests, without any constraint on firm actions. So, the equilibrium effort schedule given in Proposition 1 (which is a positive, strictly decreasing function of effort cost) obtains in this case, automatically generating $\gamma = \frac{1}{2}$. Yet, with only the slightest (strict monotone likelihood ratio) difference in cost distributions favoring group 1, our characterization given above of the unique D1 equilibrium under category-blind redistribution with representation target $\gamma = \frac{1}{2}$ implies zero effort for all agents. This discontinuity of the equilibrium effort supply schedule, as a function of the cost distributions, when population proportionality is the affirmative action target is curious, and it is not an artifact of our having imposed the D1 refinement. For (per the argument just given) in any equilibrium, when firms see an effort level e they (in effect) place some value $V(e)$ on a worker with that effort level, hiring from any pair of workers the one whose value is greater. Moreover, $V(e)$ must be strictly increasing on the set of efforts observed by firms in equilibrium, and $e^*(c)$ must be non-decreasing, otherwise workers could not be best-responding. All of which implies $V(e^*(c))$ must be non-increasing in any equilibrium, which, in light of Definition 1 implies that the target $\gamma = \frac{1}{2}$ can only be met in equilibrium if the set E is a singleton. So, population proportionality as a target together with strict cost distribution differences between the groups, however small, requires a pooling equilibrium with all workers taking the same effort level. Imposing D1 merely forces that pooled effort level to be zero. Hence, the aforementioned discontinuity does not depend on imposing D1.

pin down four parameters: (i) $\widehat{c}$; (ii) $\widehat{e}$; (iii) ϕ; and (iv) $e^*(c)$ for all $c \in [c_{\min}, \widehat{c}]$. Under group-blind affirmative action, the effort incentives at the margin (for both groups) of those not in the pool are identical to the marginal incentives facing workers in the laissez-faire equilibrium. Using Proposition 1, it follows directly that $e^*(c) - \widehat{e} = \omega \int_c^{\widehat{c}} \frac{dF(y)}{y}$ for all $c \in [c_{\min}, \widehat{c})$. Thus, we are left with three equations (the workers' and firms' indifference conditions and the affirmative action constraint,) and three unknowns $(\widehat{c}, \widehat{e}, \phi)$. Consider, first, the affirmative action constraint (which requires that the probability a group 1 worker wins when matched against a group 2 worker just equals γ.) In the *D1-equilibrium* being asserted here, this amounts to:

$$\int_{c_{\min}}^{\widehat{c}} dF_1(c)\left[1 - F_2(c)\right] + \frac{1}{2}\left[1 - F_1(\widehat{c})\right] \cdot \left[1 - F_2(\widehat{c})\right] = \gamma \tag{13}$$

Hence, the cost cut-off $\widehat{c}$ solves equation 13, which pins down (i).

Now, consider the workers' indifference condition. The worker with cost $\widehat{c}$ must be indifferent between exerting effort $\widehat{e}$ and effort 0. If he exerts $\widehat{e}$ he beats all workers in the pool and incurs the cost $\widehat{c}\widehat{e}$; if he invests 0, he ties all workers in the pool – winning with a probability of $\frac{1}{2}$ when matched against any one of them, but paying zero effort costs. The indifference condition implies that

$$\widehat{e} = \frac{\frac{\omega}{2}\left[1 - F(\widehat{c})\right]}{\widehat{c}},$$

which pins down (ii).

Finally, to establish the equilibrium shadow price of diversity, ϕ, consider the firm's indifference condition:

$$\widehat{e} + \phi\widehat{\xi} = \phi\xi_{pool},$$

where ξ_{pool} is the probability of a randomly drawn worker in the pool being disadvantaged. Obviously, this implies,

$$\phi = \frac{\widehat{e}}{\xi_{pool} - \widehat{\xi}},$$

which establishes the desired result.

4 Conclusion

Understanding the theoretical trade-offs of affirmative action in winner-take-all markets will take us a considerable way in better understanding a contentious and oft misunderstood set of policies. This paper opens new directions in the study of affirmative action and tournament theory by deriving the equilibrium handicapping strategy of employers under group-sighted and group-blind constraints.

In winner-take-all markets, group-blind affirmative action is inherently inefficient. Under group-sighted affirmative action an employer is allowed to narrowly tailor her policies for agents in disadvantaged groups in order to achieve her diversity goal – focusing exclusively on handicapping in mixed group contests. Insistence on blindess forces her to implement a policy across all contests. This is especially inefficient when the disadvantaged group is a small share of the population. Indeed, we show that under group-blind affirmative action, there is a nontrivial measure of workers that supply zero effort. And, this pool increases with the aggressiveness of the diversity goal. In the extreme case where the employer strives for group parity, the only D1-equilibrium involves all workers supplying zero effort and the employer picking at random between them.

References

[1] Akerlof, George. 1978. "The Economics of Tagging as Applied to Optimal Income Tax, Welfare Programs, and Manpower Planning." *American Economic Review*, 68: 1, 8-19.

[2] Cho, I-K. and D. Kreps. 1987. "Signaling Games and Stable Equilibria." *Quarterly Journal Economics,* May: 179-221.

[3] Fryer, Roland., and Loury, Glenn. 2003. "Categorical Redistribution in Winner-Take-All Markets." NBER Working paper No. XXXX.

[4] Green, Jerry R. and Stokey, Nancy L., "A Comparison of Tournaments and Contracts," *Journal of Political Economy*, June 1983, 91, 349-65.

[5] Lazear, Edward P. and Rosen, Sherwin, "Rank Order Tournaments as Optimum Labor Contracts," *Journal of Political Economy*, October 1981, 89, 861-64.

[6] Mirrlees, James. A. 1971. "An Exploration in the Theory of Optimum Income Taxation." *Review*

of Economic Studies, vol. 38 (2), 175-208.

[7] Riley, J G. 1979. "Informational Equilibrium," *Econometrica*, 47, 331-360.

[8] Rosen, Sherwin, "Prizes and Incentives in Elimination Tournaments" *American Economic Review*, September 1986, 76, 701-15.