JOURNAL OF URBAN ECONOMICS 5, 147–153 (1978)

The Minimum Border Length Hypothesis Does Not Explain the Shape of Black Ghettos

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Received November 4, 1976; revised May 31, 1977

INTRODUCTION

While attempting to model the process of residential segregation in urban areas, some authors have advanced the hypothesis that that spatial allocation of households will emerge which provides the white majority with the least exposure to blacks [2, 4]. This notion will be referred to as the "minimum border length" hypothesis. Invesigation of this hypothesis has led to the posing of the following problem:

> *Let a circular city of fixed radius be given. Suppose a given population of blacks and whites is to be spread with uniform density over the city, but in such a way that complete segregation obtains. What spatial configuration of the black neighborhood(s) will accommodate the black population while providing a border with the white population of minimal length?*

It is the purpose of this note to show the above query to be one of dubious scientific value. We will investigate its implications and find them at variance with observation. Most areas of black residential concentration are, at least in part, located near the central business district. While some "ghettos" are wholly contained enclaves, others extend all the way to the edge of the city. In a recent paper [5] Yinger studies a somewhat less general version of the question stated above, in an attempt to determine which of these two shapes emerge in a given city. Specifically, he studies the conditions under which locating blacks in a wedge shaped area gives a common border of length less than that arising from locating blacks in a circular area in the city's center. He finds in the case of uniformly distributed population that when the fraction of

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blacks exceeds $1/\pi^2 \approx 0.101$, a wedge shaped area will be preferred by racist whites.

Yinger's analysis, while quite correct as far as it goes, is unfortunately irrelevant and misleading. The minimum border length hypothesis implies neither of the aforementioned empirically relevant shapes. Rather, as is shown below, it is never optimal from this standpoint to locate an undesired minority in the city's center. Moreover, the wedge shaped configuration is "best" only when blacks are exactly one-half the population. The complete solution is seen to involve locating the undesired racial minority at the city's fringe, in a lense-shaped area determined by the intersection of two circles, one representing the city's boundary.

A Fundamental Principle

Actually it is possible to get considerable insight into this problem through application of the following Principle of Duality:

> *A spatial configuration containing a fixed area and yielding the minimal black-white border length must also contain the maximal area over all configurations with that same border length.*

While a formal proof of the above principle may be derived from classical calculus of variations theory [1], it is quite intuitive as it stands. The basic point is that if the border minimizing configuration did not maximize enclosed area, then there would exist an alternative configuration with the same border length enclosing greater area. By perturbing the boundary of this alternative configuration slightly we can reduce its border length and remain with an area at least as large as the original configuration. This would contradict the presumed optimality of the initial configuration.

Consequences of Duality

Armed with the above observation, we can learn a number of preliminary facts about the shape of the border minimizing enclosure. First note that either the black area is bounded by a non-trivial segment of the city's boundary or it is not. If not, then it is well known that the optimal configuration must be circular. Hence we may proceed by investigating the alternative case and comparing our results to those arising for circular enclosures. Henceforth we discuss only those enclosures bordered in part by the city's boundary.

Notice now that *the optimal configuration must be a convex set.* For suppose this were not so. If a presumed optimal non-convex configuration is connected (i.e., consists of one piece) then its convex hull encloses more area with shorter boundary—a contradiction. If it were not con-

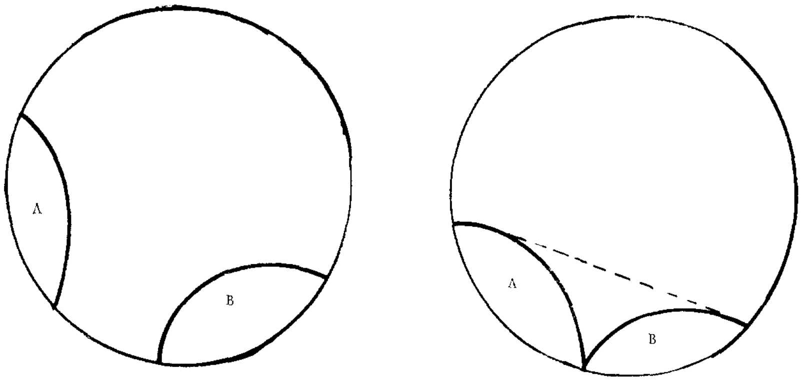

FIGURE 1

nected then the above argument shows that each segment must be convex. Slide two disjoint segments along the city's border until they just touch. Now take the convex hull of the resulting configuration. (See Fig. 1.) Again the resulting set contains greater area with a shorter border, contradicting presumed optimality.

In addition, any straight line supporting the optimal configuration at an interior border point must have an intersection with the city of length at least as great as the length of the interior border of the configuration. (That is $|CD| \geqq |AB|$ in Fig. 2.) The reason is obvious. Moreover, these lengths can be the same only if the boundary and supporting line segment coincide.

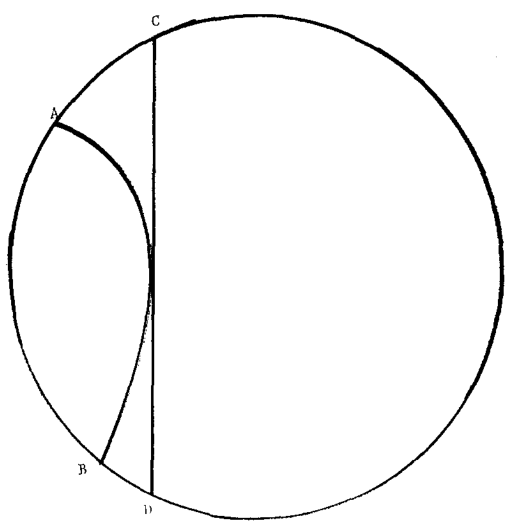

FIGURE 2

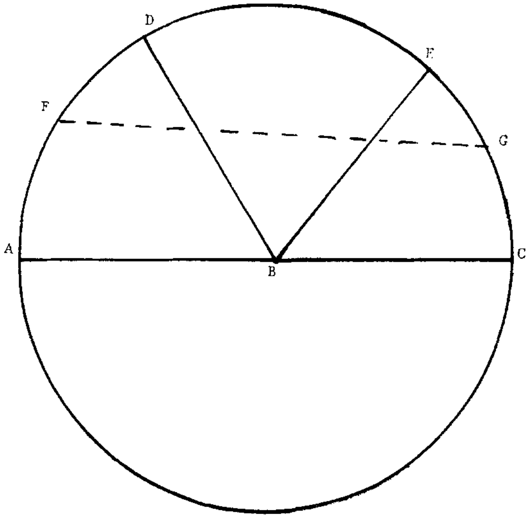

FIGURE 3

It is easily seen that the wedge shaped area studied by Yinger could only be optimal when blacks are to have exactly half the city's area. Consult Fig. 3. The supporting segment ABC has the same length as the interior boundary DBE. The segment FG is constructed so that the area above it is equal to the area of the wedge DBE. It obviously has a shorter border.

The Optimal Configuration

What then is the border minimizing configuration? Figure 4 will assist in answering this question. A and B are arbitrarily chosen points on the city's boundary. The enclosure ACBDA, a convex set, is our candidate for an optimal configuration. Since the enclosure is convex, the dashed line segment $\overline{AB}$ is contained wholly within it. Moreover, since the area above $\overline{AB}$ is given once A and B are, optimality requires that the arc ACB encloses below $\overline{AB}$ an area at least as great as that enclosed by any other arc of the same length. It is well known that this implies ACB must be a circular arc.

Yet there are many circular arcs passing through the points A and B. To see which one is appropriate note the mathematical problem before us, which is to find the function which minimizes an integral representing arc length, subject to an integral constraint specifying the area to be enclosed.[1] Necessary conditions will consist of the standard Euler-

[1] Space constraints prevent the formal treatment of this problem here. For a rigorous discussion see the author's [3].

Lagrange differential equations and, since the end points are free, a set of transversality conditions. The Euler equations will imply that the arc ACB be circular. The transversality conditions interpreted geometrically imply that the arc ACB be transversal to the arc representing the city's boundary at the points of intersection (A and B).

Consider Fig. 5. The angle θ is arbitrary. The line segments $\overline{AB}$ and $\overline{BC}$ are perpendicular to the radii OA and OC respectively, and are therefore equal. Rotate $\overline{AB}$ about B until it coincides with $\overline{BC}$, tracing out the dashed circular arc in the figure. The lense-shaped area with corners at A and C has the minimal border length over all configurations containing the same area. As θ varies from zero to π, the enclosed area in this construction varies from zero to one-half the city's area. The problem is of course symmetric. (A point missed by Yinger.) If blacks are in the majority, then the white minority must locate so that there is minimal border length. This construction represents a complete solution to the initially posed problem.

Proof of Dominance over Interior Circular Configurations

We now show that the optimal boundary using configuration always dominates the interior circular configuration when both are enclosing strictly positive area. To do this it is sufficient (from the duality considerations above) to show that the area enclosed within a border of given length is always greater for boundary using configurations.

Now the area of a circle (A) is proportional to the square of its cir-

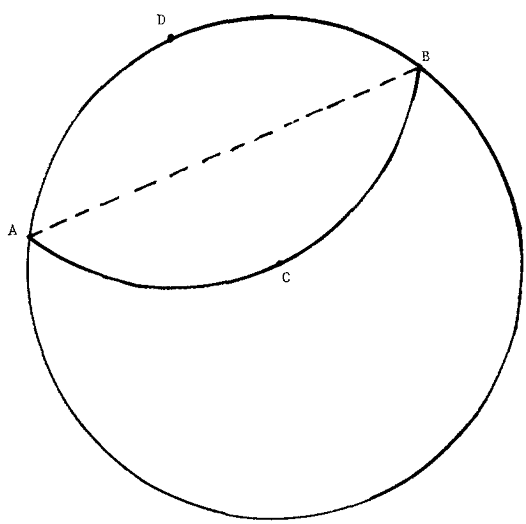

FIGURE 4

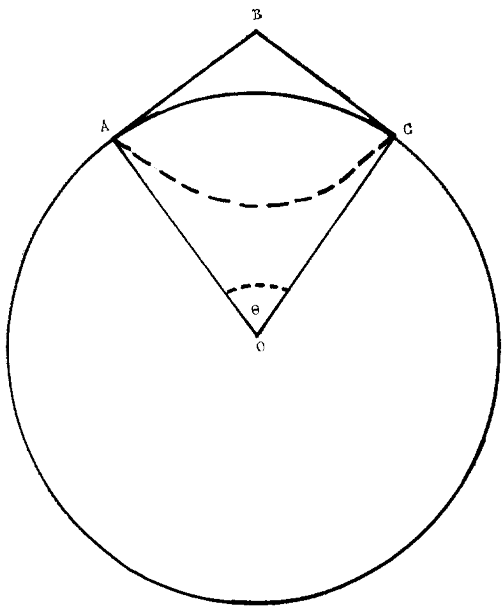

FIGURE 5

cumference (L) satisfying

$$A = L^2/4\pi. \tag{1}$$

Consider now the relationship between area (A') and border length (L') for a boundary using configuration whose interior arc is always a semicircle. The area enclosed by such a configuration is of course always greater than the area of the semi-circle itself. Thus it is readily seen that

$$A' \geqq L'^2/2\pi \tag{2}$$

with equality only if $A' = L' = 0$. Now since a semicircular arc will be optimal only in special cases, it is clear that for given L', the optimal boundary using configuration will have an area even bigger than A'. But when $L = L'$, (1) and (2) imply $A' \geqq 2A$. Thus the optimal boundary using configuration encloses more than twice as much area as the interior circular configuration.

CONCLUSION

We have shown the full implication of the minimum border length hypothesis to be counter to empirical observation. This needn't imply that the hypothesis is without value, as it might still be used to explain the historical evolution of the shapes of certain black ghettos. One rather suspects however that considerable effort will be required to rehabilitate the hypothesis via this route. It is perhaps a bit too crude a notion to successfully explain such a complex social phenomenon.

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