Memorandum

To: EC 237 Students
From: Prof. Loury
March 1, 2006

This note provides a solution to the problem I posed at the end of class on Tuesday, 2/28/06. I asked you to characterize the dynamics of the income distribution in the simple two-job, fixed human capital investment model introduced in that lecture, under the assumption that the cost of becoming a highly skilled worker would be zero if a person's ability was "high," and the would be $K > 0$ if that person's ability was "low." I asked you to assume that $\pi \in (0, 1)$ is the probability a child is born with "high" ability.

I begin by introducing some notation. Parents get utility from consumption, and from the anticipated wage of their child. Parental consumption is what's left over after making payment for a child's human capital (if any.) Let $u(c)+w$ denote parent's utility when parent's consumption is c and child's wage is w. Assume that $u(\cdot)$ is a strictly increasing, strictly concave function. With the training cost for a high skilled job being K, denote by $\phi(w)$ the utility cost to a parent with wage w of training a "low" ability child for a high skilled job. Then

$$\phi(w) = u(w) - u(w - K),$$

and $\phi(\cdot)$ is a strictly decreasing function of w. Assume that, with probability $\pi \in (0, 1)$ a child is born with "high" ability, which means that it is costless to that child's parent to train the child for skilled work.

Time is measured in terms of generations, $t = 0, 1, 2, ...,$ and x_t represents the fraction of skilled workers in generation t. The fraction of skilled parents in the starting generation ($t = 0$) is denoted $x_0 \in (0, 1)$, and is assumed to have been given by history. Our task is to find the path of the subsequent fractions of skilled workers $\{x_t : t \geq 1\}$ in the competitive equilibrium of the model. Let $w_H(x)$ be the high skilled wage in any generation when the fraction of skilled workers is x, let $w_L(x)$ be the low skilled wage, and define $\Delta w(x) \equiv w_H(x) - w_L(x)$. Then, the skilled parents of generation t have the wage $w_H(x_t)$ and the unskilled parents have the wage $w_L(x_t)$. Notice that since skill acquisition is costly, in a competitive equilibrium where any "low ability" children receive training, we must have that $\Delta w(x) \geq 0$. To avoid a trivial equilibrium (in which only those born with "high" ability become high skilled workers), we need to assume that:

$$\phi(w_H(\pi)) < \Delta w(\pi). \quad (1)$$

The result I wish to prove is that, for $\pi > 0$, no matter how small, the dynamic path of the model converges to a unique steady state. To prove this I will need to assume a stability condition (SC), which can be stated as follows:

$$\left| \frac{d}{dx}[\Delta w(x)] \right| > \frac{d}{dx}[\phi(w_H(x))]. \quad (2)$$

This condition states that the graph of $\{\Delta w(x)\}$ is declining more quickly with increasing x than is the graph of $\{\phi(w_H(x))\}$ rising with increasing x. [This assumption guarantees that we do not have cobweb instability in the difference equation (6) below.]

Now, suppose that in some generation, t, the fraction of skilled workers, $x(t)$, is such that the following condition holds:

$$\phi(w_H(x_t)) < \Delta w(\pi + (1-\pi)x_t) < \phi(w_L(x_t)) \tag{3}$$

Under this condition, then, it would be a rational expectation to anticipate that all high skilled parents with low ability children are training their children for high skilled jobs, but that no low skilled parents with low ability children are doing so. It follows that, in any competitive equilibrium, at any generation t for which condition (3) holds, the fraction of skilled workers in the next generation, $t+1$, must satisfy:

$$x_{t+1} = \pi + (1-\pi)x_t. \tag{4}$$

So, whenever condition (3) above holds, $x_{t+1} > x_t$, and the sequence $\{x : \tau > t\}$ continues to increase until condition (3) fails to hold.

Suppose, next, that (3) above fails, but that the following condition holds:

$$\Delta w(\pi + (1-\pi)x_t) < \phi(w_H(x_t)) \tag{5}$$

Now it is no longer a rational expectation to anticipate that all high skilled parents with low ability children are training their children for high skilled jobs. On the other hand, in light of (1), it would also not be a rational expectation to anticipate that no high skilled parents with low ability children are training their children for high skilled jobs. Hence, under condition (5), in any competitive equilibrium we must have that some high skilled parents are training their "low" ability children, and some are not. This requires that the high skilled parents of generation t be indifferent between training and not training their "low" ability children, which in turn implies that:

$$\Delta w(x_{t+1}) = \phi(w_H(x_t)). \tag{6}$$

Hence, whenever condition (5) holds we have that the sequence $\{x_\tau\colon \tau > t\}$ follows the difference equation (6), which implies a cobweb-type cycle, since $\Delta w(x)$is a decreasing function and $\phi(w_H(x))$ is an increasing function of x. The stability condition (2) implies that this cycle dampens in amplitude until $\{x_\tau\colon \tau > t\}$ approaches the point x^* where:

$$\Delta w(x^*) = \phi(w_H(x^*)) \tag{7}$$

[It may help you to draw a diagram at this point to convince yourself that what I have said so far is true.]

Finally, suppose that condition (3) fails in the following way:

$$\Delta w(\pi + (1-\pi)x_t) > \phi(w_L(x_t)). \tag{8}$$

Now it is no longer a rational expectation to anticipate that none of the low skilled parents with low ability children are training their children for high skilled jobs. On the other hand, it is impossible in any competitive equilibrium for all low skilled parents to train their low ability children for high skilled employment, since then we would have $x_{t+1} = 1$, and the skilled-unskilled wage premium, $\Delta w(x_{t+1})$, would be negative. Hence, under condition (8), in any competitive equilibrium, we must have that some low skilled parents are training their "low" ability children, and some are not. This requires that the low skilled parents of generation t be indifferent between training and not training their "low" ability children, which in turn implies that:

$$\Delta w(x_{t+1}) = \phi(w_L(x_t)). \tag{9}$$

Hence, whenever condition (8) holds we have that $x_{t+1} > x_t$ [indeed, $x_{t+1} > \pi + (1-\pi)x_t$], and the sequence $\{x_\tau \colon \tau > t\}$ continues to rise until condition (8) fails to hold, and either condition (3) or condition (5) is met.

Of course, we have already shown that under condition (3) the sequence $\{x_t\}$ also rises [following the linear difference equation (4)], until condition (5) is met, at which point the cobweb-stable difference equation (6) takes over. Therefore, we can conclude that, no matter where the system starts, under the assumptions stated above the fraction of skilled workers in any competitive equilibrium must eventually approach the point x^* defined in equation (7), as was to be proved. $\square$ GL