Search as a social process: The Lucas model

Lucas (2009) developed a related model in which search is a social process; people learn not by sampling from some abstract set of possibilities but by talking with each other. More specifically, each period, each firm talks (that is, the owner of each firm talks) with a random sample of n other firms, where n is a given constant. When firms i and j talk, i learns how to produce with j's productivity A_j and j learns how to produce with A_i. The firm with the higher of the two productivities will obviously choose to continue producing with its old productivity, but the other firm will move up to the higher productivity.

Average productivity rises each period because some firms move up and no firm moves down. More specifically, the cumulative probability distribution of productivity across firms at each date t, $F_t(A) = Prob\{A_{i,t} \leq A\}$ evolves according to:

$$F_{t+1}(A) = [F_t(A)]^{1+n} = [F_0(A)]^{(1+n)^{t+1}} \tag{1}$$

That is, a randomly chosen firm's productivity at $t+1$ will be the maximum of its productivity at t and those of the n other firms it talks with; so its new productivity will be less than A if and only if these $1+n$ productivities at t were all less than A, which occurs with probability $[F_t(A)]^{1+n}$.

In order for growth to be sustained, the support of the initial distribution of productivities must be unbounded, because no one can ever achieve a productivity that was not available to some firm at time 0. Moreover, in order for the growth rate to remain bounded away from zero, the initial distribution must have enough mass in its upper tail. One such distribution is the Fréchet distribution, which has the form:

$$F_t(A) = \exp\left\{-\lambda_t A^{-1/\theta}\right\} \tag{2}$$

with the two parameters $\theta \in (0,1)$ and $\lambda_t > 0$. Lucas supposes that initial distribution F_0 is Fréchet. Then F_t will always be Fréchet because,[1] according to (1) and (2) for $t = 0$:

$$F_{t+1}(A) = \exp\left\{-\lambda_0 (1+n)^{t+1} A^{-1/\theta}\right\}$$

which is Fréchet with parameters θ and:

$$\lambda_{t+1} = \lambda_0 (1+n)^{t+1}$$

Average productivity at period t is the mean of the Fréchet distribution (2), which is

[1] The assumption that F_0 is Fréchet is not as restrictive as it might seem. For it follows from standard results in extreme value theory (see for example Galambos, 1987) that F_t will converge asymptotically to a Fréchet distribution as long as F_0 has fat enough tails to sustain positive growth.

proportional[2] to $(\lambda_t)^\theta$:

$$A_t = (\lambda_t)^\theta \overline{A} = \left[(1+n)^\theta\right]^t (\lambda_0)^\theta \overline{A} \tag{3}$$

where $\overline{A}$ is a constant. So the growth rate g of average productivity depends positively on the frequency of encounters n and the parameter θ according to:

$$g = (1+n)^\theta .$$

References

Lucas, Robert E. Jr. "Growth and Ideas." *Economica* 76 (2009): 1–19.

Galambos, J. *The Asymptotic Theory of Extreme Order Statistics*. Melbourne, Florida: Robert E. Krieger Publishing Company, 1987.

[2] Using a change of variables from A to $y = \lambda^{-1} A^{1/\theta}$, we can express the mean of the evolving Fréchet distribution as:

$$\begin{aligned} A_t &= \int_0^\infty A F_t'(A)\, dA \\ &= \int_0^\infty A\,(1/\theta)\,\lambda_t A^{-1/\theta-1} \exp\left\{-\lambda_t A^{-1/\theta}\right\} dA \\ &= \int_0^\infty (1/\theta)\, y^{-1} \exp\left(-y^{-1}\right) (dA/dy)^{-1}\, dy \\ &= (\lambda_t)^\theta \int_0^\infty y^{\theta-2} \exp\left(-y^{-1}\right) dy \end{aligned}$$

which verifies (3) with the constant:

$$\overline{A} = \int_0^\infty y^{\theta-2} \exp\left(-y^{-1}\right) dy$$