Money and Bonds in Organized Exchange*

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Abstract

A model of fiat money is constructed in which money is used in equilibrium as the universal medium of exchange despite the presence of a negotiable and perfectly divisible outside bond that dominates money in rate of return. All trade takes place in facilities organized by competitive entrepreneurs. There are setup costs of operating a trade facility, as well as a Baumol-Tobin cost of trading money for bonds.

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1 Introduction

One test of a monetary theory is whether it permits equilibria in which outside money is used as a medium of exchange despite the existence of other assets that dominate it in rate of return. This can be done by putting money into the utility function or by invoking legal restrictions, but the former begs more questions than it answers and the latter is difficult to justify as a general principle. So the real test is whether the theory permits equilibria with rate of return dominance even when money is assigned no special role a priori in preferences or regulations.

The present paper shows that this test is passed by a version of the theory developed by Starr and Stinchcombe (1998, 1999), Howitt and Clower (2000) and Howitt (2005). In this theory all trade takes place in facilities that are organized by competitive merchants who incur an irreducible setup cost with each facility created. The theory is closely related to the familiar cash-in-advance model of money, but it allows for the possibility of bonds being used as a medium of exchange. The main difference from the similar papers by Krishna (2005) and Wallace (2004), which reaches different conclusions, is that I allow for a setup cost to visiting the bond market, a la Baumol-Tobin.

The basic idea is that without this setup cost, shops offering to trade goods for money would be unable to compete against shops offering to trade goods for bonds, because people selling goods for money would bear an interest-opportunity cost of holding their receipts until the beginning of next period (as in any cash-in-advance model) whereas people selling for bonds could avoid this cost because their receipts would bear interest. But when there is a setup cost to visiting the bond market and we are in a situation where all other shops trade goods for money instead of for bonds, the advantage of avoiding the interest-opportunity cost when selling for bonds is counteracted by the disadvantage of receiving payment in a less liquid form than when selling for money. When the rate of interest is low enough this disadvantage outweighs the advantage, and shops that conform to the convention of trading goods for money but not for bonds cannot be driven out by those deviating from the convention.

The model analyzed below is an example of what is commonly called a "directed search" model. Other examples include the models in Moen (1997), Rocheteau and Wright (2005) and Head and Kumar (2005). Unlike these other models, it is assumed here that there is a cost to creating a market; Howitt (2005) shows that this setup cost is necessary for the existence of a monetary equilibrium when entrepreneurs have the alternative of opening shops trading commodities directly for each other. Unlike these other directed-search models it is also assumed below that entrepreneurs undertake all the costly marketing activities, with no residual search needed by households.

Other recent papers that have included a Baumol-Tobin cost include Jovanovic (1982), Romer (1986), Chatterjee and Corbae (1992) and Alvarez, Atkeson and Kehoe (2002). These papers do not however explain why bonds are not used as a medium of exchange.

The outline of the paper is as follows. Section 2 below outlines the basic structure of the model, in which activities take place according to a specific sequence each period. Section 3 analyzes the strategic decisions at each stage under the assumption of an exogenously given function determining each household's continuation value. Section 4 defines a Stationary Monetary Equilibrium, derives the continuation value from the structure of the model, and shows that a Stationary Monetary Equilibrium does not exist when the cost of visiting the bond market is zero but it does exist for low enough rates of interest when there the cost is positive. Section 5 offers some brief concluding remarks.

2 Basics

2.1 Preferences and endowments

Time is discrete, consisting of an infinite sequence of periods. There are three different kinds of tradeable objects, all of them perfectly divisible: (1) a large number n of distinct commodities, none storable from one period to another; (2) money, a perfectly durable object and (3) government bonds, which are perpetuities paying $r > 0$ unit of money each period forever. There are three groups of transactors: households, entrepreneurs and the government.

Each household has a type (i, j), with $i \neq j$. The household has a constant endowment of y units per period of commodity i (is an i-maker) and wants to consume commodity j (is a j-eater). Households live forever and have a utility function:

$$\sum_{t=0}^{\infty} \beta^t \left(\ln(x_t) - \widetilde{\delta}_t \right), \ \beta = \frac{1}{1+r}$$

where x_t is consumption and $\widetilde{\delta}_t$ is a "Baumol-Tobin" cost - a lumpy cost that must be paid each time the household visits the bond market to trade bonds for money:

$$\widetilde{\delta}_t = \left\{ \begin{array}{ll} \delta \geq 0 & \text{if the household visits the bond market at } t \\ 0 & \text{otherwise} \end{array} \right\}$$

There is a continuum of households, with a mass $\frac{1}{n-1}$ of each type.

Each entrepreneur has a type i, where i indexes a commodity. The entrepreneur has the ability to trade i, for either money or bonds. In order to use this ability the entrepreneur must set up a "shop" in which i

can be traded for money (a "money-shop") or bonds (a "bond-shop"). By assumption the shop cannot trade both money and bonds. Each entrepreneur lives for only one period. For each i there are two entrepreneurs each period. An entrepreneur consumes only the commodity he trades, and has a utility function equal to:

$$\pi - s$$

where π is his consumption and s is a setup cost of operating his shop, with $s = \sigma > 0$ if the entrepreneur opens a shop and $s = 0$ otherwise. To make it possible for shops to operate profitably in equilibrium I assume:

$$\sigma < y \tag{1}$$

The government issues money and bonds, and distributes the amount $m^s > 0$ and $b^s > 0$ to each household at the beginning of the first period. It also imposes a tax on all sales from a household to a shop, collected from the shop, at the rate τ per unit of commodity. To avoid having money assume a special role because of legal restrictions, I suppose the tax is payable in either money or bonds. In an effort to keep the outstanding stocks of money and bonds constant the government sets the tax rate at:

$$\tau = rb^s/y \tag{2}$$

I assume the bond issue is small enough that this tax can be paid each period in cash:

$$rb^s < m^s \tag{3}$$

2.2 Trading logistics

The logistical constraints imposed on people are similar to those described in more detail in my earlier paper (Howitt, 2005). People meet anonymously except for the government, and this anonymity rules out private debt. All trade must take place through shops. In order for a shop to compete, it must put up a sign indicating what commodity it deals in, whether it uses money or bonds, its retail price p (either in money per unit of commodity or bonds per unit of commodity) and its wholesale price w (in terms of money or bonds per unit of commodity). The wholesale price w is quoted net of taxes. All posted prices must be strictly positive, a constraint which for notational simplicity I will leave implicit throughout the analysis. Each period, trade proceeds in the following order:

0. Interest is paid on outstanding bonds
1. Entrepreneurs set up their shops, posting their prices

2. The bond market convenes
3. Households visit shops to place their buy and sell orders
4. Entrepreneurs execute these orders
5. Consumption takes place

Stage 1 consists of two meta-stages. In the first one, one of the entrepreneurs for each commodity i decides whether to open a money shop, and if so what prices to post. Since I am only looking for a monetary equilibrium I assume for simplicity that this first entrepreneur does not have the option of opening a bond-shop. In the second meta-stage the other entrepreneur decides whether to set up a bond shop, or a money shop, and at what prices. If an entrepreneur enters in the first meta-stage his prices are known to the other entrepreneurs in the second meta-stage.

In stage 2 the government stands ready to trade bonds for money, at an exchange rate of unity. Since the coupon rate r on the bonds equals the households' rate of time preference, this exchange rate will be consistent with a stationary equilibrium with no inflation. If a household is going to trade money for bonds then it must send a member to the bond market in this stage and must therefore incur the Baumol-Tobin cost δ.

In stage 3, each household can see which shops have entered, trading which objects, and posting which prices. The household sends one member (worker) to sell endowment and one (shopper) to purchase consumption. Each of these members can visit only one shop during the week, and must have in her possession enough of the object she is offering to sell (commodities in the case of the worker and money or bonds in the case of the shopper) to honour all orders that the entrepreneurs will accept in stage 4.

In stage 4, entrepreneurs have received their orders, and are free to refuse any of them, although by this time it is too late for them to avoid paying the psychic setup cost σ. I assume however that they are committed to the prices they have posted.

I restrict attention to stationary monetary equilibria, in which aggregate events repeat themselves each period. In such an equilibrium only one shop enters in each market, in the first meta-stage of stage 1; all open shops are money shops; and all shops post the same pair of prices $(\overline{w}, \overline{p})$; and no open shop refuses all buy orders or all sell orders. The solution concept will be a variant of stationary Markov equilibrium, although each infinitesimal household will take all other agents' choices as being independent of its own decision.[1]

[1] The setup is an example of what is commonly called "directed search." Other examples include Moen (1997), Rocheteau and Wright (2005) and Head and Kumar (2005). Unlike these papers it is assumed here that there is a cost to creating a market; Howitt (2005) shows that this setup cost is necessary for the existence of a monetary equilibrium when entrepreneurs have the alternative of opening shops trading commodities directly for each other. Unlike these other papers it is also assumed below that entrepreneurs undertake all the costly marketing activities, with no residual search needed by households.

3 Equilibrium choices.

3.1 Stage 4: Acceptance rates

In stage 4 each entrepreneur has received buy and sell orders from households, and must decide how much of each order to accept for execution. All accepted orders must be executed at the prices that were posted in stage 1. How the acceptances are distributed across customers and suppliers is a matter of complete indifference to the entrepreneur, who is at the end of his economic life. I assume accordingly that he will ration all buy orders in the same proportion and ration all sell orders in the same proportion.

Consider first an entrepreneur who has opened a money shop posting prices (w, p) and has received orders (s, d). The entrepreneur will choose acceptance rates $(a_s, a_d) \in [0,1]^2$ so as to maximize

$$a_s s - a_d d - \sigma$$

subject to the two material balance constraints:

$$a_s s - a_d d \geq 0$$

and

$$p a_d d - (w + \tau) a_s s \geq 0$$

The quantity $a_s s - a_d d$ is the entrepreneur's consumption, since he will receive in delivery the fraction a_s of the s units of commodity i offered to him by his suppliers (i-makers) but will have to pay out the fraction a_d of the d units ordered by his customers (i-eaters). He must also incur the setup cost σ which was sunk in stage 1. The first constraint says that consumption cannot be negative while the second constraint says that the money paid to suppliers $w a_s s$ plus the money paid to the government $\tau a_s s$ cannot exceed the money received from customers $p a_d d$.

We can assume at this point that $p > w + \tau$, since otherwise the entrepreneur would not have entered in stage 1. Define

$$R = \min \{(w + \tau) s, pd\}$$

If $R = pd$ then sales revenue is a binding constraint on the entrepreneur, which will limit his ability to accept sell orders, whereas if $R = (w + \tau) s$ then the entrepreneur will have no incentive to accept buy orders beyond those needed to pay for the sell orders received (recall that the entrepreneur consumes only the commodity that he trades). Formally, one solution to his decision problem during this stage is:

$$(a_s, a_d) = \left\{ \begin{array}{ll} \left(\frac{R}{(w+\tau)s}, \frac{R}{pd}\right) & \text{if } s > 0 \text{ and } d > 0 \\ (0,0) & \text{otherwise} \end{array} \right\} \tag{4}$$

According to (4) the entrepreneur must receive orders on both sides in order to operate. He will ration customers if they are demanding more than enough to pay for all supplies and will ration suppliers if he cannot afford all their sales orders, but he will never ration both sides if both sides have made orders. The solution is unique in the case where $s > 0$ and $d > 0$ but not otherwise. However, the solution (4) will give the correct signal to households in stage 3, namely that a shop to which nothing has been delivered on one side of the market will not be accepting any deliveries on either side of the market. So this is the one I assume is chosen by all entrepreneurs.

Consider next an entrepreneur who has opened a bond shop posting prices $\left(w^b, p^b\right)$ and has received orders (s, d). These prices commit the entrepreneur to paying w^b bonds for each unit of i accepted in delivery and to charging p^b units of bonds per unit for each buy order executed. This entrepreneur will have to pay taxes in the form of bonds, in which case the per unit rate is:

$$\tau^b = \tau / (1 + r) \tag{5}$$

The rate (5) represents a discount for paying taxes in the form of bonds. However it will be a matter of indifference to the government in which form the tax is paid, since paying in bonds reduces the government's debt service charge.

Other than this the decision problem facing the entrepreneur is the same as that facing an entrepreneur who has opened a money shop, so his acceptance rates will be given by the analogous formula:

$$\left(a_s^b, a_d^b\right) = \left\{ \begin{array}{ll} \left(\frac{R^b}{(w^b+\tau^b)s}, \frac{R^b}{p^b d}\right) & \text{if } s > 0 \text{ and } d > 0 \\ (0,0) & \text{otherwise} \end{array} \right\} \tag{6}$$

where

$$R^b = \min\left\{\left(w^b + \tau^b\right) s, p^b d\right\}.$$

3.2 Stages 2 and 3: Household decisions

Each household of type (i, j) enters stage 2 holding the stocks (m^s, b^s) of money and bonds. At this point I take as given the function $V(m', b')$, continuous and increasing in both arguments, that determines the household's continuation value, where (m', b') are the stocks held at the beginning of next period, including interest receipts. I also take as given the universal pair of money prices $(\overline{w}, \overline{p})$ assumed to prevail at all future dates. The household takes as given the prices being charged by all shops trading i and j. The household is also aware of the orders currently being placed by all others, knows that acceptance rates will be given by the functions (4) and (6), and hence can perfectly anticipate the acceptance rates that will be charged in stage 3 by all shops. Being infinitessimal, the household takes these acceptance rates as given.

In looking for a stationary equilibrium I can restrict my analysis of household decisions to situations in which there is at most one commodity for which the universal pattern (one money shop, posting $(\overline{w}, \overline{p})$ with strictly positive acceptance rates, and no bond shop) does not prevail.

3.2.1 Money shops

Suppose that all commodities other than the household's consumption commodity conform to the universal pattern, and that the household plans to send its shopper to a money shop posting a retail price p, whose acceptance rate on buy orders will be a_d. Its worker will be going to a money shop posting the equilibrium wholesale price $\overline{w}$ with an acceptance rate on sell orders equal to $a_s > 0$. Then the household will choose order quantities $(s, d) \geq 0$ and next-periods stocks $(m', b') \geq 0$ so as to maximize

$$\ln(a_d d) - \widetilde{\delta}(b' - b^s) + \beta V(m', b')$$

subject to:

$$pa_d d + b' \leq m^s + b^s,$$

$$a_s s \leq y$$

and

$$m' + b' = m^s + b^s - pa_d d + \overline{w} a_s s + rb'$$

where the function $\widetilde{\delta}$ is defined as:

$$\widetilde{\delta}(\Delta) = \left\{ \begin{array}{ll} \delta & \text{if } \Delta \neq 0 \\ 0 & \text{if } \Delta = 0 \end{array} \right\}$$

The first constraint says that after the bond market has convened, the shopper must have enough cash on hand to pay for all accepted buy orders she has placed, the second says that the worker must have enough commodity i to make delivery on all accepted sell orders she has placed, and the third says that the household will start next period with an amount of money m' equal to the initial holdings m^s, plus the proceeds of bond sales (or minus cost of bond purchases) $b^s - b'$, minus the cost of commodity purchases $pa_d d$, plus the receipts from commodity sales $\overline{w} a_s s$, plus the interest received on bonds rb'.

If the acceptance rate a_d is also strictly positive, then the household can game the rationing scheme by over-ordering. That is, the worker will offer to sell the amount $s = y/a_s$ such that all of its endowment will be sold, and the shopper will offer to buy the amount $d = x/a_d$, knowing that the amount x will actually be bought, where the choice of x is made by solving the problem:[2]

$$\left\{ \begin{array}{ll} W_d\left(p\right) = & \max\limits_{(x,b')\geq 0} \ \ln(x) - \widetilde{\delta}\left(b' - b^s\right) + \beta V\left(m^s + b^s - b' - px + \overline{w}y + rb', b'\right) \\ & \text{subject to } px + b' \leq m^s + b^s, \end{array} \right\} \tag{7}$$

The solution to this problem is not necessarily unique, because of the discontinuous function $\widetilde{\delta}$ in the objective function. Let $\xi\left(p\right)$ be the solution correspondence:

$$\xi\left(p\right) = \left\{\left(x, b'\right) \geq 0 \,|\, \left(x, b'\right) \text{ solves the problem (7)}\right\}$$

Then x will be chosen from the set:

$$\xi_x\left(p\right) = \left\{x \geq 0 \,|\, \left(x, b'\right) \in \xi\left(p\right) \text{ for some } b' \geq 0\right\} \tag{8}$$

On the other hand, if the acceptance rate a_d is zero, then the household's payoff will be independent of its choice of d, so I assume it will choose $d = 0$.

Suppose now that all commodities other than the household's endowment commodity conform to the equilibrium pattern, and that the household is planning to send its worker to a shop posting the wholesale price w with an acceptance rate a_s. Since in this case the household's consumption good conforms to the equilibrium pattern, the shopper will be going to a money shop posting the price $\overline{p}$ with a strictly positive acceptance rate. If $a_s > 0$, then by the same logic as used above the household will choose to sell all its endowment by having its worker offer $s = y/a_s$ and will choose its consumption x according to the decision problem:

$$\left\{ \begin{array}{ll} W_s\left(w\right) = & \max\limits_{(x,b')\geq 0} \ \ln(x) - \widetilde{\delta}\left(b' - b^s\right) + \beta V\left(m^s + b^s - b' - \overline{p}x + wy + rb', b'\right) \\ & \text{subject to } \overline{p}x + b' \leq m^s + b^s, \end{array} \right\} \tag{9}$$

On the other hand, if $a_s = 0$, then the household's payoff will be independent of its choice of s, so I assume

[2] Existence of a solution to each of the problems (7), (9), (10) and (12) is guaranteed by the assumed continuity of V and the compactness of the constraint set.

it will choose $s = 0$.

Because V is continuous and increasing in both arguments therefore W_d is a continuous, decreasing function of p and W_s is a continuous, increasing function of w.

3.2.2 Bond shops

Suppose again that all commodities other than the household's consumption commodity conform to the equilibrium pattern, and that the household sends its shopper to a bond shop posting a retail price $p^b > 0$, whose acceptance rate on buy orders will be a_d^b. Then the household will choose order quantities $(s, d) \geq 0$ and next-period stocks $(m', b') \geq 0$ to maximize

$$\ln\left(a_d^b d\right) - \widetilde{\delta}\left(b' + p^b a_d^b d - b^s\right) + \beta V\left(m', b'\right)$$

subject to:

$$p^b a_d^b d + b' \leq m^s + b^s,$$

$$a_s s \leq y$$

and

$$m' + b' = m^s + b^s - p^b a_d^b d + \overline{w} a_s s + r b'.$$

Notice that if the household does not go to the bond market then its next period's bond holding b' will be the initial holding b^s minus the amount $p^b a_d^b d$ spent on its consumption good. Accordingly, the number of bonds bought (or minus the amount sold) in the bond market is $b' + p^b a_d^b d - b^s$; hence the second term in the objective function. The first constraint says that the household cannot purchase more than m^s bonds in the bond market. The third constraint says that the amount of money on hand at the start of next period will be the initial holding m^s, plus the proceeds from bond sales (or cost of bond purchases) $b^s - p^b a_d^b d - b'$, plus receipts from commodity sales plus bond interest.

If the acceptance rate a_d^b is strictly positive then the household will choose order quantities $s = y/a_s$ and $d = x/a_d$, where the choice of x is governed by the decision problem:

$$\left\{\begin{array}{ll} W_d^b\left(p^b\right) = & \max\limits_{(x,b')\geq 0} \ln(x) - \widetilde{\delta}\left(b' + p^b x - b^s\right) + \beta V\left(m^s + b^s - b' - p^b x + \overline{w} y + r b', b'\right) \\ & \text{subject to } p^b x + b' \leq m^s + b^s \end{array}\right\} \tag{10}$$

whose solution correspondence is:

$$\xi^b\left(p^b\right) = \{(x,b') \geq 0 \,|\, (x,b') \text{ solves the problem (10)}\}$$

Thus x will be chosen from the set:

$$\xi_x^b\left(p^b\right) = \Big\{x \geq 0 \,|\, (x,b') \in \xi^b\left(p^b\right) \text{ for some } b' \geq 0\Big\}. \tag{11}$$

On the other hand, if $a_d^b = 0$ we may assume the household chooses $d = 0$.

Finally, suppose that all commodities other than the household's endowment commodity conform to the equilibrium pattern, and that the household sends its worker to a bond shop posting the wholesale price $w^b > 0$ and accepting the fraction a_s^b of sell orders. If $a_s^b > 0$ then the household's decision problem is equivalent to:

$$\left\{\begin{array}{ll} W_s^b\left(w^b\right) = & \max\limits_{(x,b')\geq 0} \ln(x) - \widetilde{\delta}\left(b' - w^b y - b^s\right) + \beta V\left(m^s + b^s - b' - \overline{p}x + w^b y + rb', b'\right) \\ & \text{subject to } \overline{p}x + b' - w^b y \leq m^s + b^s \text{ and } b' \geq w_y^b \end{array}\right\} \tag{12}$$

and the worker will place a sell order equal to $s = y/a_s^b$. Note that in this case the household leaves the bond market holding in stage 2 holding the amount $b' - w^b y$ of bonds, so the size of the bond purchase (minus the sale) is $b' - w^b y - b^s$, and in order to rule out private borrowing we must impose the constraint that $b' \geq w^b y$. On the other hand, if $a_s^b = 0$ then I assume the worker places no sales order: $s = 0$.

Because V is increasing in both arguments therefore W_d^b is a continuous decreasing function of p^b and W_s^b is a continuous increasing function of w^b.

3.2.3 Shop Selection

Suppose that both entrepreneurs chose to open shops trading commodity i in stage 1. The zero-activity problem of trading-post models implies that any choice of a common shop to patronize in stage 3 would be an equilibrium of the shop-selection game. This is because when there are two shops then, regardless of their posted prices, if all customers went to the first entrepreneur's shop no one would have an incentive to go to the second shop, since the second shop would have to refuse all deliveries, as indicated by the rationing functions (4) and (6) above. On the other hand if they all went to the second shop then no one would have an incentive to go to the first, for the same reason.

I deal with this problem by supposing that there is a convention of choosing the first money-shop if it has entered, unless the payoffs to i-makers and i-eaters from choosing that shop are strictly dominated by the payoffs from visiting the other shop (if it has entered), where the payoffs are defined by the functions W_d, W_s, W_d^b and W_s^b described above.[3] This way I can be sure that if no bond shop enters then it is not because of the zero activity problem but rather because there is no way that an entrepreneur can profitably open a bond shop that is attractive to the money shop's customers and suppliers.

3.3 Stage 1: Entry and price setting

3.3.1 Money shops

Consider an entrepreneur thinking of opening a money shop for commodity i, posting the money prices (w, p), when all other commodities conform to the equilibrium pattern. Suppose that all i-makers and i-eaters choose to visit this shop. It follows from our discussion in the previous section that the orders received by the shop will depend on its acceptance rates (a_s, a_d) according to:

$$s = \left\{ \begin{array}{ll} y/a_s & \text{if } a_s > 0 \\ 0 & \text{otherwise} \end{array} \right\} \text{ and } d = \left\{ \begin{array}{ll} x/a_d & \text{if } a_d > 0 \\ 0 & \text{otherwise} \end{array} \right\} \tag{13}$$

for some $x \in \xi_x(p)$. But the shop's acceptance rates in turn will depend upon (s, d) according to (4) above.

The only way conditions (4) and (13) can both hold is if there is either no "effective" rationing or 100 percent rationing. That is, it follows directly from (4) that if $(a_s, a_d) \neq (0, 0)$ then $a_s (w + \tau) s = a_d p d$; substituting in this equation for $a_s s$ and $a_d d$ using (13) implies that in order for any orders to be executed in stage 4, the shop's prices (w, p) must satisfy the existence condition:

$$(w + \tau)y = px \text{ for some } x \in \xi_x(p). \tag{14}$$

The result is an instance of the well-known difficulty of establishing an equilibrium with manipulable rationing schemes like the one we are assuming (Bénassy, 2002, pp. 21 ff.). It says that either the shop's prices must give it exactly enough revenue to execute all sell orders or else the only equilibrium will be the degenerate one in which the shop accepts no orders and no orders are forthcoming.

If this condition is satisfied, then one solution to (4) and (13) is $(a_s, a_d) = (1, 1)$, $s = y$ and $d = x$. All other solutions, except for the degenerate one of no orders or acceptances, are payoff-equivalent to this one (they involve rationing of one side only and over-ordering on that side so that accepted orders are identical),

[3] Note that if in fact the convention selected a bond shop then there is no reason to believe that the assumed equilibrium pattern of prices underlying these functions would actually prevail in every period in the future. However this does not mean that households are acting with less than rational expectations when making their selection of shops, just that the self-reinforcing convention by which everyone is collectively making the selection would not be a rational choice function for an isolated household or group of households that was in a position to dictate the selection. In fact I am assuming that no one is in such a position.

so this is the one I assume will be acted out by the transactors. Thus, provided that condition (14) is satisfied the entrepreneur's payoff will depend on his posted prices according to:

$$\Pi(w,p) = y - x - \sigma = \left(\frac{p - w - \tau}{p}\right) y - \sigma. \tag{15}$$

On the other hand, if the shop's prices do not satisfy (14) then the only solution to (4) and (13) will be $s = d = a_s = a_d = 0$, and the entrepreneur's payoff will be $-\sigma < 0$. Therefore the entrepreneur will never choose prices violating (14), since he can achieve a zero payoff by choosing instead not to enter. Nor, for the same reason, will he choose prices that yield a negative profit. Formally, his choice of prices will be restricted to the set of feasible prices, defined as follows:

Definition 1 *The money prices (w,p) are "feasible" if and only if they satisfy (14) and $\Pi(w,p) \geq 0$.*

3.3.2 Bond shops

Next, consider an entrepreneur thinking of opening a bond shop for commodity i, posting prices $\left(w^b, p^b\right)$, when all other commodities conform to the equilibrium pattern. Suppose that all i-makers and i-eaters choose to visit this shop. It follows from our discussion in the previous section that the orders received by this shop will depend on its acceptance rates $\left(a_s^b, a_d^b\right)$ according to:

$$s = \left\{ \begin{array}{ll} y/a_s^b & \text{if } a_s^b > 0 \\ 0 & \text{otherwise} \end{array} \right\} \text{ and } d = \left\{ \begin{array}{ll} x/a_d^b & \text{if } a_d^b > 0 \\ 0 & \text{otherwise} \end{array} \right\} \tag{16}$$

for some $x \in \xi_x^b\left(p^b\right)$. But the shop's acceptance rates in turn will depend upon (s,d) according to (6) above. It then follows from the same reasoning used to establish the existence condition (14) above that the shop will accept all orders if the existence condition:

$$(w^b + \tau^b)y = p^b x \text{ for some } x \in \xi_x^b\left(p^b\right) \tag{17}$$

is satisfied, and otherwise it will accept no orders.

So if (17) is satisfied, then the outcome will be $\left(a_s^b, a_d^b\right) = (1,1)$, $s = y$ and $d = x$, and the entrepreneur's payoff will be:

$$\Pi^b\left(w^b, p^b\right) = y - x - \sigma = \left(\frac{p^b - w^b - \tau^b}{p^b}\right) y - \sigma \tag{18}$$

Again, the entrepreneur will restrict his choice of prices to those that are not dominated by choosing not to enter; that is, prices that satisfy the following definition of feasibility:

Definition 2 *The bond prices* $\left(w^b, p^b\right)$ *are "feasible" if and only if they satisfy (17) and* $\Pi^b\left(w^b, p^b\right) \geq 0$.

3.3.3 Threat of entry

In addition, the first entrepreneur will enter only at prices that are immune to entry. That is, there must be no possibility for the second entrepreneur to break even by opening a shop with prices that dominate those of the first entrepreneur's shop. For if that were the case then the second entrepreneur would do so, resulting in the second entrepreneur's shop being selected and the first entrepreneur getting a payoff of $-\sigma < 0$. Formally, I define "immune to entry" in two parts as follows:

Definition 3 *The prices* (w, p) *are "immune to money entry" if and only if there exists no feasible set of money prices* (w', p') *such that* $w' \geq w$ *and* $p' \leq p$ *with at least one strict inequality.*

Definition 4 *The prices* (w, p) *are "immune to bond entry" if and only if there exists no feasible set of bond prices* $\left(w^b, p^b\right)$ *such that* $W_s^b\left(w^b\right) \geq W_s(w)$ *and* $W_d^b\left(p^b\right) \geq W_d(p)$ *with at least one strict inequality.*

4 Stationary Monetary Equilibrium

The first entrepreneur in each shop, who will be the only one to open in a stationary monetary equilibrium, will post prices that maximize the payoff $\Pi(w, p)$ over the set of feasible money prices that are immune to money and bond entry. A pair of prices satisfying this condition is however not necessarily a stationary monetary equilibrium, because for stationarity we require that it equal the pair $(\overline{w}, \overline{p})$ which we have until now taken as given in deriving the entrepreneur's profit function.

We also require that the excess demand for bonds be zero, since otherwise the asset holdings (m^s, b^s) would be changing over time rather than stationary. Because in general there is no demand function for bonds, just a correspondence, we need that there be a point in the solution correspondence that is consistent with both the existence condition (14) required for feasibility and also with the stationarity condition that $b' = b^s$.

Accordingly we define a Stationary Monetary Equilibrium as follows:

Definition 5 *A Stationary Monetary Equilibrium is a pair of money prices* $(\overline{w}, \overline{p})$ *such that (a)* $(\overline{w}, \overline{p})$ *maximizes* $\Pi(w, p)$ *over the set of feasible money prices* (w, p) *that are immune to both money and bond entry, and (b)* $((\overline{w} + \tau) y/\overline{p}, b^s) \in \xi(\overline{p})$.

Before examining the issue of the existence of a Stationary Monetary Equilibrium we need to characterize the continuation function V to ensure that it is consistent with the specification of the model and to verify that it is indeed continuous and increasing in both arguments as assumed above.

4.1 The continuation function

In a stationary monetary equilibrium, each commodity is traded each period by a single shop, each one posting the same constant prices $(\overline{w}, \overline{p})$ and each accepting all deliveries. The continuation function $V(m, b)$ is the value function of a household starting any period with asset holdings (m, b) not necessarily equal to the per-capita supplies (m^s, b^s), facing the constant prices $(\overline{w}, \overline{p})$ each period from now until forever. Thus V is defined in the same way as W_d was defined in (7) above, except with the restriction that $p = \overline{p}$ and without the restriction that $(m, b) = (m^s, b^s)$:

$$\left\{ \begin{array}{ll} V(m,b) = & \max\limits_{(x,b')\geq 0} \ln(x) - \widetilde{\delta}(b'-b) + \beta V(m+b-b'-\overline{p}x+\overline{w}y+rb', b') \\ & \text{subject to } \overline{p}x + b' \leq m+b \end{array} \right\} \tag{19}$$

It is straightforward to establish that this function is indeed continuous and increasing in both arguments.

A closed form solution can be found in the case where the Baumol-Tobin cost δ equals zero. For then V depends only on the sum of asset holdings $m+b$ and can be expressed as: $V(m,b) = V_0(m+b)$, where:

$$\left\{ \begin{array}{ll} V_0(m+b) = & \max\limits_{(x\geq 0)} \ln(x) + \beta V_0((1+r)(m+b-\overline{p}x) + \overline{w}y) \\ & \text{subject to } \overline{p}x \leq m+b \end{array} \right\} \tag{20}$$

because the fact that V is an increasing function of b implies that the constraint $b' \leq m+b-\overline{p}x$ will always be binding when $\delta = 0$.

Straightforward analysis of the problem defined by (20) implies that its unique solution is: $x = (m+b)/\overline{p}$ if $m+b < \overline{w}y$ and $x = (r(m+b)+\overline{w}y)/((1+r)\overline{p})$ otherwise. Accordingly:

$$V_0(m+b) = \frac{1+r}{r}\ln\left(\frac{r(m+b)+\overline{w}y}{(1+r)\overline{p}}\right) \text{ if } m+b \geq \overline{w}y \tag{21}$$

$$V_0(m+b) = \ln\left(\frac{m+b}{\overline{p}}\right) + \frac{1}{r}\ln\left(\frac{\overline{w}y}{\overline{p}}\right) \quad \text{if } m+b < \overline{w}y \tag{22}$$

In the general case, whether or not $\delta = 0$, we have:

$$V(m,b) \leq V_0(m+b), \tag{23}$$

because adding a cost of trading cannot make the household better off, and:

$$V(m^s, b^s) = W_d(\overline{p}) = W_s(\overline{w})$$

4.2 Non-existence with costless bond trading

Suppose there is no cost to visiting the bond-money market: $\delta = 0$. Then it is easy to show that a Stationary Monetary Equilibrium does not exist, for essentially the same reasons as in the analyses of Krishna (2005) and Wallace (2004).

The function (7) can be re-expressed in terms of the continuation function (20) as:

$$\left\{ \begin{array}{ll} W_{d0}(p) = & \max\limits_{(x \geq 0)} \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - px\right) + \overline{w}y\right) \\ & \text{subject to } px \leq m^s + b^s, \end{array} \right\} \tag{24}$$

and (10) can be re-expressed as:

$$\left\{ \begin{array}{ll} W_{d0}^b(p) = & \max\limits_{(x \geq 0)} \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - p^b x\right) + \overline{w}y\right) \\ & \text{subject to } p^b x \leq m^s + b^s \end{array} \right\} \tag{25}$$

where in both cases I make use of the fact that the constraint in the original formulation of the problem will be binding when $\delta = 0$.

Since V_0 is given by (21) and (22), it is continuous and concave, so it follows immediately from (24) and (25) that:

Result 1 *When the Baumol-Tobin cost δ is zero, there is a unique solution $\left(\widetilde{x}_0(p), \widetilde{b}_0(p)\right)$ to problem (7) and a unique solution $\left(\widetilde{x}_0^b\left(p^b\right), \widetilde{b}_0^b\left(p^b\right)\right)$ to problem (10).*

Moreover, inspection of (24) and (25) reveals that when $p = p^b$, these two problems are identical and have identical solutions:

$$\text{For all } p > 0,\ W_{d0}^b(p) = W_{d0}(p) \ \text{ and } \widetilde{x}_0^b(p) = \widetilde{x}_0(p). \tag{26}$$

This is because when there is no cost to accessing the bond market it makes no difference to the household whether it has to pay for consumption using a certain amount of money or an equal valued amount of bonds.

Likewise I can rewrite (9) as:

$$\left\{ \begin{array}{ll} W_{s0}(w) = & \max\limits_{(x \geq 0)} \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - \overline{p}x\right) + wy\right) \\ & \text{subject to } \overline{p}x \leq m^s + b^s, \end{array} \right\} \tag{27}$$

and rewrite (12) as:[4]

$$\left\{ \begin{array}{ll} W_{s0}^{b}\left(w^{b}\right) = & \max\limits_{(x\geq 0)} \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - \overline{p}x + w^b y\right)\right) \\ & \text{subject to } \overline{p}x \leq m^s + b^s \end{array} \right\} \tag{28}$$

Inspection of (27) and (28) shows that the two problems are identical when $w = (1+r)\,w^b$:

$$\text{For all } w > 0,\ W_{s0}^{b}(w) = W_{s0}\left((1+r)\,w\right) \tag{29}$$

This is because when the household is paid in an equivalent amount of bonds, it will receive interest on its sales proceeds that it would not have received if it had been paid in the form of money. In effect, sending the worker to a bond shop is a means of avoiding the interest-opportunity cost of holding sales proceeds in non-interest-bearing money.

This last point is the key to our non-existence result. A Stationary Monetary Equilibrium does not exist when there is costless access to the bond market because an entrepreneur operating a bond shop can offer to its suppliers the possibility of avoiding the interest-opportunity cost of holding cash receipts, and hence it can offer its suppliers and customers terms that dominate those offered by any money shop. Formally we have:

Proposition 1 *When the Baumol-Tobin cost δ equals zero there exists no Stationary Monetary Equilibrium.*

Proof. Consider any feasible money prices (w, p). I show that there is a feasible pair of bond prices $\left(w^b, p^b\right)$ such that:

$$W_{d0}^{b}(p) = W_{d0}(p) \tag{30}$$

and

$$W_{s0}^{b}\left(w^{b}\right) > W_{s0}(w)\,. \tag{31}$$

Consider the pair of bond prices:

$$\left(\overline{w}^{b}, \overline{p}^{b}\right) = \left(w + \tau - \tau^{b}, p\right) \tag{32}$$

This pair is feasible because (26) and the assumed feasibility of (w, p) imply:

$$(\overline{w}^{b} + \tau^{b})y = (w+\tau)\,y = p\widetilde{x}_0(p) = \overline{p}^{b}\widetilde{x}_0^{b}\left(\overline{p}^{b}\right)$$

[4]Note that when the first constraint in (12) is binding, which again it must be when $\delta = 0$, then the second constraint is equivalent to the constraint in (28).

and

$$\Pi^b\left(\overline{w}^b,\overline{p}^b\right)=\left(\frac{\overline{p}^b-\overline{w}^b-\tau^b}{\overline{p}^b}\right)y-\sigma=\left(\frac{p-w-\tau}{p}\right)y-\sigma=\Pi\left(w,p\right)\geq 0$$

Also it follows immediately from (26) that $\overline{p}^b=p$ satisfies condition (30). Finally it follows from (5) that

$$\overline{w}^b=w+\frac{\tau r}{(1+r)}>w>\frac{w}{1+r}$$

This and the fact that W^b_{s0} is an increasing function implies:

$$W^b_{s0}\left(\overline{w}^b\right)>W^b_{s0}\left(\frac{w}{1+r}\right)$$

which together with (29) implies that $\overline{w}^b$ satisfies condition (31). It follows that there is no feasible pair of money prices that is immune to bond entry. Therefore there is no pair of money prices that satisfy the immunity condition in part (a) of the definition of Stationary Monetary Equilibrium (Definition 5 above). Therefore there is no Stationary Monetary Equilibrium. ■

4.3 Existence with a positive Baumol-Tobin cost and a small enough rate of interest

When there is a positive Baumol-Tobin cost, then a bond shop can no longer automatically offer terms that dominate those of any money shop. This is because in order to convert the sales receipts into cash that can be used to purchase consumption in the future the household will have to either incur the cost of visiting the bond market or be content with consuming just the interest on the bonds received. If the rate of interest is small enough then this disadvantage will outweigh the advantage of avoiding the interest-opportunity cost. In effect, the supplier may value the extra liquidity yield of the cash receipts more highly than the extra pecuniary yield of the bond receipts.

Suppose therefore that $\delta>0$. My candidate for a Stationary Monetary Equilibrium is the pair of money prices $(\widehat{w},\widehat{p})$ defined as:

$$\widehat{w}=\frac{m^s}{y}-\tau \tag{33}$$

and

$$\widehat{p}=\frac{m^s}{y-\sigma}. \tag{34}$$

which are also the prices I showed to constitute a unique stationary monetary equilibrium in my earlier (2005) paper, where by assumption there are no government bonds. Note that assumptions (2) and (3) imply that $\widehat{w}$ is strictly positive, and assumption (1) implies that $\widehat{p}$ is strictly positive.

These prices are obvious candidates for a stationary equilibrium because in a stationary equilibrium the cash-in-advance constraint should bind, with households spending all their money each period: $px = m^s$. Under these circumstances the feasibility condition (14) requiring that a shop's receipts be just enough to cover its wage and tax costs would require a wholesale price equal to $\widehat{w}$. And under the same circumstances the zero profit condition: $y - \sigma - m^s/p = 0$ which must be satisfied for a shop to survive the Bertrand-like entrepreneurial competition of stage 1, would require a retail price equal to $\widehat{p}$.

I will need to show that the conditions defining a Stationary Monetary Equilibrium are satisfied under the assumption that:

$$(\overline{w}, \overline{p}) = (\widehat{w}, \widehat{p})$$

which I now maintain for the rest of the analysis.

First I show that under this assumption each household will indeed spend all its money each period, and that moreover it will always have a zero excess demand for bonds. Because it never visits the bond market the household will never incur the Baumol-Tobin cost and will therefore attain the same payoff as if δ were equal to zero, which according to (24) is the function $\widehat{W}_{d0}$ defined by:

$$\left\{ \begin{array}{ll} \widehat{W}_{d0}\left(p\right) = & \max\limits_{(x \geq 0)} \ \ln(x) + \beta V_0 \left(\left(1+r\right) \left(m^s + b^s - px\right) + \widehat{w} y \right) \\ & \text{subject to } px \leq m^s + b^s, \end{array} \right\} \tag{35}$$

That is, I will establish:

Result 2 *When* $(\overline{w}, \overline{p}) = (\widehat{w}, \widehat{p})$ *then, for all* $p > 0$, *(a)* $W_d\left(p\right) = \widehat{W}_{d0}\left(p\right)$ *and (b) the solution to the household's decision problem (7) is uniquely given by* $x = m^s/p$ *and* $b' = b^s$. *This result holds even when the Baumol-Tobin cost* δ *is strictly positive.*

Proof. Consider a household whose shopper is patronizing a money shop posting the retail price p while all other shops, now and in the future, are posting the pair $(\widehat{w}, \widehat{p})$.

Suppose first that $p = \widehat{p}$. Then $W_d\left(p\right)$ is just the the continuation function V evaluated at (m^s, b^s), and $\widehat{W}_{d0}\left(p\right)$ is just $V_0\left(m^s + b^s\right)$. So part (a) of Result 2 is equivalent to:

$$V\left(m^s, b^s\right) = V_0\left(m^s + b^s\right) \tag{36}$$

To see that (36) must hold, note that the pair $(x, b') = (m^s/\widehat{p}, b^s)$ is feasible for the problem (19) defining

$V(m^s, b^s)$, so we have:

$$V(m^s, b^s) \geq \ln\left(\frac{m^s}{\widehat{p}}\right) + \beta V\left(m^s + b^s - b^s - m^s + \widehat{w}y + rb^s, b^s\right)$$

But the assumption (2) that the tax rate is set just high enough to pay the interest on b^s bonds, together with the definition (33) of $\widehat{w}$, implies that:

$$\widehat{w}y + rb^s = m^s \tag{37}$$

Therefore:

$$V(m^s, b^s) \geq \ln\left(\frac{m^s}{\widehat{p}}\right) + \beta V(m^s, b^s)$$

From this and the fact that $\beta = (1+r)^{-1}$ we get:

$$V(m^s, b^s) \geq \frac{1+r}{r} \ln\left(\frac{m^s}{\widehat{p}}\right)$$

However, since (37) implies that $m^s + b^s \geq \widehat{w}y$, therefore the closed-form solution (21) for $V_0(m^s + b^s)$ applies, so:

$$V_0(m^s + b^s) = \frac{1+r}{r} \ln\left(\frac{r(m^s + b^s) + \widehat{w}y}{(1+r)\widehat{p}}\right)$$

which together with (37) implies:

$$V_0(m^s + b^s) = \frac{1+r}{r} \ln\left(\frac{m^s}{\widehat{p}}\right) \tag{38}$$

So I have established that:

$$V(m^s, b^s) \geq V_0(m^s + b^s)$$

This and the inequality (23) stating that the continuation value cannot be greater than if the Baumol-Tobin cost were zero establish the equality (36).

Next consider the general case where p is not necessarily equal to $\widehat{p}$. I will show first that:

$$W_d(p) = \widehat{W}_{d0}(p) = \ln\left(\frac{m^s}{p}\right) + \frac{1}{r} \ln\left(\frac{m^s}{\widehat{p}}\right), \tag{39}$$

which will establish part (a) of Result 2 in the general case. First note that because $\delta \geq 0$, and because (23) implies that the continuation value in (7) is never greater than that in (35), therefore:

$$W_d(p) \leq \widehat{W}_{d0}(p) \tag{40}$$

According to (37) and the closed-form solution (21) we can rewrite (35) as:

$$\left\{\begin{array}{ll}\widehat{W}_{d0}(p) = & \max\limits_{(x\geq 0)} \ln(x) + (1/r)\ln\left(\left(m^s + r\left(m^s - px\right)\right)/\widehat{p}\right) \\ & \text{subject to } px \leq m^s + b^s,\end{array}\right\}$$

which has the unique solution: $x = m^s/p$, yielding:

$$\widehat{W}_{d0}(p) = \ln\left(\frac{m^s}{p}\right) + \frac{1}{r}\ln\left(\frac{m^s}{\widehat{p}}\right)$$

which is the second equality of (39).

Next, note that because the choice $(x, b') = (m^s/p, b^s)$ is feasible for the problem (7) we have:

$$W_d(p) \geq \ln\left(\frac{m^s}{p}\right) + \beta V\left(m^s + b^s - b^s - m^s + \widehat{w}y + rb^s, b^s\right)$$

So (37) implies:

$$W_d(p) \geq \ln\left(\frac{m^s}{p}\right) + \beta V\left(m^s, b^s\right)$$

which together with (36) and (38) implies:

$$W_d(p) \geq \ln\left(\frac{m^s}{p}\right) + \frac{1}{r}\ln\left(\frac{m^s}{\widehat{p}}\right)$$

Together with (40) and the second equality of (39) this implies the first equality.

The above argument establishes that $(m^s/p, b^s)$ provides one solution to the problem defined by (7). Together with Result 1 it also shows that m^s/p is the unique solution to (35). To complete the demonstration of Result 2 we need to show that the solution to (7) is unique. So consider any $(x, b') \geq 0$ in the constraint set of (7). Because $b' + px \leq m^s + b^s$ therefore x is in the constraint set of (35). This and the fact that m^s/p is the unique solution to (35) implies:

$$\widehat{W}_{d0}(p) \geq \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - px\right) + \widehat{w}y\right) \text{ with strict inequality if } x \neq m^s/p$$

This and part (a) of Result 2 imply that:

$$W_d(p) \geq \ln(x) + \beta V_0\left((1+r)\left(m^s + b^s - px\right) + \widehat{w}y\right) \text{ with strict inequality if } x \neq m^s/p$$

Also, because $b' + px \leq m^s + b^s$ therefore:

$$(1+r)(m^s + b^s - px) + \widehat{w}y \geq m^s + b^s - px + \widehat{w}y + rb'$$

which, together with the fact that V_0 is an increasing function, implies:

$$W_d(p) \geq \ln(x) + \beta V_0(m^s + b^s - px + \widehat{w}y + rb') \text{ with strict inequality if } x \neq m^s/p$$

From this, the inequality (23) and the fact that $\delta > 0$ we have:

$$W_d(p) \geq \ln(x) - \widetilde{\delta}(b' - b^s) + \beta V(m^s + b^s - b' - px + \widehat{w}y + rb', b')$$
$$\text{with strict inequality if either } x \neq m^s/p \text{ or } b' \neq b^s$$

Therefore (x, b') does not solve the problem (7) if it is not equal to $(m^s/p, b^s)$, which is therefore the unique solution. This establishes part (b) of Result 2. ■

With Result 2 in hand we can now demonstrate, as suggested above, that $(\widehat{w}, \widehat{p})$ is the only set of money prices that are both feasible and immune to money entry:

Result 3 *The money prices (w, p) are feasible and immune to money entry if and only if $w = \widehat{w}$ and $p = \widehat{p}$.*

Proof. To demonstrate this result, consider any money prices (w, p). Since Result 2 implies that m^s/p is the only element of $\xi_x(p)$ therefore (w, p) satisfies the first condition (14) in the definition of feasibility (Definition 1 above) if and only if

$$(w + \tau)y = m$$

which, according to the definition (33), is true if and only if $w = \widehat{w}$. Using this and the definition (15) of Π, the prices (w, p) satisfy both conditions of feasibility in Definition 1 if and only if $w = \widehat{w}$ and

$$\left(\frac{p - \widehat{w} + \tau}{p}\right) y - \sigma \geq 0$$

which, according to the definition (34), is true if and only if $p \geq \widehat{p}$. It follows from this and Definition 3 that the prices (w, p) are feasible immune to money entry if and only if $(w, p) = (\widehat{w}, \widehat{p})$. ■

It follows immediately from part (b) of Result 2 that the candidate prices $(\widehat{w}, \widehat{p})$ satisfy the second condition in the above definition of Stationary Monetary Equilibrium (Definition 5), i.e., that there be no excess demand for bonds. It then follows from this and Result 3 that $(\widehat{w}, \widehat{p})$ constitutes a Stationary Monetary Equilibrium if it is immune to bond entry, for then it will satisfy not only the second but also the first condition of Definition 5. The rest of this section will show that this will happen if the rate of interest is low enough. That is:

Proposition 2 *For any Baumol-Tobin cost $\delta > 0$ the prices $(\widehat{w}, \widehat{p})$ constitute a Stationary Monetary Equilibrium when the rate of interest r is small enough.*

Proof. As indicated above, I just need to demonstrate immunity to bond entry for small enough $r > 0$. I do this in three (very unequal) steps:

1. I find a number $\varepsilon > 0$ such that $W_s^b\left(w^b\right) \geq W_s\left(\widehat{w}\right)$ implies $w^b > (1+\varepsilon)\,\widehat{w}$ for small enough $r > 0$.
2. I show that if $\left(w^b, p^b\right)$ is a feasible pair of bond prices and $w^b > (1+\varepsilon)\,\widehat{w}$ then $p^b > \widehat{p}$ for small enough $r > 0$.
3. I show that if $p^b > \widehat{p}$ then $W_d^b\left(p^b\right) < W_d\left(\widehat{p}\right)$.

It follows immediately from these three results and Definition 4 that $(\widehat{w}, \widehat{p})$ is immune to bond entry for small enough $r > 0$, which will complete the demonstration of Proposition 2.

1. Take any value of ε such that:

$$0 < \varepsilon < \min\left\{2\ln(2) - 1, \delta\right\} \tag{41}$$

By construction:

$$W_s\left(\widehat{w}\right) = V\left(m^s, b^s\right)$$

From this and the results (36) and (38):

$$W_s\left(\widehat{w}\right) = V_0\left(m^s + b^s\right) = \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right) \tag{42}$$

So given that W_s^b is an increasing function I just need to demonstrate that

$$W_s^b\left((1+\varepsilon)\,\widehat{w}\right) < \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right) \text{ for small enough } r > 0. \tag{43}$$

Note that:

$$W_s^b\left((1+\varepsilon)\,\widehat{w}\right) = \max\left\{W_{s-}^{b\varepsilon}, W_{s+}^{b\varepsilon}\right\} \tag{44}$$

where $W_{s-}^{b\varepsilon}$ is the maximum payoff a household could obtain by supplying to a bond shop at the price $w^b = (1+\varepsilon)\,\widehat{w}$ if the household was constrained not to visit the bond market this period or next, and $W_{s+}^{b\varepsilon}$ is the payoff that the same household could get if it was constrained to visit the bond market at least once during these first two periods.

Consider first $W_{s-}^{b\varepsilon}$. By definition:

$$\left\{\begin{array}{lll} W_{s-}^{b\varepsilon} = & \max\limits_{(x_0,x_1,m_1,m_2,b_1,b_2)\geq 0} & \ln(x_0) + \beta\ln(x_1) + \beta^2 V(m_2, b_2) \\ & \text{subject to} & \widehat{p}x_0 \leq m^s, \\ & & \widehat{p}x_1 \leq m_1, \\ & & m_1 = m^s - \widehat{p}x_0 + rb_1 \\ & & b_1 = b^s + (1+\varepsilon)\,\widehat{w}y \\ & & m_2 = m_1 - \widehat{p}x_1 + \widehat{w}y + rb_2 \\ & & b_2 = b_1 \end{array}\right\} \tag{45}$$

So

$$W_{s-}^{b\varepsilon} \leq \overline{W}_{s-}^{b\varepsilon},$$

where:

$$\left\{\begin{array}{lll} \overline{W}_{s-}^{b\varepsilon} = & \max\limits_{(x_0,x_1,m_1,m_2,b_1,b_2)\geq 0} & \ln(x_0) + \beta\ln(x_1) + \beta^2 V_0(m_2 + b_2) \\ & \text{subject to} & \widehat{p}x_1 \leq m_1, \\ & & m_1 = m^s - \widehat{p}x_0 + rb_1 \\ & & b_1 = b^s + (1+\varepsilon)\,\widehat{w}y \\ & & m_2 = m_1 - \widehat{p}x_1 + \widehat{w}y + rb_2 \\ & & b_2 = b_1 \end{array}\right\} \tag{46}$$

because the absence of the constraint $\widehat{p}x_0 \leq m^s$ makes the constraint set in (46) a superset of the constraint set in (45) and the fact that $V_0(m_2 + b_2) \geq V(m_2, b_2)$ makes the objective function in (46) never smaller than the objective function in (45). Using the last four constraints to replace m_1, m_2, b_1 and b_2 we can rewrite (46) as:

$$\left\{\begin{array}{ll} \overline{W}_{s-}^{b\varepsilon} = & \max\limits_{(x_0,x_1)\geq 0} \ln(x_0) + \beta\ln(x_1) + \beta^2 V_0\left(m^s - \widehat{p}(x_0 + x_1) + \widehat{w}y + (1+2r)\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)\right) \\ & \text{subject to } \widehat{p}(x_0 + x_1) \leq m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right) \end{array}\right\}$$

From this and the closed-form solution (21) for V_0:

$$\left\{\begin{array}{lll} \overline{W}_{s-}^{b\varepsilon} = & \max\limits_{(x_0,x_1,x_2)\geq 0} & \ln(x_0) + \beta\ln(x_1) + \beta^2\frac{1+r}{r}\ln(x_2) \\ & \text{subject to} & \widehat{p}(x_0 + x_1) \leq m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right) \\ & \text{and} & x_2 = \left(r\left(m^s - \widehat{p}(x_0 + x_1) + \widehat{w}y + (1+2r)\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)\right) + \widehat{w}y\right)/\left((1+r)\,\widehat{p}\right) \end{array}\right\}$$

The following argument shows that the first inequality constraint in this problem will bind for small enough $r > 0$. Suppose on the contrary that the constraint is not binding. Then some tedious but elementary calculations yield the solution to the problem:

$$x_0 = (1+r)^{-1} X/\widehat{p},\ x_1 = (1+r)^{-2} X/\widehat{p} \text{ and } x_2 = (1+r)^{-3} X/\widehat{p}$$

where

$$X = r\left(m^s + (1+2r)\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)\right) + (1+r)\,\widehat{w}y$$

Now consider what happens as we send r to zero, keeping m^s and b^s constant, but allowing τ to adjust to keep (2) satisfied, so that, by (37) $\widehat{w}y \to m^s$. Therefore: $X \to m^s$, so:

$$\begin{aligned} & m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right) - \widehat{p}\,(x_0 + x_1) \\ = \quad & m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right) - \left((1+r)^{-1} + (1+r)^{-2}\right) X \\ \to \quad & m^s - 2m^s \\ < \quad & 0 \end{aligned}$$

So for small enough $r > 0$ the solution will violate the first inequality constraint, a contradiction. Therefore for small enough $r > 0$ the first inequality constraint will bind and routine calculation shows that:

$$x_0 = \frac{m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)}{(2+r)\,\widehat{p}},$$

$$x_1 = \frac{m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)}{(1+r)\,(2+r)\,\widehat{p}}$$

and

$$x_3 = \frac{\widehat{w}y + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)}{\widehat{p}},$$

so:

$$\overline{W}^{b\varepsilon}_{s-} = (1+\beta)\ln\left(\frac{m^s + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)}{(2+r)\,\widehat{p}}\right) - \beta\ln(1+r) + \beta^2\frac{1+r}{r}\ln\frac{\widehat{w}y + r\left(b^s + (1+\varepsilon)\,\widehat{w}y\right)}{\widehat{p}}$$

Using (37) and the fact that $\beta = (1+r)^{-1}$, we have:

$$\begin{aligned}
& \overline{W}^{b\varepsilon}_{s-} - \frac{1+r}{r} \ln\left(\frac{m^s}{\widehat{p}}\right) \\
= & \frac{2+r}{1+r} \left[\ln\left(m^s + r\left(b^s + (1+\varepsilon)\left(m^s - rb^s\right)\right)\right) - \ln(2+r)\right] - \frac{1}{1+r}\ln(1+r) \\
& + \frac{1}{1+r}\frac{1}{r}\ln\left(m^s + r(1+\varepsilon)\left(m^s - rb^s\right)\right) - \frac{1+r}{r}\ln\left(m^s\right)
\end{aligned}$$

Define

$$f(r) = r(1+r)\left(\overline{W}^{b\varepsilon}_{s-} - \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right)\right)$$

Then:

$$\begin{aligned}
f(r) \quad = \quad & r(2+r)\ln\left(m^s + r\left(b^s + (1+\varepsilon)\left(m^s - rb^s\right)\right)\right) - r(2+r)\ln(2+r) \\
& - r\ln(1+r) + \ln\left(m^s + r(1+\varepsilon)\left(m^s - rb^s\right)\right) - (1+r)^2\ln\left(m^s\right)
\end{aligned}$$

$$f(0) = 0$$

and

$$f'(0) = 1 + \varepsilon - 2\ln(2) < 0.$$

Therefore for small enough $r > 0$ we have $f(r) < 0$, hence $\overline{W}^{b\varepsilon}_{s-} - \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right) < 0$ and hence

$$W^{b\varepsilon}_{s-} < \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right) \text{ for small enough } r > 0. \tag{47}$$

Next, consider $W^{b\varepsilon}_{s+}$. In this thought experiment the discounted sum of Baumol-Tobin costs is at least equal to $\beta\delta$. So $W^{b\varepsilon}_{s+}$ is no greater than what the household could get from paying the cost $\beta\delta$ and then having unlimited access, at no additional cost, to the bond market forever. That is

$$W^{b\varepsilon}_{s+} \leq W^b_{s0}\left((1+\varepsilon)\widehat{w}\right) - \beta\delta,$$

where the function W^b_{s0} is the function W^b_s defined by (28) above. By (29) we have:

$$W^b_{s0}\left((1+\varepsilon)\widehat{w}\right) = W_{s0}\left((1+r)(1+\varepsilon)\widehat{w}\right)$$

where W_{s0} is defined by (27) above. From (27) and the close-form solution (21):

$$\left\{ \begin{array}{ll} W_{s0}\left((1+r)(1+\varepsilon)\widehat{w}\right) = & \max\limits_{(x\geq 0)} \ln(x) + \frac{1}{r}\ln\left(\frac{r((1+r)(m^s+b^s-\widehat{p}x)+(1+r)(1+\varepsilon)\widehat{w}y)+\widehat{w}y}{(1+r)\widehat{p}}\right) \\ & \text{subject to } \widehat{p}x \leq m^s + b^s, \end{array} \right\}$$

Therefore:

$$W_{s0}\left((1+r)(1+\varepsilon)\widehat{w}\right) \leq \max_{(x\geq 0)} \ln(x) + \frac{1}{r}\ln\left(\frac{r\left((1+r)(m^s+b^s-\widehat{p}x)+(1+r)(1+\varepsilon)\widehat{w}y\right)+\widehat{w}y}{(1+r)\widehat{p}}\right)$$

or, solving for the maximization problem on the right-hand side:

$$W_{s0}\left((1+r)(1+\varepsilon)\widehat{w}\right) \leq \frac{1+r}{r}\ln\left(\frac{r(m^s+b^s)+\left(r(1+\varepsilon)+\frac{1}{1+r}\right)\widehat{w}y}{(1+r)\widehat{p}}\right)$$

Putting all this together yields:

$$W_{s+}^{b\varepsilon} \leq \overline{W}_{s+}^{b\varepsilon}$$

where:

$$\overline{W}_{s+}^{b\varepsilon} \equiv \frac{1+r}{r}\ln\left(\frac{r(m^s+b^s)+\left(r(1+\varepsilon)+\frac{1}{1+r}\right)\widehat{w}y}{(1+r)\widehat{p}}\right) - \beta\delta$$

Define

$$h(r) = \frac{r}{1+r}\left(\overline{W}_{s+}^{b\varepsilon} - \frac{1+r}{r}\ln\left(\frac{m^s}{\widehat{p}}\right)\right)$$

Then:

$$h(r) = \ln\left(r(m^s+b^s)+\left(r(1+\varepsilon)+\frac{1}{1+r}\right)\widehat{w}y\right) - \ln(1+r) - \frac{r}{1+r}\beta\delta - \ln(m^s)$$

Using (37) we have:

$$h(r) = \ln\left(r(m^s+b^s)+\left(r(1+\varepsilon)+\frac{1}{1+r}\right)(m^s-rb^s)\right) - \ln(1+r) - \frac{r}{1+r}\beta\delta - \ln(m^s)$$

so:

$$h(0) = 0$$

and

$$h'(0) = \varepsilon - \delta < 0$$

So for small enough $r > 0$ we have $h(r) < 0$, hence $\overline{W}^{b\varepsilon}_{s+} < \frac{1+r}{r} \ln\left(\frac{m^s}{\widehat{p}}\right)$ and hence

$$W^{b\varepsilon}_{s+} < \frac{1+r}{r} \ln\left(\frac{m^s}{\widehat{p}}\right) \text{ for small enough } r > 0. \tag{48}$$

The results (47) and (48) together with (44) imply (43) and this establishes step 1 of the proof.

2. To establish step 2, consider any feasible bond prices $\left(w^b, p^b\right)$ with

$$w^b > (1+\varepsilon)\,\widehat{w}$$

By the definition of feasibility (Definition 2 above) and the definition (18) of Π^b, we have:

$$\left(\frac{p^b - w^b - \tau^b}{p^b}\right) y - \sigma \geq 0$$

so:

$$\left(\frac{p^b - \widehat{w}\,(1+\varepsilon) - \tau^b}{p^b}\right) y - \sigma > 0$$

or:

$$p^b > \frac{\varepsilon \widehat{w} y}{y-\sigma} + \frac{\widehat{w} y - \tau^b y}{y - \sigma}$$

As $r \rightarrow 0$ equations (1), (2), (5), (34) and (37) imply that

$$\frac{\varepsilon \widehat{w} y}{y-\sigma} + \frac{\widehat{w} y - \tau^b y}{y - \sigma} \rightarrow (1+\varepsilon)\,\widehat{p} > \widehat{p}$$

Therefore, for small enough $r > 0$ we have $p^b > \widehat{p}$. The establishes step 2.

3. Result (26) above establishes that

$$\widehat{W}_{d0}(p) = \widehat{W}^b_{d0}(p) \text{ for all } p > 0$$

where $\widehat{W}_{d0}$ and $\widehat{W}^b_{d0}$ are defined respectively by (24) and (25) with $\overline{w} = \widehat{w}$. Result (39) establishes that:

$$W_d(p) = \widehat{W}_{d0}(p) \text{ for all } p > 0$$

and the fact that $\delta > 0$ implies that:

$$\widehat{W}^b_{d0}(p) \geq W^b_d(p) \text{ for all } p > 0$$

Hence:

$$W_d(p) \geq W_d^b(p) \text{ for all } p > 0$$

Step 3 follows from this and the fact that W_d is a decreasing function.

This finishes the proof. ■

5 Conclusion

I have shown that with a Baumol-Tobin cost there exists a stationary equilibrium in which all trades use non-interest-bearing money and non involve bonds. This does not mean however that there does not also exist a stationary equilibrium in which all trades use bonds and not money. Instead what the Baumol-Tobin cost does is to establish superior liquidity for whatever is dictated by convention to be the medium of exchange, and hence it aids in making that convention self-sustaining. The more interesting question is whether or nor society would be any better off under a convention of using only interest-bearing bonds as the universal medium of exchange. This is the subject of ongoing research

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