The supremum argument in the new approach to the existence of equilibrium in vector lattices

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Received 31 August 1998; accepted 14 January 1999

Abstract

In this note, we present conditions under which the supremum argument of Mas-Colell and Richard (1991) is not needed in the proof of the second fundamental theorem of welfare economics, and thereby in the proof of the existence of competitive equilibrium in infinite-dimensional commodity spaces. © 1999 Elsevier Science S.A. All rights reserved.

Keywords: Pareto optimality; Valuation equilibrium; Competitive equilibrium; Uniform properness

JEL classification: D50; D61

1. Introduction

There are at least two approaches to the proof of the second fundamental theorem of welfare economics in the set-theoretic setting.\textsuperscript{1} The first is due to Arrow (1951) and Debreu (1951, 1954), and involves the sum of the individual better-than sets.\textsuperscript{2} The second involves the Cartesian product of these better-than sets. In the first approach, the sum of the better-than sets is separated from the aggregate endowment, and a decentralization argument is used to show that the supporting hyperplane also supports the individual sets. In the second approach, separation in the product space furnishes as many separating hyperplanes as there are agents, so to speak, and the valuation equilibrium price is obtained from this list of prices. For example, Khan and Vohra (1987) appeal to feasibility considerations to show that the resulting individual prices are all identical. Indeed, the fact that

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\textsuperscript{1}As opposed to treatments emphasizing calculus, as in Samuelson (1947) and his references.
\textsuperscript{2}Strictly speaking, these should be referred to as ‘better-than-\( \bar{x}_i \) sets’ where \( \bar{x}_i \) is the Pareto optimal bundle of consumer \( i \). We shall also have occasion to consider other consumption bundles. We hope that the reader will see the shorter phrase as an acceptable abbreviation.

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PII: S0165-1765(99)00015-4
marginal rates of substitution at individual Pareto optimal bundles are all identical is precisely the essential (neoclassical) content of the second welfare theorem.\(^3\)

In their new approach to the price equilibrium existence problem in vector lattices, Richard (1989) and Mas-Colell and Richard (1991) rely on the second welfare theorem as an essential step in their existence proof\(^4\), and prove this theorem by working with the Cartesian product of the better-than sets. As underscored by Aliprantis (1997), Podczeck (1996), Tourky (1998), and others, the remarkable contribution of Mas-Colell and Richard is their use of the lattice structure of the price space to show the supportability of weakly optimal Pareto allocations. They simply take the supremum of the individual marginal rates of substitution at the Pareto optimal bundles;\(^5\) such an argument is in keeping with their identification of Riesz spaces (vector lattices), with lattice ordered topological duals, as the appropriate mathematical setting for infinite dimensional general equilibrium analysis.

In light of this work, it is natural to ask whether there are any infinite-dimensional settings for which one does not need the supremum argument of Richard (1989) and Mas-Colell and Richard (1991), but can rely instead on the more elementary feasibility considerations. The fact that this is so when the commodity space is an ordered topological vector space with order unit is already evident from Khan and Vohra (1988); what is possibly surprising is that the supremum argument can be also dispensed with in the Mas-Colell (1986a) original setting, where the commodity space is a locally convex-solid Riesz space, consumption sets coincide with the positive cone, and preferences are \(\omega\)-uniformly proper. The object of this note is to substantiate this observation.

It is important to point out that the theorems of Richard (1989) and Mas-Colell and Richard (1991) hold even when the positive cone has no internal points or when the price space is smaller than its solid hull and the topology of the commodity space is not locally solid. All that is needed is that the price space be a sublattice of the order dual of the commodity space, which in turn is a Riesz space. In particular, Richard (1989) proves two theorems. The first is set in a locally convex-solid Riesz space, and it is evident from this note that it can be proved using the Cartesian product approach, but with a direct appeal to feasibility considerations and without invoking the supremum argument. However, whether this approach of showing the equality of the marginal rates of substitutions, is also extendible to Riesz commodity spaces that are not locally solid, and thereby to Richard’s second theorem, remains an interesting open question. In view of Aliprantis and Burkinshaw (1991), it seems that such a venture must somehow use the lattice structure of the price space.

2. The model and the result

Our commodity space \(E\) is an ordered topological vector space. The price space is the topological dual of \(E\), which is denoted \(E^*\).\(^6\)

Let the set of consumers be given by \(I = \{1, \ldots, m\}\), where \(m\) is a positive integer greater than or equal to one. We assume that consumption sets coincide with the positive cone \(E_+\) of \(E\). The point \(\omega_i \in E_+\) is the \(i\)th consumer’s endowment, and we assume that \(\omega_i > 0\), for all \(i \in I\). Let \(\omega = \sum_{i=1}^{m} \omega_i\).

\(^{3}\)It is by now well understood that when these rates are not singletons, equality in the statement of the second welfare theorem is replaced by intersection; see Khan and Vohra (1987) and their references.


\(^{5}\)The supremum argument in this connection was in fact first used in Mas-Colell (1986b).

\(^{6}\)See Aliprantis et al. (1987) for undefined terms and notation.
Consumer preferences \( \succeq_i \subseteq E_+ \times E_+ \) are assumed to be locally non-satiated, monotone, total preorderings. Define \( P_i : E_+ \rightarrow 2^{E_+} \) by the rule \( P_i(x) = \{ z \in E_+ : z \succ x \} \); and assume it to be convex for all \( x \in E_+ \).

We shall distinguish some points and sets in the Cartesian products \( E^m \) and \( (E')^m \) by setting them in bold face characters, and their \( m \) individual coordinates by subscripts. Let \( \mathbf{w} = (\omega, \ldots, \omega) \in E^m \). Let \( \mathbf{F} = \{ z \in E^m : \sum_{i=1}^m z_i = \omega \} \) and \( \mathbf{F}_+ = \mathbf{F} \cap E^m_+ \), which is the set of feasible allocations. An allocation \( \mathbf{x} \in E^m_+ \) is weakly Pareto optimal iff \( \mathbf{x} \in \mathbf{F}_+ \) and \( \prod_{i=1}^m P_i(x_i) \cap \mathbf{F}_+ = \emptyset \).

A pseudo-valuation equilibrium is a pair \((\mathbf{x}, \mathbf{p})\), where \( \mathbf{x} \in \mathbf{F}_+ \) and \( \mathbf{p} \in E' \), such that \( y \in P_i(x_i) \) implies \( p(y) \geq p(x_i) \), for all \( i \in I \); and \( p(\omega) > 0 \).

We shall consider two settings for our economy.

**A1.** \( \omega \) is an interior point of \( E_+ \).

Our next assumption is taken from Mas-Colell (1986a).

**A2.** \( E \) is a locally convex-solid Riesz space, and for all \( i \in I \), \( \succeq_i \) is \( \omega \)-uniformly proper; which is to say that there is an open cone \( \Gamma_i \) with vertex \( 0 \) containing \( -\omega \) such that for all \( x \in E_+ \), \( \{ \Gamma_i + \{ x \} \} \cap \{ y \in E_+ : y \succeq x \} = \emptyset \).

We shall prove the following well-known theorem. Though our proof centers around a separating hyperplane argument in the space of allocations we do not use the supremum argument of Mas-Colell and Richard (1991).

**Theorem.** Suppose that either A1 or A2 hold. For every weakly Pareto optimal allocation \( \mathbf{x} \), there is \( \mathbf{p} \in E' \) such that \((\mathbf{x}, \mathbf{p})\) is a pseudo-valuation equilibrium.

The statement of the theorem conceals the following stronger result that can be used, along with assumptions of compactness and continuity, to prove the existence of equilibrium.

**Corollary.** Suppose that either A1 or A2 hold. There is a fixed weak* compact convex set \( \Delta \subset E' \) disjoint from \( \{0\} \) such that for every weakly Pareto optimal allocation \( \mathbf{x} \), there is \( \mathbf{p} \in \Delta \) such that \((\mathbf{x}, \mathbf{p})\) is a pseudo-valuation equilibrium.

### 3. Proof of the result

The following proposition is due to Khan and Vohra (1987, 1988).

**Proposition 1.** If \( \mathbf{p} \in (E')^m \) supports \( \mathbf{F} \) at some \( \mathbf{x} \in \mathbf{F} \) then \( \forall i, j \in I, \ p_i = p_j \).

**Proof.** Suppose that \( p_1 \neq p_2 \) then for some \( y \in E \) we have \( p_1(y) + p_2(-y) > 0 \). Take the point \( \mathbf{z} \in \mathbf{F} \) defined by

\[
\mathbf{z}_i = \begin{cases} 
  y = x_1 & \text{if } i = 1; \\
  -y + x_2 & \text{if } i = 2; \\
  x_i & \text{otherwise.}
\end{cases}
\]

Then \( \sum_{i=1}^m p_i(z_i) > \sum_{i=1}^m p_i(x_i) \), which is a contradiction. □

In the presence of local non-satiation, this is equivalent to saying that \( y \succeq x_i \Rightarrow p(y) \geq p(x_i) \), for all \( i \in I \).
Let \( \tilde{x} \) be the weakly Pareto optimal allocation. We know that \( \tilde{x} \in F \) and that \( \tilde{x} \in \bigtimes_{i=1}^{m} P_i(\tilde{x}) \). It is therefore clear from Proposition 1 that for any \( p \in (E')^m \), \( p(w) > 0 \), that separates \( \bigtimes_{i=1}^{m} P_i(\tilde{x}) \) and \( F \), the pair \( (\tilde{x}, P_i) \) is a pseudo-valuation equilibrium. Thus, all we need is that there be a convex set \( P \subset E^m \) such that (cf., Tourky (1998), Definition 1)

(i) \( \tilde{x} + w \) is an interior point of \( P \);
(ii) \( \bigtimes_{i=1}^{m} P_i(\tilde{x}) \subset P \);
(iii) \( P \cap F = \emptyset \).

Our next two propositions can be seen as furnishing such a set \( P \).

**Proposition 2.** Assume A1. Then \( \bigtimes_{i=1}^{m} P_i(\tilde{x}) \cap F = \emptyset \).

**Proof.** The proof follows trivially from the definition of weak Pareto optimality. \( \Box \)

We prove the next proposition using the, now classical, argument of Mas-Colell (1986a). For \( i \in I \), let \( I_i \) be the cones from uniform properness, and let \( V \subset \bigcap_{i=1}^{m} (I_i + \omega) \) be a non-empty solid open convex neighborhood of zero.

**Proposition 3.** Assume A2. Then \( \bigtimes_{i=1}^{m} P_i(\tilde{x}) + \bigcup_{\alpha > 0} \alpha(V + m\omega) \cap F = \emptyset \).

**Proof.** Suppose not; then for each \( i \in I \), there exist \( x_i > \tilde{x}_i \), \( v_i \in V \), \( \alpha_i > 0 \), such that

\[
\sum_{i=1}^{m} \left( x_i + \alpha_i (v_i + m\omega) \right) = \sum_{i=1}^{m} \omega_i
\]

Let \( \alpha = \Sigma_{i=1}^{m} \alpha_i \) and let \( u = \Sigma_{i=1}^{m} (\alpha_i/\alpha) v_i \). We can now rewrite Eq. (1) as

\[
\sum_{i=1}^{m} (x_i - \omega_i) + \alpha m\omega = -\alpha u
\]

Since \( V \) is convex then \( u \in V \).

Since \( 0 \leq (\Sigma_{i=1}^{m} (x_i - \omega_i) + \alpha m\omega)^+ \leq \Sigma_{i=1}^{m} (x_i + \alpha \omega) \), we can appeal to the Riesz decomposition property\(^8\) to assert for each \( i \in I \), there exists \( u_i \in E^+ \) such that \( 0 \leq u_i \leq x_i + \alpha \omega \) and \( \Sigma_{i=1}^{m} u_i = (\Sigma_{i=1}^{m} (x_i - \omega_i) + \alpha m\omega)^+ \).

For all \( i \in I \), let \( y_i = x_i + \alpha \omega - u_i \geq 0 \). Then

\[
\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i + \alpha m\omega - \sum_{i=1}^{m} u_i = \sum_{i=1}^{m} x_i + \alpha m\omega - \left( \sum_{i=1}^{m} (x_i - \omega_i) + \alpha m\omega \right)^+
\]

\[
\leq \sum_{i=1}^{m} x_i + \alpha m\omega - \sum_{i=1}^{m} (x_i - \omega_i) - \alpha m\omega = \sum_{i=1}^{m} \omega_i
\]

We can now use Eq. (2) to assert that

\(^8\) Let \( 0 \leq x \leq \sum_{i=1}^{m} y_i \) hold in a Riesz space, then there exists \( x_1, \ldots, x_m \) satisfying \( x = x_1 + \cdots + x_m \) and \( 0 \leq x_i \leq y_i \) for each \( i = 1, \ldots, m \); see, for example, Aliprantis et al. (1987).
\[ 0 \leq u_i \leq \left( \sum_{j=1}^{m} (x_j - \omega_j) + \alpha m \omega \right)^+ \leq \left| \sum_{j=1}^{m} (x_j - \omega_j) + \alpha m \omega \right| = \alpha |v| \]

Since \( V \) is solid, this implies that \( u_i \in \alpha V \).

Hence for all \( i \in I, x_i = y_i - \alpha \omega + u_i \) is contained in \( I_i + y_i \) and therefore by uniform properness \( x \not\preceq y \). Since preferences are complete and transitive, \( y \succeq x \), for all \( i \in I \). Because preferences are monotone we now have a contradiction to the weak Pareto optimality of \( \bar{x} \). \( \square \)

**Proof of the Theorem.** Under A1 or A2, and by respective appeals to Propositions 2 and 3, we can assert the existence of a non-zero list of prices \( p \in (E')^m \) that separates \( \times_{i=1}^{m} P_i(\bar{x}_i) \) and \( F \), such that \( \sum_{i=1}^{m} p_i(\omega) > 0 \), given the interiority of \( w \).

Since \( \bar{x} \in \times_{i=1}^{m} P_i(\bar{x}_i) \) then \( p_i[P_i(\bar{x}_i)] \geq p_i(\bar{x}_i) \), for all \( i \in I \). Also, \( p \) supports \( F \) at \( \bar{x} \). Thus, by Proposition 1, for all \( i, j \in I, p_i = p_j = \pi \).

Since \( \pi(\omega) = 1/m \sum_{i=1}^{m} p_i(\omega) > 0 \), this completes the proof that \( (\bar{x}, \pi) \) is a pseudo-valuation equilibrium. \( \square \)

**Acknowledgements**

This work was initiated while Khan was a Visiting Fellow at the Australian National University, and a Visitor at the University of Melbourne; he would like to thank Adrian Pagan and Peter Bardsley for making these visits possible, and for the hospitality and excellent facilities extended to him at both institutions.

**References**


