EN224: Linear Elasticity

Homework 2: Potential representations, Fourier Transforms, Singular Solutions
Due Friday March 4, 2005

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1. The simplest possible derivation of the Kelvin State. Assume that the solution is to be generated from Papkovich-Neuber potentials $\Psi, \quad \phi$ satisfying

$$\Psi_{i,kk}(\mathbf{x}) = -b_i \delta(\mathbf{x} - \xi) \qquad \phi_{,kk}(\mathbf{x}) = -x_j b_j \delta(\mathbf{x} - \xi)$$

Here, we have chosen the body force to be a Dirac delta sequence centered at ξ. By taking Fourier transforms of the governing equations for $\Psi, \quad \phi$ above, find the Papkovich-Neuber potentials that generate the required solution.

2. Papkovich-Neuber potentials for the Doublet states. Let $\Psi^{(k)}, \quad \phi^{(k)}$ denote the Papkovich-Neuber potentials for the normalized Kelvin state, i.e. a point force of unit magnitude acting in the $\mathbf{e}_k$ direction at the origin. Let $S^{(k,l)} = [\mathbf{u}^{(k,l)}, \varepsilon^{(k,l)}, \sigma^{(k,l)}]$ denote the doublet states, i..e.

$$u_i^{(k,l)} = u_i^{(k)}{}_{,l} \qquad \varepsilon_{ij}^{(k,l)} = \varepsilon_{ij}^{(k)}{}_{,l} \qquad \sigma_{ij}^{(k,l)} = \sigma_{ij}^{(k)}{}_{,l}$$

Show that $S^{(k,l)}$ may be generated from Papkovich-Neuber potentials

$$\Psi^{(kl)}{}_i = \Psi_i^{(k)}{}_{,l} \qquad \phi^{(kl)} = \phi^{(k)}{}_{,l} - \Psi_l^{(k)}$$

Hence, verify that the Papkovich Neuber potentials for the doublet states centered at the origin are

$$\Psi_i^{(k,l)} = -\frac{1}{4\pi}\frac{\delta_{ki} x_l}{r^3} \qquad \phi^{(k,l)} = -\frac{1}{4\pi}\frac{\delta_{kl}}{r}$$

3. Center of Compression. Using the results of the preceding section, find the displacement, strain and stress fields associated with a center of compression at the origin, i.e., find

$$S^{(k,k)} = [\mathbf{u}^{(k,k)}, \varepsilon^{(k,k)}, \sigma^{(k,k)}]$$

4. Center of compression in a sphere. Using superposition and the results of problem (3), find the displacement fields induced by a center of compression at the center of a sphere of radius *a.* Assume that the surface of the sphere is free of traction.

5. Dilatation at the center of a sphere due to arbitrary surface traction. Using the result of problem (4), show that the dilatation at the center of a sphere of radius *a* due to a self-equilibrating distribution of traction **t** acting on its surface is

$$\varepsilon_{kk} = \frac{3(1-2\nu)}{8\pi(1+\nu)\mu \ a^3} \int_B \mathbf{t} \cdot \mathbf{r} dA$$

where **r** is the position vector of a point on the sphere's surface relative to the origin, and *B* denotes the surface of the sphere. Verify the predictions that were made in our proof of Saint-Venants principle. What happens if the tractions act tangent to the surface of the sphere?