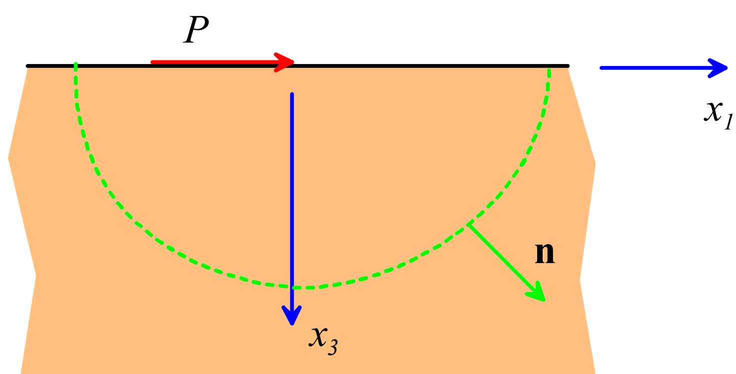
EN224: Linear Elasticity

Homework 3: Half space problems, Eshelby Inclusions,
Due Friday March 18, 2005

Division of Engineering
Brown University

1. Tangentially loaded half-space.



(1.1) State the appropriate boundary conditions for a half-space subjected to a point force acting tangent to its surface, as shown in the figure.

(1.2) Examine the list of Boussinesq potentials given at the end of Sect 3.1 of the online lecture notes. Consider solutions A, B and E, which generate stresses from harmonic potentials $\hat{\phi}$, ω and Ω as follows:

Solution A: $\sigma_{ij} = \hat{\phi}_{,ij}$

Solution B: $\sigma_{ij} = x_3\omega_{,ij} - (1-2\nu)(\delta_{i3}\omega_{,j} + \delta_{j3}\omega_{,i}) - 2\nu\omega_{,3}\,\delta_{ij}$

Solution E: $\sigma_{ij} = e_{ik3}\Omega_{,kj} + e_{jk3}\Omega_{,ki}$

Following the procedure outlined in Sect 3.7 for a normally loaded half-space, find a way to combine solutions A, B, and E so as to generate an elastostatic state which automatically satisfies $\sigma_{33} = \sigma_{23} = 0$ on the surface of the half-space. You should find you can generate the required solution from a single harmonic potential Θ. Set up the boundary conditions that represent a point force at the origin.

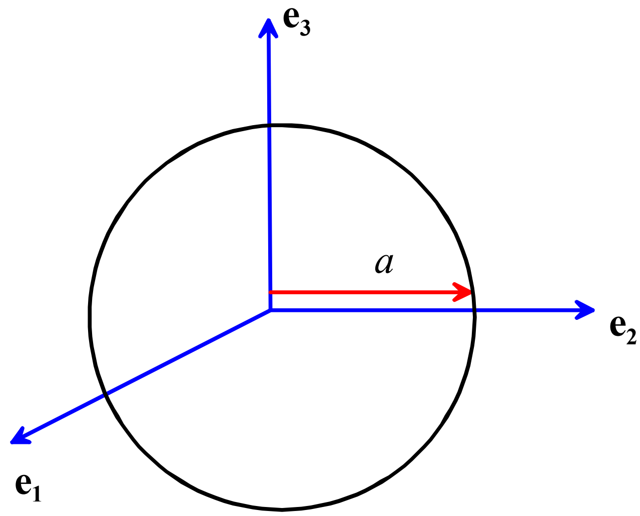
(1.3) Take Fourier transforms of the governing equations and boundary conditions found in (1.2). Hence, deduce that the transform of the required potential satisfies

$$\bar{\Theta}_{,3} = -\frac{F}{p^2}\exp(-px_3)$$

Use the result for the normally loaded half-space to write down $\Theta_{,3}$ and hence deduce the potential Θ

(1.4) Determine the displacement field for the tangentially loaded half-space.

2. Spherical inhomogeneity. Suppose that an infinite solid with shear modulus and Poisson's ratio μ_0, ν_0 contains a spherical inclusion with shear modulus μ_1, ν_1. The solid is loaded in uniaxial tension σ_{11}^{∞} at infinity. Calculate the stress, strain and displacement in the inclusion (take the displacement to be zero at the origin)



3. Energetics of Eigenstrains. Consider a homogeneous, stress free, linear elastic solid with elastic constants C_{ijkl}. Suppose that an eigenstrain distribution ε_{ij}^* is introduced into a bounded subregion of the solid *B*. Let $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^*$ denote the total strain distribution in the solid.

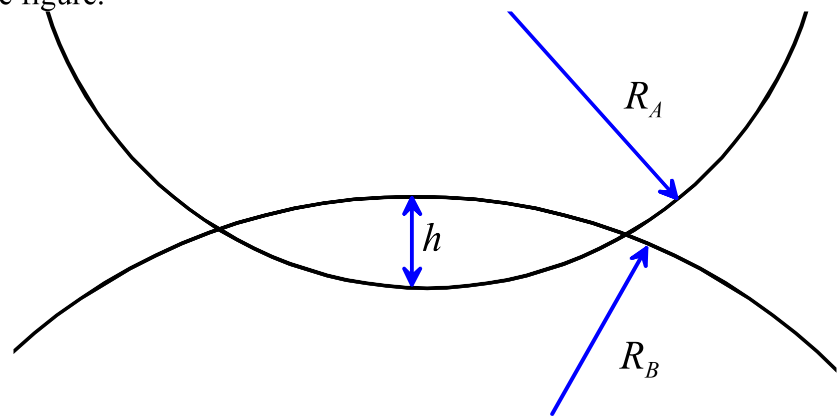
Show that the total strain energy of the solid is

$$E = -\frac{1}{2}\int_B C_{ijkl}(\varepsilon_{ij} - \varepsilon_{ij}^*)\varepsilon_{ij}^* \quad dV$$

To do this, begin by writing down the strain energy within *B*. Then, write down an expression for the total strain energy outside *B* in terms of the traction acting on the boundary of *B*. Then find a way to rearrange the sum of these two terms into the form given above.

4. General Axisymmetric Contact. Suppose that two isotropic, linear elastic spheres with radii $R_A, \ R_B$ and moduli and Poisson's ratios $\mu_A, \quad \nu_{A,} \qquad \mu_B, \quad \nu_{B,}$ are pressed into contact.

Assume that if the two spheres did not deform, they would overlap by a distance h as shown in the figure.



Make the following assumptions:

(1) The radius of the contact area between the two spheres is much smaller than the radius of either sphere.
(2) Both spheres deform as though they were infinite half-spaces. That is to say, the radial displacement of a point on the surface of sphere A due to a point force acting a distance r away on its surface is

$$u = \frac{1-\nu_A}{2\pi\mu_A \ r} \qquad\qquad r << R_A$$

(3) Approximate the profile of each sphere by a parabola.

Write down an integral equation for the contact pressure distribution acting between the spheres, in terms of the sphere radii and the elastic constants.

Compare the result with the integral equation that governs the pressure distribution acting between a rigid sphere and an elastic half-space. Hence, find expressions for the radius of the contact area between the spheres, the contact pressure distribution and the relationship between the load P and the approach of remote points on the spheres h.