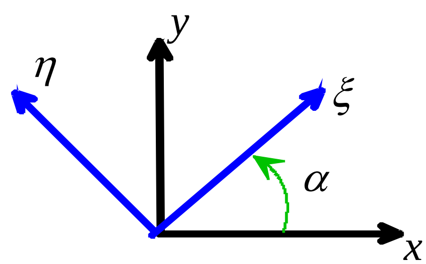
Division of Engineering
Brown University

EN224: Linear Elasticity

Homework 5: Complex variable methods for plane problems

Due Friday April 22, 2005



1. Let $\Omega(z),\ \omega(z)$ be two complex potentials that generate stresses and displacements according to the usual formulation (no continuation)

$$2\mu(u_x + iu_y) = (3-4\nu)\Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}$$

$$\sigma_{xx} + \sigma_{yy} = 2\left\{\Omega'(z) + \overline{\Omega'(z)}\right\}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2\left\{z\overline{\Omega''(z)} + \overline{\omega'(z)}\right\}$$

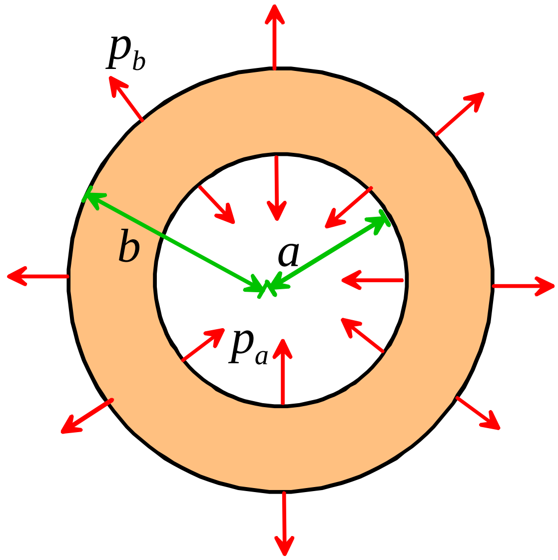
Show that the displacement and stress components in the (ξ,η) basis shown in the figure can be calculated as

$$2\mu(u_\xi + iu_\eta) = \left[(3-4\nu)\Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}\right]e^{-i\alpha}$$

$$\sigma_{\xi\xi} + \sigma_{\eta\eta} = 2\left\{\Omega'(z) + \overline{\Omega'(z)}\right\}$$

$$\sigma_{\xi\xi} - \sigma_{\eta\eta} + 2i\sigma_{\xi\eta} = -2\left\{z\overline{\Omega''(z)} + \overline{\omega'(z)}\right\}e^{-2i\alpha}$$

2. Complex variable solution to a pressurized cylinder.



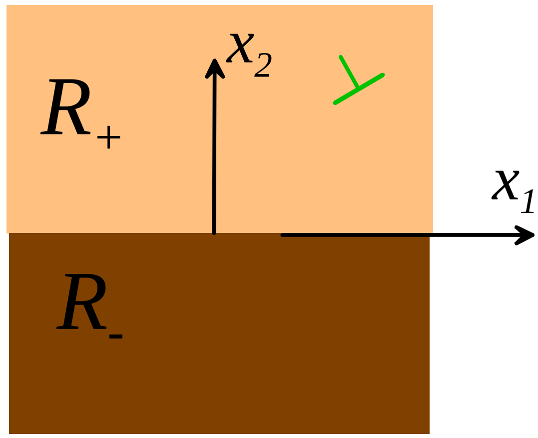
2.1 Using the results of the preceding problem, show that the complex potentials that generate stress and displacement fields in a pressurized cylinder (see above) must satisfy

$$\Omega'(z)+\overline{\Omega'(z)}-\left(z\overline{\Omega''(z)}+\overline{\omega'(z)}\right)e^{-2i\theta}=p_a \qquad z=ae^{i\theta}$$

$$\Omega'(z)+\overline{\Omega'(z)}-\left(z\overline{\Omega''(z)}+\overline{\omega'(z)}\right)e^{-2i\theta}=p_b \qquad z=be^{i\theta}$$

2.2 By expanding $\Omega(z)$ and $\omega(z)$ as Laurent series, find the potentials that solve this problem. To simplify the algebra, note that $e^{-2i\theta}=r^2/z^2 \quad \bar{z}=r^2/z \quad on \quad z=re^{i\theta}$, and assume that the solution can be generated from terms in the series that after substitution in the boundary conditions, are independent of *z*.

3. Dislocation near a rigid interface



3.1 Begin by finding an analytic continuation that automatically satisfies *D=0* on $z=\bar{z}$. To do this, start with the standard complex variable formulation

$$2\mu(u_1+iu_2)=(3-4\nu)\Omega(z)-z\overline{\Omega'(z)}-\overline{\omega(z)}$$

$$\sigma_{11}+\sigma_{22}=2\left\{\Omega'(z)+\overline{\Omega'(z)}\right\}$$

$$\sigma_{11}-\sigma_{22}+2i\sigma_{12}=-2\left\{z\overline{\Omega''(z)}+\overline{\omega'(z)}\right\}$$

Express the boundary condition in terms of Ω and ω defined in $R+$. Next, express the boundary condition in terms of potentials $\overline{\Omega(\bar{z})}$ $\overline{\omega(\bar{z})}$ which are analytic in $R-$. Use the result to show that

$$(3-4\nu)\lim_{z\to L+}\overline{\Omega(z)}=\lim_{z\to L-}\left\{\bar{z}\Omega'(\bar{z})+\omega(\bar{z})\right\}$$

where L denotes the real axis. Hence, conclude that this implies that

$$\Omega(z)=\begin{cases}\Omega(z) & z\in R+\\ \left(z\overline{\Omega'(\bar{z})}+\overline{\omega(\bar{z})}\right)/(3-4\nu) & z\in R-\end{cases}$$

is analytic in R. Use this to calculate an expression for $\omega(z)$ $\quad z\in R+$, and hence show that a solution with *D=0* on $z=\bar{z}$ can be generated by finding a single potential $\Omega(z)$ that is analytic in R, and calculating displacements and stresses from

$$2\mu(u_1+iu_2)=(3-4\nu)\left\{\Omega(z)-\Omega(\bar{z})\right\}+(\bar{z}-z)\overline{\Omega'(z)}$$

$$\sigma_{11}+\sigma_{22}=2\left\{\Omega'(z)+\overline{\Omega'(z)}\right\}$$

$$\sigma_{11}-\sigma_{22}+2i\sigma_{12}=-2\left\{(3-4\nu)\Omega'(\bar{z})-\overline{\Omega'(z)}+(z-\bar{z})\overline{\Omega''(z)}\right\}$$

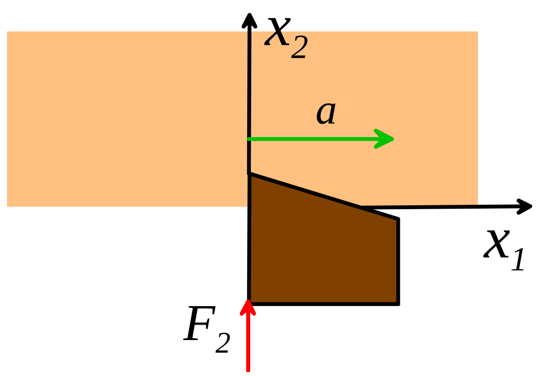
3.2 Let $\Omega_0(z) = i\dfrac{b_1 + ib_2}{8\pi(1-\nu)}\log(z - z_0) \qquad \omega_0(z) = -i\dfrac{b_1 - ib_2}{8\pi(1-\nu)}\log(z - z_0)$ generate the solution for a dislocation at position z_0 in an infinite solid. Deduce that, to satisfy *D=0* on *L*, we must superpose a second potential $\Omega(z)$ satisfying

$$(3-4\nu)\left(\lim_{z\to L+}\Omega(z) - \lim_{z\to L-}\Omega(z)\right) = -\left(3-4\nu\right)\Omega_0(x_1) + x_1\overline{\Omega_0{}'(x_1)} + \overline{\omega_0(x_1)}$$

and calculating stresses and displacements from this potential using the formulation in 3.1.

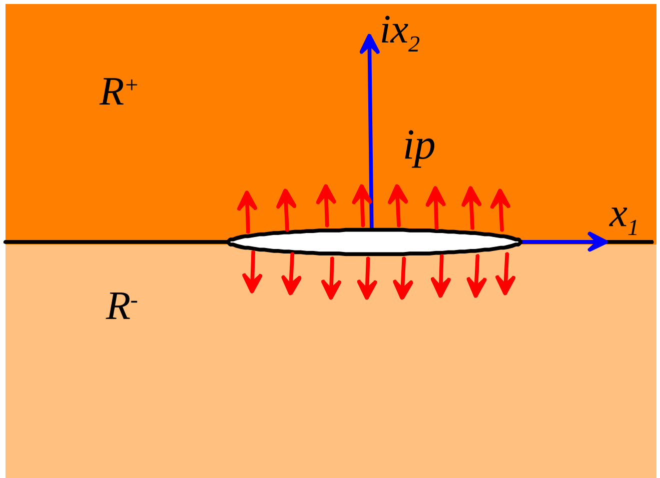
3.3 Using (3.1) as a guide, write down the potential $\Omega(z)$ in terms of $\Omega_0(z)$ and $\omega_0(z)$

4. Stress induced by indentation with a rigid wedge



Suppose that an elastic half-space is indented by a rigid frictionless wedge, with profile $f(x_1) = \varepsilon x_1$. Calculate the potential $\Omega'(z)$ that generates the stress field in the solid in terms of a and F_2. Hence, determine the contact pressure distribution and the slope of the surface for $x_1 > 0$. Use the conditions that the contact pressure cannot be tensile, and the two solids cannot overlap to deduce the relationship between contact width *a* and the force applied to the punch.

5. An alternative solution for the pressurized crack



There is often more than one choice of analytic continuation for a particular boundary value problem. To illustrate this, in this problem we will devise an alternative procedure to solve the pressurized crack problem that was discussed in class.

Consider a crack that is subjected to equal and opposite tractions $t_1 + it_2 = ip$ on its faces. Symmetry conditions imply that $u_2 = 0 \quad \sigma_{12} = 0$ on $z = \overline{z}$ outside the crack. Moreover, it is evidently sufficient to find a solution in the upper half-plane, since the solution in the lower half-plane follows by symmetry.

5.1 Starting with the standard complex variable formulation

$$2\mu(u_1 + iu_2) = (3 - 4\nu)\Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}$$

$$\sigma_{11} + \sigma_{22} = 2\left\{\Omega'(z) + \overline{\Omega'(z)}\right\}$$

$$\sigma_{22} - i\sigma_{12} = \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{\omega'(z)}$$

show that setting $\omega(z) = \Omega(z) - z\Omega'(z)$ will automatically satisfy $\sigma_{12} = 0$ on $z = \overline{z}$.

5.2 With this choice of $\omega(z)$, show that the condition that $u_2 = 0$ on $z = \overline{z}$ implies that

$$\lim_{z \to L+} \Omega(z) = \lim_{z \to L-} \overline{\Omega(\overline{z})}$$

showing that

$$\theta(z) = \begin{cases} \Omega(z) & z \in R+ \\ \overline{\Omega(\overline{z})} & z \in R- \end{cases}$$

is continuous outside the crack, and analytic in the whole plane. Deduce that $\overline{\Omega(z)} = \theta(\overline{z})$.

5.3 Hence show that traction boundary condition on the crack faces leads to a Hilbert problem for $\theta(z)$

5.4 Write down the general solution to the Hilbert problem.

5.5 Hence, find an expression for the stress intensity factors induced at the right hand crack tip by a pair of equal and opposite point forces acting on the crack faces.